This is Volume 3 of the five-volume book Mathematical Inequalities, which introduces and develops the main types of elementary inequalities. The first three volumes are a great opportunity to look into many old and new inequalities, as well as elementary procedures for solving them: Volume 1 - Symmetric Polynomial Inequalities, Volume 2 - Symmetric Rational and Nonrational Inequalities, Volume 3 - Cyclic and Noncyclic Inequalities. As a rule, the inequalities in these volumes are increasingly ordered according to the number of variables: two, three, four, ... , n-variables. The last two volumes (Volume 4 – Extensions and Refinements of Jensen's Inequality, Volume 5 – Other Recent Methods for Creating and Solving Inequalities) present beautiful and original methods for solving inequalities, such as Half/Partial convex function method, Equal variables method, Arithmetic compensation method, Highest coefficient cancellation method, pqr method etc. The book is intended for a wide audience: advanced middle school students, high school students, college and university students, and teachers. Many problems and methods can be used as group projects for advanced high school students.

The author, Vasile Cirtoaje, is a Professor at the Department of Automatic Control and Computers from the University of Ploiesti, Romania. He is the author of many well-known interesting and delightful inequalities, as well as strong methods for creating and proving mathematical inequalities.

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Cyclic and Noncyclic Inequalities

Cirtoaje

Vasile Cirtoaje

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Chapter 1

Cyclic Inequalities

1.1 Applications

1.1. If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then

$$ab^2 + bc^2 + ca^2 \leq 4.$$

1.2. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$(ab + bc + ca)(ab^2 + bc^2 + ca^2) \leq 9.$$

1.3. If a, b, c are nonnegative real numbers such that $a^2 + b^2 + c^2 = 3$, then

(a) $$ab^2 + bc^2 + ca^2 \leq abc + 2;$$

(b) $$\frac{a}{b+2} + \frac{b}{c+2} + \frac{c}{a+2} \leq 1.$$

1.4. If $a, b, c \geq 1$, then

(a) $$2(ab^2 + bc^2 + ca^2) + 3 \geq 3(ab + bc + ca);$$

(b) $$ab^2 + bc^2 + ca^2 + 6 \geq 3(a + b + c).$$

1.5. If a,b,c are nonnegative real numbers such that

$$a+b+c=3, \qquad a \ge b \ge c,$$

then

(a) $$a^2b+b^2c+c^2a \ge ab+bc+ca;$$

(b) $$8(ab^2+bc^2+ca^2)+3abc \le 27;$$

(c) $$\frac{18}{a^2b+b^2c+c^2a} \le \frac{1}{abc}+5.$$

1.6. If a,b,c are nonnegative real numbers such that

$$a^2+b^2+c^2=3, \qquad a \ge b \ge c,$$

then

$$ab^2+bc^2+ca^2 \le \frac{3}{4}(ab+bc+ca+1).$$

1.7. If a,b,c are nonnegative real numbers such that $a^2+b^2+c^2=3$, then

$$a^2b^3+b^2c^3+c^2a^3 \le 3.$$

1.8. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

$$a^4b^2+b^4c^2+c^4a^2+4 \ge a^3b^3+b^3c^3+c^3a^3.$$

1.9. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

(a) $$ab^2+bc^2+ca^2+abc \le 4;$$

(b) $$\frac{a}{4-b}+\frac{b}{4-c}+\frac{c}{4-a} \le 1;$$

(c) $$ab^3+bc^3+ca^3+(ab+bc+ca)^2 \le 12;$$

(d) $$\frac{ab^2}{1+a+b}+\frac{bc^2}{1+b+c}+\frac{ca^2}{1+c+a} \le 1.$$

1.10. If a,b,c are positive real numbers, then

$$\frac{1}{a(a+2b)}+\frac{1}{b(b+2c)}+\frac{1}{c(c+2a)} \ge \frac{3}{ab+bc+ca}.$$

1.11. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a}{b^2+2c}+\frac{b}{c^2+2a}+\frac{c}{a^2+2b} \geq 1.$$

1.12. If a, b, c are positive real numbers such that $a+b+c \geq 3$, then

$$\frac{a-1}{b+1}+\frac{b-1}{c+1}+\frac{c-1}{a+1} \geq 0.$$

1.13. If a, b, c are positive real numbers such that $a+b+c=3$, then

(a) $$\frac{1}{2ab^2+1}+\frac{1}{2bc^2+1}+\frac{1}{2ca^2+1} \geq 1;$$

(b) $$\frac{1}{ab^2+2}+\frac{1}{bc^2+2}+\frac{1}{ca^2+2} \geq 1.$$

1.14. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{ab}{9-4bc}+\frac{bc}{9-4ca}+\frac{ca}{9-4ab} \leq \frac{3}{5}.$$

1.15. If a, b, c are positive real numbers such that $a+b+c=3$, then

(a) $$\frac{a^2}{2a+b^2}+\frac{b^2}{2b+c^2}+\frac{c^2}{2c+a^2} \geq 1;$$

(b) $$\frac{a^2}{a+2b^2}+\frac{b^2}{b+2c^2}+\frac{c^2}{c+2a^2} \geq 1.$$

1.16. Let a, b, c be positive real numbers such that $a+b+c=3$. Then,

$$\frac{1}{a+b^2+c^3}+\frac{1}{b+c^2+a^3}+\frac{1}{c+a^2+b^3} \leq 1.$$

1.17. If a, b, c are positive real numbers, then

$$\frac{1+a^2}{1+b+c^2}+\frac{1+b^2}{1+c+a^2}+\frac{1+c^2}{1+a+b^2} \geq 2.$$

1.18. If a, b, c are nonnegative real numbers, then

$$\frac{a}{4a+4b+c}+\frac{b}{4b+4c+a}+\frac{c}{4c+4a+b}\le\frac{1}{3}.$$

1.19. If a, b, c are positive real numbers, then

$$\frac{a+b}{a+7b+c}+\frac{b+c}{b+7c+a}+\frac{c+a}{c+7a+b}\ge\frac{2}{3}.$$

1.20. If a, b, c are positive real numbers, then

$$\frac{a+b}{a+3b+c}+\frac{b+c}{b+3c+a}+\frac{c+a}{c+3a+b}\ge\frac{6}{5}.$$

1.21. If a, b, c are positive real numbers, then

$$\frac{2a+b}{2a+c}+\frac{2b+c}{2b+a}+\frac{2c+a}{2c+b}\ge 3.$$

1.22. If a, b, c are positive real numbers, then

$$\frac{a(a+b)}{a+c}+\frac{b(b+c)}{b+a}+\frac{c(c+a)}{c+b}\le\frac{3(a^2+b^2+c^2)}{a+b+c}.$$

1.23. If a, b, c are real numbers, then

$$\frac{a^2-bc}{4a^2+b^2+4c^2}+\frac{b^2-ca}{4b^2+c^2+4a^2}+\frac{c^2-ab}{4c^2+a^2+4b^2}\ge 0.$$

1.24. If a, b, c are real numbers, then

(a) $$a(a+b)^3+b(b+c)^3+c(c+a)^3\ge 0;$$

(b) $$a(a+b)^5+b(b+c)^5+c(c+a)^5\ge 0.$$

1.25. If a, b, c are real numbers, then

$$3(a^4+b^4+c^4)+4(a^3b+b^3c+c^3a)\ge 0.$$

1.26. If a, b, c are positive real numbers, then

$$\frac{(a-b)(2a+b)}{(a+b)^2}+\frac{(b-c)(2b+c)}{(b+c)^2}+\frac{(c-a)(2c+a)}{(c+a)^2}\ge 0.$$

1.27. If a, b, c are positive real numbers, then

$$\frac{(a-b)(2a+b)}{a^2+ab+b^2}+\frac{(b-c)(2b+c)}{b^2+bc+c^2}+\frac{(c-a)(2c+a)}{c^2+ca+a^2}\ge 0.$$

1.28. If a, b, c are positive real numbers, then

$$\frac{(a-b)(3a+b)}{a^2+b^2}+\frac{(b-c)(3b+c)}{b^2+c^2}+\frac{(c-a)(3c+a)}{c^2+a^2}\ge 0.$$

1.29. Let a, b, c be positive real numbers such that $abc = 1$. Then,

$$\frac{1}{1+a+b^2}+\frac{1}{1+b+c^2}+\frac{1}{1+c+a^2}\le 1.$$

1.30. Let a, b, c be positive real numbers such that $abc = 1$. Then,

$$\frac{a}{(a+1)(b+2)}+\frac{b}{(b+1)(c+2)}+\frac{c}{(c+1)(a+2)}\ge \frac{1}{2}.$$

1.31. If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then

$$(a+2b)(b+2c)(c+2a)\ge 27.$$

1.32. If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then

$$\frac{a}{a+a^3+b}+\frac{b}{b+b^3+c}+\frac{c}{c+c^3+a}\le 1.$$

1.33. If a, b, c are positive real numbers such that $a \ge b \ge c$ and $ab + bc + ca = 3$, then

$$\frac{1}{a+2b}+\frac{1}{b+2c}+\frac{1}{c+2a}\ge 1.$$

1.34. If $a, b, c \in [0, 1]$, then

$$\frac{a}{4b^2+5}+\frac{b}{4c^2+5}+\frac{c}{4a^2+5} \ge \frac{1}{3}.$$

1.35. If $a, b, c \in \left[\frac{1}{3}, 3\right]$, then

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} \ge \frac{7}{5}.$$

1.36. If $a, b, c \in \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$, then

$$\frac{3}{a+2b}+\frac{3}{b+2c}+\frac{3}{c+2a} \ge \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}.$$

1.37. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{4abc}{ab^2+bc^2+ca^2+abc}+\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 2.$$

1.38. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\frac{1}{ab^2+8}+\frac{1}{bc^2+8}+\frac{1}{ca^2+8} \ge \frac{1}{3}.$$

1.39. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\frac{ab}{bc+3}+\frac{bc}{ca+3}+\frac{ca}{ab+3} \le \frac{3}{4}.$$

1.40. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

(a) $$\frac{a}{b^2+3}+\frac{b}{c^2+3}+\frac{c}{a^2+3} \ge \frac{3}{4};$$

(b) $$\frac{a}{b^3+1}+\frac{b}{c^3+1}+\frac{c}{a^3+1} \ge \frac{3}{2}.$$

1.41. Let a, b, c be positive real numbers, and let

$$x = a + \frac{1}{b} - 1, \quad y = b + \frac{1}{c} - 1, \quad z = c + \frac{1}{a} - 1.$$

Prove that

$$xy + yz + zx \geq 3.$$

1.42. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - \frac{1}{b} - \sqrt{2}\right)^2 + \left(b - \frac{1}{c} - \sqrt{2}\right)^2 + \left(c - \frac{1}{a} - \sqrt{2}\right)^2 \geq 6.$$

1.43. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left|1 + a - \frac{1}{b}\right| + \left|1 + b - \frac{1}{c}\right| + \left|1 + c - \frac{1}{a}\right| > 2.$$

1.44. If a, b, c are different positive real numbers, then

$$\left|1 + \frac{a}{b-c}\right| + \left|1 + \frac{b}{c-a}\right| + \left|1 + \frac{c}{a-b}\right| > 2.$$

1.45. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(2a - \frac{1}{b} - \frac{1}{2}\right)^2 + \left(2b - \frac{1}{c} - \frac{1}{2}\right)^2 + \left(2c - \frac{1}{a} - \frac{1}{2}\right)^2 \geq \frac{3}{4}.$$

1.46. Let

$$x = a + \frac{1}{b} - \frac{5}{4}, \quad y = b + \frac{1}{c} - \frac{5}{4}, \quad z = c + \frac{1}{a} - \frac{5}{4},$$

where $a \geq b \geq c > 0$. Prove that

$$xy + yz + zx \geq \frac{27}{16}.$$

1.47. Let a, b, c be positive real numbers, and let

$$E = \left(a + \frac{1}{a} - \sqrt{3}\right)\left(b + \frac{1}{b} - \sqrt{3}\right)\left(c + \frac{1}{c} - \sqrt{3}\right);$$

$$F = \left(a + \frac{1}{b} - \sqrt{3}\right)\left(b + \frac{1}{c} - \sqrt{3}\right)\left(c + \frac{1}{a} - \sqrt{3}\right).$$

Prove that $E \geq F$.

1.48. If a,b,c are positive real numbers such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=5$, then

$$\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\ge\frac{17}{4}.$$

1.49. If a,b,c are positive real numbers, then

(a) $$1+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 2\sqrt{1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}\,;$$

(b) $$1+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\ge\sqrt{1+16\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)}\,;$$

(c) $$3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 2\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\,.$$

1.50. If a,b,c are positive real numbers, then

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+15\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\ge 16\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right).$$

1.51. If a,b,c are positive real numbers such that $abc=1$, then

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge a+b+c;$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{3}{2}(a+b+c-1);$$

(c) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+2\ge\frac{5}{3}(a+b+c).$$

1.52. If a,b,c are positive real numbers such that $a^2+b^2+c^2=3$, then

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 2+\frac{3}{ab+bc+ca};$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{9}{a+b+c}.$$

1.53. If a,b,c are positive real numbers such that $a^2+b^2+c^2=3$, then

$$6\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+5(ab+bc+ca)\ge 33.$$

1.54. If a, b, c are positive real numbers such that $a+b+c=3$, then

(a) $$6\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3 \geq 7(a^2+b^2+c^2);$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq a^2+b^2+c^2.$$

1.55. If a, b, c are positive real numbers, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+2 \geq \frac{14(a^2+b^2+c^2)}{(a+b+c)^2}.$$

1.56. Let a, b, c be positive real numbers such that $a+b+c=3$, and let

$$x=3a+\frac{1}{b}, \quad y=3b+\frac{1}{c}, \quad z=3c+\frac{1}{a}.$$

Prove that

$$xy+yz+zx \geq 48.$$

1.57. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a+1}{b}+\frac{b+1}{c}+\frac{c+1}{a} \geq 2(a^2+b^2+c^2).$$

1.58. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+3 \geq 2(a^2+b^2+c^2).$$

1.59. If a, b, c are positive real numbers, then

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+2(ab+bc+ca) \geq 3(a^2+b^2+c^2).$$

1.60. If a, b, c are positive real numbers such that $a^4+b^4+c^4=3$, then

(a) $$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq 3;$$

(b) $$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \geq \frac{3}{2}.$$

1.61. If a, b, c are positive real numbers, then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \ge \frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}.$$

1.62. If a, b, c are positive real numbers, then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+a+b+c \ge 2\sqrt{(a^2+b^2+c^2)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}.$$

1.63. If a, b, c are positive real numbers, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+32\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right) \ge 51.$$

1.64. Find the greatest positive real number K such that the inequalities below hold for any positive real numbers a, b, c:

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3 \ge K\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\right);$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3+K\left(\frac{a}{2a+b}+\frac{b}{2b+c}+\frac{c}{2c+a}-1\right) \ge 0.$$

1.65. If $a, b, c \in \left[\frac{1}{2}, 2\right]$, then

(a) $$8\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge 5\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+9;$$

(b) $$20\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge 17\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

1.66. If a, b, c are positive real numbers such that $a \le b \le c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge \frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}.$$

1.67. Let a, b, c be positive real numbers such that $abc = 1$.

(a) If $a \le b \le c$, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge a^{3/2} + b^{3/2} + c^{3/2};$$

(b) If $a \le 1 \le b \le c$, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge a^{\sqrt{3}} + b^{\sqrt{3}} + c^{\sqrt{3}}.$$

1.68. If k and a, b, c are positive real numbers, then

$$\frac{1}{(k+1)a+b} + \frac{1}{(k+1)b+c} + \frac{1}{(k+1)c+a} \ge \frac{1}{ka+b+c} + \frac{1}{kb+c+a} + \frac{1}{kc+a+b}.$$

1.69. If a, b, c are positive real numbers, then

(a) $$\frac{a}{\sqrt{2a+b}} + \frac{b}{\sqrt{2b+c}} + \frac{c}{\sqrt{2c+a}} \le \sqrt{a+b+c};$$

(b) $$\frac{a}{\sqrt{a+2b}} + \frac{b}{\sqrt{b+2c}} + \frac{c}{\sqrt{c+2a}} \ge \sqrt{a+b+c}.$$

1.70. Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that

$$a\sqrt{\frac{a+2b}{3}} + b\sqrt{\frac{b+2c}{3}} + c\sqrt{\frac{c+2a}{3}} \le 3.$$

1.71. If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then

$$a\sqrt{1+b^3} + b\sqrt{1+c^3} + c\sqrt{1+a^3} \le 5.$$

1.72. If a, b, c are positive real numbers such that $abc = 1$, then

(a) $$\sqrt{\frac{a}{b+3}} + \sqrt{\frac{b}{c+3}} + \sqrt{\frac{c}{a+3}} \ge \frac{3}{2};$$

(b) $$\sqrt[3]{\frac{a}{b+7}} + \sqrt[3]{\frac{b}{c+7}} + \sqrt[3]{\frac{c}{a+7}} \ge \frac{3}{2}.$$

1.73. If a,b,c are positive real numbers, then

$$\left(1+\frac{4a}{a+b}\right)^2+\left(1+\frac{4b}{b+c}\right)^2+\left(1+\frac{4c}{c+a}\right)^2\ge 27.$$

1.74. If a,b,c are positive real numbers, then

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+a}}\le 3.$$

1.75. If a,b,c are nonnegative real numbers, then

$$\sqrt{\frac{a}{4a+5b}}+\sqrt{\frac{b}{4b+5c}}+\sqrt{\frac{c}{4c+5a}}\le 1.$$

1.76. If a,b,c are positive real numbers, then

$$\frac{a}{\sqrt{4a^2+ab+4b^2}}+\frac{b}{\sqrt{4b^2+bc+4c^2}}+\frac{c}{\sqrt{4c^2+ca+4a^2}}\le 1.$$

1.77. If a,b,c are positive real numbers, then

$$\sqrt{\frac{a}{a+b+7c}}+\sqrt{\frac{b}{b+c+7a}}+\sqrt{\frac{c}{c+a+7b}}\ge 1.$$

1.78. If a,b,c are nonnegative real numbers, no two of which are zero, then

(a) $$\sqrt{\frac{a}{3b+c}}+\sqrt{\frac{b}{3c+a}}+\sqrt{\frac{c}{3a+b}}\ge\frac{3}{2};$$

(b) $$\sqrt{\frac{a}{2b+c}}+\sqrt{\frac{b}{2c+a}}+\sqrt{\frac{c}{2a+b}}\ge\sqrt[4]{8}.$$

1.79. If a,b,c are positive real numbers such that $ab+bc+ca=3$, then

(a) $$\frac{1}{(a+b)(3a+b)}+\frac{1}{(b+c)(3b+c)}+\frac{1}{(c+a)(3c+a)}\ge\frac{3}{8};$$

(b) $$\frac{1}{(2a+b)^2}+\frac{1}{(2b+c)^2}+\frac{1}{(2c+a)^2}\ge\frac{1}{3}.$$

1.80. If a, b, c are nonnegative real numbers, then

$$a^4 + b^4 + c^4 + 15(a^3b + b^3c + c^3a) \geq \frac{47}{4}(a^2b^2 + b^2c^2 + c^2a^2).$$

1.81. If a, b, c are nonnegative real numbers such that $a + b + c = 4$, then

$$a^3b + b^3c + c^3a \leq 27.$$

1.82. Let a, b, c be nonnegative real numbers such that

$$a^2 + b^2 + c^2 = \frac{10}{3}(ab + bc + ca).$$

Prove that

$$a^4 + b^4 + c^4 \geq \frac{82}{27}(a^3b + b^3c + c^3a).$$

1.83. If a, b, c are positive real numbers, then

$$\frac{a^3}{2a^2 + b^2} + \frac{b^3}{2b^2 + c^2} + \frac{c^3}{2c^2 + a^2} \geq \frac{a + b + c}{3}.$$

1.84. If a, b, c are positive real numbers, then

$$\frac{a^4}{a^3 + b^3} + \frac{b^4}{b^3 + c^3} + \frac{c^4}{c^3 + a^3} \geq \frac{a + b + c}{2}.$$

1.85. If a, b, c are positive real numbers such that $abc = 1$, then

(a) $$3\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right) + 4\left(\frac{b}{a^2} + \frac{c}{b^2} + \frac{a}{c^2}\right) \geq 7(a^2 + b^2 + c^2);$$

(b) $$8\left(\frac{a^3}{b} + \frac{b^3}{c} + \frac{c^3}{a}\right) + 5\left(\frac{b}{a^3} + \frac{c}{b^3} + \frac{a}{c^3}\right) \geq 13(a^3 + b^3 + c^3).$$

1.86. If a, b, c are positive real numbers, then

$$\frac{ab}{b^2 + bc + c^2} + \frac{bc}{c^2 + ca + a^2} + \frac{ca}{a^2 + ab + b^2} \leq \frac{a^2 + b^2 + c^2}{ab + bc + ca}.$$

1.87. If a, b, c are positive real numbers, then

$$\frac{a-b}{b(2b+c)}+\frac{b-c}{c(2c+a)}+\frac{c-a}{a(2a+b)} \ge 0.$$

1.88. If a, b, c are positive real numbers, then

(a) $$\frac{a^2+6bc}{ab+2bc}+\frac{b^2+6ca}{bc+2ca}+\frac{c^2+6ab}{ca+2ab} \ge 7;$$

(b) $$\frac{a^2+7bc}{ab+bc}+\frac{b^2+7ca}{bc+ca}+\frac{c^2+7ab}{ca+ab} \ge 12.$$

1.89. If a, b, c are positive real numbers, then

(a) $$\frac{ab}{2b+c}+\frac{bc}{2c+a}+\frac{ca}{2a+b} \le \frac{a^2+b^2+c^2}{a+b+c};$$

(b) $$\frac{ab}{b+c}+\frac{bc}{c+a}+\frac{ca}{a+b} \le \frac{3(a^2+b^2+c^2)}{2(a+b+c)};$$

(c) $$\frac{ab}{4b+5c}+\frac{bc}{4c+5a}+\frac{ca}{4a+5b} \le \frac{a^2+b^2+c^2}{3(a+b+c)}.$$

1.90. If a, b, c are positive real numbers, then

(a) $$a\sqrt{b^2+8c^2}+b\sqrt{c^2+8a^2}+c\sqrt{a^2+8b^2} \le (a+b+c)^2;$$

(b) $$a\sqrt{b^2+3c^2}+b\sqrt{c^2+3a^2}+c\sqrt{a^2+3b^2} \le a^2+b^2+c^2+ab+bc+ca.$$

1.91. If a, b, c are positive real numbers, then

(a) $$\frac{1}{a\sqrt{a+2b}}+\frac{1}{b\sqrt{b+2c}}+\frac{1}{c\sqrt{c+2a}} \ge \sqrt{\frac{3}{abc}};$$

(b) $$\frac{1}{a\sqrt{a+8b}}+\frac{1}{b\sqrt{b+8c}}+\frac{1}{c\sqrt{c+8a}} \ge \sqrt{\frac{1}{abc}}.$$

1.92. If a, b, c are positive real numbers, then

$$\frac{a}{\sqrt{5a+4b}}+\frac{b}{\sqrt{5b+4c}}+\frac{c}{\sqrt{5c+4a}} \le \sqrt{\frac{a+b+c}{3}}.$$

1.93. If a, b, c are positive real numbers, then

(a) $$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}} \geq \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{2}};$$

(b) $$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}} \geq \sqrt[4]{\frac{27(ab+bc+ca)}{4}}.$$

1.94. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\sqrt{3a+b^2}+\sqrt{3b+c^2}+\sqrt{3c+a^2} \geq 6.$$

1.95. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$\sqrt{a^2+b^2+2bc}+\sqrt{b^2+c^2+2ca}+\sqrt{c^2+a^2+2ab} \geq 2(a+b+c).$$

1.96. If a, b, c are nonnegative real numbers, then

$$\sqrt{a^2+b^2+7bc}+\sqrt{b^2+c^2+7ca}+\sqrt{c^2+a^2+7ab} \geq 3\sqrt{3(ab+bc+ca)}.$$

1.97. If a, b, c are positive real numbers, then

$$\frac{a^2+3ab}{(b+c)^2}+\frac{b^2+3bc}{(c+a)^2}+\frac{c^2+3ca}{(a+b)^2} \geq 3.$$

1.98. If a, b, c are positive real numbers, then

$$\frac{a^2b+1}{a(b+1)}+\frac{b^2c+1}{b(c+1)}+\frac{c^2a+1}{c(a+1)} \geq 3.$$

1.99. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\sqrt{a^3+3b}+\sqrt{b^3+3c}+\sqrt{c^3+3a} \geq 6.$$

1.100. If a, b, c are positive real numbers such that $abc=1$, then

$$\sqrt{\frac{a}{a+6b+2bc}}+\sqrt{\frac{b}{b+6c+2ca}}+\sqrt{\frac{c}{c+6a+2ab}} \geq 1.$$

1.101. If a, b, c are positive real numbers such that $abc = 1$, then

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2 \ge 6(a+b+c-1).$$

1.102. If a, b, c are positive real numbers, then

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a} \ge \frac{a+b+c}{a+b+c-\sqrt[3]{abc}}.$$

1.103. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$a\sqrt{b^2+b+1}+b\sqrt{c^2+c+1}+c\sqrt{a^2+a+1} \le 3\sqrt{3}.$$

1.104. If a, b, c are positive real numbers, then

$$\frac{1}{b(a+2b+3c)^2}+\frac{1}{c(b+2c+3a)^2}+\frac{1}{a(c+2a+3b)^2} \le \frac{1}{12abc}.$$

1.105. Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that

(a) $$\frac{a^2+9b}{b+c}+\frac{b^2+9c}{c+a}+\frac{c^2+9a}{a+b} \ge 15;$$

(b) $$\frac{a^2+3b}{a+b}+\frac{b^2+3c}{b+c}+\frac{c^2+3a}{c+a} \ge 6.$$

1.106. If $a, b, c \in [0,1]$, then

(a) $$\frac{bc}{2ab+1}+\frac{ca}{2bc+1}+\frac{ab}{2ca+1} \le 1.$$

(b) $$\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1} \le \frac{3}{2}.$$

1.107. If a, b, c are nonnegative real numbers, then

$$a^4+b^4+c^4+5(a^3b+b^3c+c^3a) \ge 6(a^2b^2+b^2c^2+c^2a^2).$$

1.108. If a, b, c are positive real numbers, then

$$a^5 + b^5 + c^5 - a^4b - b^4c - c^4a \geq 2abc(a^2 + b^2 + c^2 - ab - bc - ca).$$

1.109. If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} \geq \frac{3}{2}.$$

1.110. If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then

$$a\sqrt{a+b} + b\sqrt{b+c} + c\sqrt{c+a} \geq 3\sqrt{2}.$$

1.111. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$\frac{a}{2b^2+c} + \frac{b}{2c^2+a} + \frac{c}{2a^2+b} \geq 1.$$

1.112. If a, b, c are positive real numbers such that $a + b + c = ab + bc + ca$, then

$$\frac{1}{a^2+b+1} + \frac{1}{b^2+c+1} + \frac{1}{c^2+a+1} \leq 1.$$

1.113. If a, b, c are positive real numbers, then

$$\frac{1}{(a+2b+3c)^2} + \frac{1}{(b+2c+3a)^2} + \frac{1}{(c+2a+3b)^2} \leq \frac{1}{4(ab+bc+ca)}.$$

1.114. If a, b, c are positive real numbers, then

$$\sqrt{\frac{a}{a+b+2c}} + \sqrt{\frac{b}{b+c+2a}} + \sqrt{\frac{c}{c+a+2b}} \leq \frac{3}{2}.$$

1.115. If a, b, c are positive real numbers, then

$$\sqrt{\frac{5a}{a+b+3c}} + \sqrt{\frac{5b}{b+c+3a}} + \sqrt{\frac{5c}{c+a+3b}} \leq 3.$$

1.116. If $a, b, c \in [0, 1]$, then

$$ab^2 + bc^2 + ca^2 + \frac{5}{4} \ge a + b + c.$$

1.117. If a, b, c are nonnegative real numbers such that

$$a + b + c = 3, \quad a \le b \le 1 \le c,$$

then

$$a^2b + b^2c + c^2a \le 3.$$

1.118. Let a, b, c be nonnegative real numbers such that

$$a + b + c = 3, \quad a \le 1 \le b \le c.$$

Prove that

(a) $$a^2b + b^2c + c^2a \ge ab + bc + ca;$$

(b) $$a^2b + b^2c + c^2a \ge abc + 2;$$

(c) $$\frac{1}{abc} + 2 \ge \frac{9}{a^2b + b^2c + c^2a};$$

(d) $$ab^2 + bc^2 + ca^2 \ge 3.$$

1.119. If a, b, c are nonnegative real numbers such that

$$a + b + c = 3, \quad a \le 1 \le b \le c,$$

then

(a) $$\frac{5-2a}{1+b} + \frac{5-2b}{1+c} + \frac{5-2c}{1+a} \ge \frac{9}{2};$$

(b) $$\frac{3-2b}{1+a} + \frac{3-2c}{1+b} + \frac{3-2a}{1+c} \le \frac{3}{2}.$$

1.120. If a, b, c are nonnegative real numbers such that

$$ab + bc + ca = 3, \quad a \le 1 \le b \le c,$$

then

(a) $$a^2b + b^2c + c^2a \ge 3;$$

(b) $$ab^2 + bc^2 + ca^2 + 3(\sqrt{3} - 1)abc \ge 3\sqrt{3}.$$

1.121. If a, b, c are nonnegative real numbers such that

$$a^2+b^2+c^2=3, \qquad a \le 1 \le b \le c,$$

then

(a) $$a^2b+b^2c+c^2a \ge 2abc+1;$$

(b) $$2(ab^2+bc^2+ca^2) \ge 3abc+3.$$

1.122. If a, b, c are nonnegative real numbers such that

$$ab+bc+ca=3, \qquad a \le b \le 1 \le c,$$

then

$$ab^2+bc^2+ca^2+3abc \ge 6.$$

1.123. If a, b, c are nonnegative real numbers such that

$$a^2+b^2+c^2=3, \qquad a \le b \le 1 \le c,$$

then

$$2(a^2b+b^2c+c^2a) \le 3abc+3.$$

1.124. If a, b, c are nonnegative real numbers such that

$$a^2+b^2+c^2=3, \qquad a \le b \le 1 \le c,$$

then

$$2(a^3b+b^3c+c^3a) \le abc+5.$$

1.125. If a,b,c are real numbers, then

$$(a^2+b^2+c^2)^2 \ge 3(a^3b+b^3c+c^3a).$$

1.126. If a,b,c are real numbers, then

$$a^4+b^4+c^4+ab^3+bc^3+ca^3 \ge 2(a^3b+b^3c+c^3a).$$

1.127. If a, b, c are positive real numbers, then

(a) $$\frac{a^2}{ab+2c^2}+\frac{b^2}{bc+2a^2}+\frac{c^2}{ca+2b^2}\ge 1;$$

(b) $$\frac{a^3}{a^2b+2c^3}+\frac{b^3}{b^2c+2a^3}+\frac{c^3}{c^2a+2b^3}\ge 1.$$

1.128. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1}\ge \frac{3}{2}.$$

1.129. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a}{3a+b^2}+\frac{b}{3b+c^2}+\frac{c}{3c+a^2}\le \frac{3}{2}.$$

1.130. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a}{b^2+c}+\frac{b}{c^2+a}+\frac{c}{a^2+b}\ge \frac{3}{2}.$$

1.131. If a, b, c are positive real numbers such that $abc=1$, then

$$\frac{a}{b^3+2}+\frac{b}{c^3+2}+\frac{c}{a^3+2}\ge 1.$$

1.132. Let a, b, c be positive real numbers such that

$$a^m+b^m+c^m=3,$$

where $m>0$. Prove that

$$\frac{a^{m-1}}{b}+\frac{b^{m-1}}{c}+\frac{c^{m-1}}{a}\ge 3.$$

1.133. If a, b, c are positive real numbers, then

(a) $$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\ge 3\left(\frac{1}{3a+b}+\frac{1}{3b+c}+\frac{1}{3c+a}\right);$$

(b) $$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{a+3b}+\frac{1}{b+3c}+\frac{1}{c+3a}\ge 2\left(\frac{1}{3a+b}+\frac{1}{3b+c}+\frac{1}{3c+a}\right).$$

1.134. If a, b, c are positive real numbers such that $a^6 + b^6 + c^6 = 3$, then

$$\frac{a^5}{b} + \frac{b^5}{c} + \frac{c^5}{a} \geq 3.$$

1.135. If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then

$$\frac{a^3}{a + b^5} + \frac{b^3}{b + c^5} + \frac{c^3}{c + a^5} \geq \frac{3}{2}.$$

1.136. If a, b, c are real numbers such that $a^2 + b^2 + c^2 = 3$, then

$$a^2b + b^2c + c^2a + 9 \geq 4(a + b + c).$$

1.137. If a, b, c are real numbers such that $a^2 + b^2 + c^2 = 3$, then

$$a^2b + b^2c + c^2a + 3 \geq a + b + c + ab + bc + ca.$$

1.138. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$\frac{12}{a^2b + b^2c + c^2a} \leq 3 + \frac{1}{abc}.$$

1.139. If a, b, c are positive real numbers such that $a + b + c = 3$, then

$$\frac{24}{a^2b + b^2c + c^2a} + \frac{1}{abc} \geq 9.$$

1.140. Let a, b, c be nonnegative real numbers such that

$$2(a^2 + b^2 + c^2) = 5(ab + bc + ca).$$

Prove that

(a) $$8(a^4 + b^4 + c^4) \geq 17(a^3b + b^3c + c^3a);$$

(b) $$16(a^4 + b^4 + c^4) \geq 34(a^3b + b^3c + c^3a) + 81abc(a + b + c).$$

1.141. Let a, b, c be nonnegative real numbers such that

$$2(a^2+b^2+c^2)=5(ab+bc+ca).$$

Prove that

(a) $$2(a^3b+b^3c+c^3a)\geq a^2b^2+b^2c^2+c^2a^2+abc(a+b+c);$$

(b) $$11(a^4+b^4+c^4)\geq 17(a^3b+b^3c+c^3a)+129abc(a+b+c);$$

(c) $$a^3b+b^3c+c^3a\leq \frac{14+\sqrt{102}}{8}(a^2b^2+b^2c^2+c^2a^2).$$

1.142. If a, b, c are real numbers such that

$$a^3b+b^3c+c^3a\leq 0,$$

then

$$a^2+b^2+c^2\geq k(ab+bc+ca),$$

where

$$k=\frac{1+\sqrt{21+8\sqrt{7}}}{2}\approx 3.7468.$$

1.143. If a, b, c are real numbers such that

$$a^3b+b^3c+c^3a\geq 0,$$

then

$$a^2+b^2+c^2+k(ab+bc+ca)\geq 0,$$

where

$$k=\frac{-1+\sqrt{21+8\sqrt{7}}}{2}\approx 2.7468.$$

1.144. If a, b, c are real numbers such that

$$k(a^2+b^2+c^2)=ab+bc+ca,\qquad k\in\left(\frac{-1}{2},1\right),$$

then

$$\alpha_k\leq \frac{a^3b+b^3c+c^3}{(a^2+b^2+c^2)^2}\leq \beta_k,$$

where

$$27\alpha_k=1+13k-5k^2-2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}},$$

$$27\beta_k=1+13k-5k^2+2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}}.$$

1.145. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\frac{a^2}{4a+b^2}+\frac{b^2}{4b+c^2}+\frac{c^2}{4c+a^2} \geq \frac{3}{5}.$$

1.146. If a, b, c are positive real numbers, then

$$\frac{a^2+bc}{a+b}+\frac{b^2+ca}{b+c}+\frac{c^2+ab}{c+a} \leq \frac{(a+b+c)^3}{3(ab+bc+ca)}.$$

1.147. If a, b, c are positive real numbers such that $a+b+c=3$, then

$$\sqrt{ab^2+bc^2}+\sqrt{bc^2+ca^2}+\sqrt{ca^2+ab^2} \leq 3\sqrt{2}.$$

1.148. If a, b, c are positive real numbers such that $a^5+b^5+c^5=3$, then

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \geq 3.$$

1.149. Let $P(a, b, c)$ be a cyclic homogeneous polynomial of degree three. The inequality

$$P(a, b, c) \geq 0$$

holds for all $a, b, c \geq 0$ if and only if the following two conditions are fulfilled:

(a) $P(1,1,1) \geq 0$;

(b) $P(0, b, c) \geq 0$ for all $b, c \geq 0$.

1.150. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$8(a^2b+b^2c+c^2a)+9 \geq 11(ab+bc+ca).$$

1.151. If a, b, c are nonnegative real numbers such that $a+b+c=6$, then

$$a^3+b^3+c^3+8(a^2b+b^2c+c^2a) \geq 166.$$

1.152. If a, b, c are nonnegative real numbers, then

$$a^3+b^3+c^3-3abc \geq \sqrt{9+6\sqrt{3}}\,(a-b)(b-c)(c-a).$$

1.153. If a,b,c are nonnegative real numbers, no two of which are zero, then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+7\ge\frac{17}{3}\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right).$$

1.154. Let a,b,c be nonnegative real numbers, no two of which are zero. If $0\le k\le 5$, then

$$\frac{ka+b}{a+c}+\frac{kb+c}{b+a}+\frac{kc+a}{c+b}\ge\frac{3}{2}(k+1).$$

1.155. Let a,b,c be nonnegative real numbers, no two of which are zero. If $k\le\dfrac{23}{8}$, then

$$\frac{ka+b}{2a+c}+\frac{kb+c}{2b+a}+\frac{kc+a}{2c+b}\ge k+1.$$

1.156. Let a,b,c be nonnegative real numbers. Prove that

(a) if $k\le 1-\dfrac{2}{5\sqrt{5}}$, then

$$\frac{ka+b}{2a+b+c}+\frac{kb+c}{a+2b+c}+\frac{kc+a}{a+b+2c}\ge\frac{3}{4}(k+1).$$

(b) if $k\ge 1+\dfrac{2}{5\sqrt{5}}$, then

$$\frac{ka+b}{2a+b+c}+\frac{kb+c}{a+2b+c}+\frac{kc+a}{a+b+2c}\le\frac{3}{4}(k+1).$$

1.157. If a,b,c are positive real numbers such that $a\le b\le c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\ge 2\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right).$$

1.158. If $a\ge b\ge c\ge 0$, then

$$\frac{3a+b}{2a+c}+\frac{3b+c}{2b+a}+\frac{3c+a}{2c+b}\ge 4.$$

1.159. Let a, b, c be nonnegative real numbers such that

$$a \geq b \geq 1 \geq c, \qquad a + b + c = 3.$$

Prove that

$$\frac{1}{a^2+3} + \frac{1}{b^2+3} + \frac{1}{c^2+3} \leq \frac{3}{4}.$$

1.160. Let a, b, c be nonnegative real numbers such that

$$a \geq 1 \geq b \geq c, \qquad a + b + c = 3.$$

Prove that

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \geq 1.$$

1.161. Let a, b, c be real numbers such that

$$a \geq b \geq 1 \geq c \geq -5, \qquad a + b + c = 3.$$

Prove that

$$\frac{6}{a^3+b^3+c^3} + 1 \geq \frac{8}{a^2+b^2+c^2}.$$

1.162. If $a \geq 1 \geq b \geq c > -3$ such that $ab + bc + ca = 3$, then

$$\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} \geq 1.$$

1.163. If $a \geq b \geq 1 \geq c \geq 0$ such that $a + b + c = 3$, then

$$\frac{1}{a^2+ab+b^2} + \frac{1}{b^2+bc+c^2} + \frac{1}{c^2+ca+a^2} \leq \frac{3}{ab+bc+ca}.$$

1.164. If a, b, c are positive real numbers such that

$$a \geq 1 \geq b \geq c, \qquad abc = 1,$$

then

$$\frac{1-a}{3+a^2} + \frac{1-b}{3+b^2} + \frac{1-c}{3+c^2} \geq 0.$$

1.165. If a,b,c are positive real numbers such that

$$a \ge 1 \ge b \ge c, \qquad abc = 1,$$

then

$$\frac{1}{\sqrt{3a+1}} + \frac{1}{\sqrt{3b+1}} + \frac{1}{\sqrt{3c+1}} \ge \frac{3}{2}.$$

1.166. If a,b,c are positive real numbers such that

$$a \ge 1 \ge b \ge c, \qquad abc = 1,$$

then

$$\frac{1}{a^2+4ab+b^2} + \frac{1}{b^2+4bc+c^2} + \frac{1}{c^2+4ca+a^2} \ge \frac{1}{2}.$$

1.167. Let $a \ge 1 \ge b \ge c \ge 0$ such that

$$a+b+c = 3, \qquad ab+bc+ca = q,$$

where $q \in [0,3]$ is a fixed number. Prove that the product $r = abc$ is maximal for $b = c$, and minimal for $b = 1$ or $c = 0$.

1.168. Let p and q be fixed real numbers such that there exist three real numbers a,b,c satisfying

$$a \ge 1 \ge b \ge c \ge 0, \qquad a+b+c = p, \qquad ab+bc+ca = q.$$

Prove that

(a) the product $r = abc$ is maximal for $b = c$;

(b) the product $r = abc$ is minimal for $a = 1$ or $b = 1$ or $c = 0$.

1.169. Let p and q be fixed real numbers such that there exist three real numbers a,b,c satisfying

$$a \ge b \ge c \ge 1, \qquad a+b+c = p, \qquad ab+bc+ca = q.$$

Prove that

(a) the product $r = abc$ is maximal for $b = c$;

(b) the product $r = abc$ is minimal for $a = b$ or $c = 1$.

1.170. Let $a \geq b \geq 1 \geq c \geq 0$ such that

$$a+b+c=3, \qquad ab+bc+ca=q,$$

where $q \in [0,3]$ is a fixed number. Prove that the product $r = abc$ is maximal for $b=1$, and minimal for $a=b$ or $c=0$.

1.171. Let p and q be fixed real numbers such that there exist three real numbers a, b, c satisfying

$$a \geq b \geq 1 \geq c \geq 0, \qquad a+b+c=p, \qquad ab+bc+ca=q.$$

Prove that

(a) the product $r = abc$ is maximal for $b=1$ or $c=1$;

(b) the product $r = abc$ is minimal for $a=b$ or $c=0$.

1.172. Let p and q be fixed real numbers such that there exist three real numbers a, b, c satisfying

$$1 \geq a \geq b \geq c \geq 0, \qquad a+b+c=p, \qquad ab+bc+ca=q.$$

Prove that

(a) the product $r = abc$ is maximal for $b=c$ or $a=1$;

(b) the product $r = abc$ is minimal for $a=b$ or $c=0$.

1.173. If $a \geq 1 \geq b \geq c \geq 0$ such that $a+b+c=3$, then

$$abc + \frac{9}{ab+bc+ca} \geq 4.$$

1.174. If $a \geq 1 \geq b \geq c \geq 0$ such that $a+b+c=3$, then

$$abc + \frac{2}{ab+bc+ca} \geq \frac{5}{a^2+b^2+c^2}.$$

1.175. If $a \geq b \geq 1 \geq c > 0$ such that $a+b+c=3$, then

$$\frac{1}{abc} + 2 \geq \frac{9}{ab+bc+ca}.$$

1.176. If $a \ge b \ge 1 \ge c > 0$ such that $a+b+c=3$, then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+11 \ge 4(a^2+b^2+c^2).$$

1.177. If $a \ge b \ge 1 \ge c > 0$ such that $a+b+c=3$, then

$$\frac{1}{abc}+\frac{2}{a^2+b^2+c^2} \ge \frac{5}{ab+bc+ca}.$$

1.178. If $a \ge b \ge 1 \ge c \ge 0$ such that $a+b+c=3$, then

$$\frac{9}{a^3+b^3+c^3}+2 \le \frac{15}{a^2+b^2+c^2}.$$

1.179. If $a \ge b \ge 1 \ge c \ge 0$ such that $a+b+c=3$, then

$$\frac{36}{a^3+b^3+c^3}+9 \le \frac{65}{a^2+b^2+c^2}.$$

1.180. If $a \ge b \ge c \ge 0$ and $ab+bc+ca=2$, then

$$\sqrt{a+ab}+\sqrt{b+bc}+\sqrt{c+ca} \ge 3.$$

1.181. If $a \ge b \ge c$ are nonnegative numbers such that $ab+bc+ca=3$, then

$$\sqrt{a+2ab}+\sqrt{b+2bc}+\sqrt{c+2ca} \ge 4.$$

1.182. If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then

$$\sqrt{a+3b}+\sqrt{b+3c}+\sqrt{c+3a} \ge 6.$$

1.183. If a,b,c are the lengths of the sides of a triangle, then

$$10\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) > 9\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

1.184. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{a}{3a+b-c}+\frac{b}{3b+c-a}+\frac{c}{3c+a-b}\geq 1.$$

1.185. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{a^2-b^2}{a^2+bc}+\frac{b^2-c^2}{b^2+ca}+\frac{c^2-a^2}{c^2+ab}\leq 0.$$

1.186. If a, b, c are the lengths of the sides of a triangle, then

$$a^2(a+b)(b-c)+b^2(b+c)(c-a)+c^2(c+a)(a-b)\geq 0.$$

1.187. If a, b, c are the lengths of the sides of a triangle, then

$$a^2b+b^2c+c^2a\geq\sqrt{abc(a+b+c)(a^2+b^2+c^2)}.$$

1.188. If a, b, c are the lengths of the sides of a triangle, then

$$a^2\left(\frac{b}{c}-1\right)+b^2\left(\frac{c}{a}-1\right)+c^2\left(\frac{a}{b}-1\right)\geq 0.$$

1.189. If a, b, c are the lengths of the sides of a triangle, then

(a) $$a^3b+b^3c+c^3a\geq a^2b^2+b^2c^2+c^2a^2;$$

(b) $$3(a^3b+b^3c+c^3a)\geq(ab+bc+ca)(a^2+b^2+c^2);$$

(c) $$\frac{a^3b+b^3c+c^3}{3}\geq\left(\frac{a+b+c}{3}\right)^4.$$

1.190. If a, b, c are the lengths of the sides of a triangle, then

$$2\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)\geq\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}+3.$$

1.191. If a,b,c are the lengths of the sides of a triangle such that $a<b<c$, then

$$\frac{a^2}{a^2-b^2}+\frac{b^2}{b^2-c^2}+\frac{c^2}{c^2-a^2}\le 0.$$

1.192. If a,b,c are the lengths of the sides of a triangle, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\ge 2\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right).$$

1.193. Let a,b,c be the lengths of the sides of a triangle. If $k\ge 2$, then

$$a^kb(a-b)+b^kc(b-c)+c^ka(c-a)\ge 0.$$

1.194. Let a,b,c be the lengths of the sides of a triangle. If $k\ge 1$, then

$$3(a^{k+1}b+b^{k+1}c+c^{k+1}a)\ge(a+b+c)(a^kb+b^kc+c^ka).$$

1.195. Let a,b,c,d be positive real numbers such that $a+b+c+d=4$. Prove that

$$\frac{a}{3+b}+\frac{b}{3+c}+\frac{c}{3+d}+\frac{d}{3+a}\ge 1.$$

1.196. Let a,b,c,d be positive real numbers such that $a+b+c+d=4$. Prove that

$$\frac{a}{1+b^2}+\frac{b}{1+c^2}+\frac{c}{1+d^2}+\frac{d}{1+a^2}\ge 2.$$

1.197. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then

$$a^2bc+b^2cd+c^2da+d^2ab\le 4.$$

1.198. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then

$$a(b+c)^2+b(c+d)^2+c(d+a)^2+d(a+b)^2\le 16.$$

1.199. If a,b,c,d are positive real numbers, then

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+a}+\frac{d-a}{a+b}\geq 0.$$

1.200. If a,b,c,d are positive real numbers, then

(a) $$\frac{a-b}{a+2b+c}+\frac{b-c}{b+2c+d}+\frac{c-d}{c+2d+a}+\frac{d-a}{d+2a+b}\geq 0;$$

(b) $$\frac{a}{2a+b+c}+\frac{b}{2b+c+d}+\frac{c}{2c+d+a}+\frac{d}{2d+a+b}\leq 1.$$

1.201. If a,b,c,d are positive real numbers such that $abcd=1$, then

$$\frac{1}{a(a+b)}+\frac{1}{b(b+c)}+\frac{1}{c(c+d)}+\frac{1}{d(d+a)}\geq 2.$$

1.202. If a,b,c,d are positive real numbers, then

$$\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+d)}+\frac{1}{d(1+a)}\geq\frac{16}{1+8\sqrt{abcd}}.$$

1.203. If a,b,c,d are nonnegative real numbers such that $a^2+b^2+c^2+d^2=4$, then

(a) $$3(a+b+c+d)\geq 2(ab+bc+cd+da)+4;$$

(b) $$a+b+c+d-4\geq(2-\sqrt{2})(ab+bc+cd+da-4).$$

1.204. Let a,b,c,d be positive real numbers.

(a) If $a,b,c,d\geq 1$, then

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{d}\right)\left(d+\frac{1}{a}\right)\geq(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right);$$

(b) If $abcd=1$, then

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{d}\right)\left(d+\frac{1}{a}\right)\geq(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

1.205. If a, b, c, d are positive real numbers, then

$$\left(1+\frac{a}{a+b}\right)^2+\left(1+\frac{b}{b+c}\right)^2+\left(1+\frac{c}{c+d}\right)^2+\left(1+\frac{d}{d+a}\right)^2>7.$$

1.206. If a, b, c, d are positive real numbers, then

$$\frac{a^2-bd}{b+2c+d}+\frac{b^2-ca}{c+2d+a}+\frac{c^2-db}{d+2a+b}+\frac{d^2-ac}{a+2b+c}\geq 0.$$

1.207. If a, b, c, d are positive real numbers such that $a \leq b \leq c \leq d$, then

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+d}}+\sqrt{\frac{2d}{d+a}}\leq 4.$$

1.208. Let a, b, c, d be nonnegative real numbers, and let

$$x=\frac{a}{b+c}, \quad y=\frac{b}{c+d}, \quad z=\frac{c}{d+a}, \quad t=\frac{d}{a+b}.$$

Prove that

(a) $$\sqrt{xz}+\sqrt{yt}\leq 1;$$

(b) $$x+y+z+t+4(xz+yt)\geq 4.$$

1.209. If a, b, c, d are nonnegative real numbers, then

$$\left(1+\frac{2a}{b+c}\right)\left(1+\frac{2b}{c+d}\right)\left(1+\frac{2c}{d+a}\right)\left(1+\frac{2d}{a+b}\right)\geq 9.$$

1.210. Let a, b, c, d be nonnegative real numbers. If $k > 0$, then

$$\left(1+\frac{ka}{b+c}\right)\left(1+\frac{kb}{c+d}\right)\left(1+\frac{kc}{d+a}\right)\left(1+\frac{kd}{a+b}\right)\geq (1+k)^2.$$

1.211. If a, b, c, d are positive real numbers such that $a+b+c+d=4$, then

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{cd}+\frac{1}{da}\geq a^2+b^2+c^2+d^2.$$

1.212. If a,b,c,d are positive real numbers, then

$$\frac{a^2}{(a+b+c)^2}+\frac{b^2}{(b+c+d)^2}+\frac{c^2}{(c+d+a)^2}+\frac{d^2}{(d+a+b)^2}\geq\frac{4}{9}.$$

1.213. If a,b,c,d are positive real numbers such that $a+b+c+d=3$, then

$$ab(b+c)+bc(c+d)+cd(d+a)+da(a+b)\leq 4.$$

1.214. If $a\geq b\geq c\geq d\geq 0$ and $a+b+c+d=2$, then

$$ab(b+c)+bc(c+d)+cd(d+a)+da(a+b)\leq 1.$$

1.215. Let a,b,c,d be nonnegative real numbers such that $a+b+c+d=4$. If $k\geq\frac{37}{27}$, then

$$ab(b+kc)+bc(c+kd)+cd(d+ka)+da(a+kb)\leq 4(1+k).$$

1.216. If a,b,c,d are nonnegative real numbers such that $a+b+c+d=4$, then

$$\sqrt{\frac{3a}{b+2}}+\sqrt{\frac{3b}{c+2}}+\sqrt{\frac{3c}{d+2}}+\sqrt{\frac{3d}{a+2}}\leq 4.$$

1.217. Let a,b,c,d be positive real numbers such that $a\leq b\leq c\leq d$. Prove that

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)\geq 4+\frac{a}{c}+\frac{c}{a}+\frac{b}{d}+\frac{d}{b}.$$

1.218. Let a,b,c,d be positive real numbers such that

$$a\leq b\leq c\leq d,\qquad abcd=1.$$

Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\geq ab+bc+cd+da.$$

1.219. Let a,b,c,d be positive real numbers such that

$$a\le b\le c\le d, \qquad abcd=1.$$

Prove that

$$4+\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\ge 2(a+b+c+d).$$

1.220. Let $A=\{a_1,a_2,a_3,a_4\}$ be a set of real numbers such that

$$a_1+a_2+a_3+a_4=0.$$

Prove that there exists a permutation $\{a,b,c,d\}$ of A such that

$$a^2+b^2+c^2+d^2+3(ab+bc+cd+da)\ge 0.$$

1.221. If a,b,c,d are nonnegative real numbers such that

$$a\ge b\ge 1\ge c\ge d, \quad a+b+c+d=3,$$

then

$$a^2+b^2+c^2+d^2+10abcd\le 5.$$

1.222. If a,b,c,d are nonnegative real numbers such that

$$a\ge b\ge 1\ge c\ge d, \quad a+b+c+d=6,$$

then

$$a^2+b^2+c^2+d^2+4abcd\le 26.$$

1.223. Let a,b,c,d be nonnegative real numbers such that

$$a\ge b\ge 1\ge c\ge d, \qquad a+b+c+d=p, \quad p\ge 2.$$

Prove that

$$\frac{p^2-4p+8}{2}\le a^2+b^2+c^2+d^2\le p^2-2p+2.$$

1.224. Let $a\ge b\ge 1\ge c\ge d\ge 0$ such that

$$a+b+c+d=4, \quad a^2+b^2+c^2+d^2=q,$$

where $q\in[4,10]$ is a fixed number. Prove that the product $r=abcd$ is maximal when $b=1$ and $c=d$.

1.225. If a, b, c, d are nonnegative real numbers such that

$$a \geq b \geq 1 \geq c \geq d, \qquad a+b+c+d=4,$$

then

$$a^2+b^2+c^2+d^2+6abcd \leq 10.$$

1.226. If a, b, c, d are nonnegative real numbers such that

$$a \geq b \geq 1 \geq c \geq d, \qquad a+b+c+d=4,$$

then

$$a^2+b^2+c^2+d^2+6\sqrt{abcd} \leq 10.$$

1.227. If a, b, c, d, e are positive real numbers, then

$$\frac{a}{a+2b+2c}+\frac{b}{b+2c+2d}+\frac{c}{c+2d+2e}+\frac{d}{d+2e+2a}+\frac{e}{e+2a+2b} \geq 1.$$

1.228. Let a, b, c, d, e be positive real numbers such that $a+b+c+d+e=5$. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a} \leq 1+\frac{4}{abcde}.$$

1.229. If a, b, c, d, e are real numbers such that $a+b+c+d+e=0$, then

$$\frac{-\sqrt{5}-1}{4} \leq \frac{ab+bc+cd+de+ea}{a^2+b^2+c^2+d^2+e^2} \leq \frac{\sqrt{5}-1}{4}.$$

1.230. Let a, b, c, d, e be positive real numbers such that

$$a^2+b^2+c^2+d^2+e^2=5.$$

Prove that

$$\frac{a^2}{b+c+d}+\frac{b^2}{c+d+e}+\frac{c^2}{d+e+a}+\frac{d^2}{e+a+b}+\frac{e^2}{a+b+c} \geq \frac{5}{3}.$$

1.231. Let a, b, c, d, e be nonnegative real numbers such that $a+b+c+d+e=5$. Prove that

$$(a^2+b^2)(b^2+c^2)(c^2+d^2)(d^2+e^2)(e^2+a^2) \leq \frac{729}{2}.$$

1.232. If $a,b,c,d,e \in [1,5]$, then

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+e}+\frac{d-e}{e+a}+\frac{e-a}{a+b}\ge 0.$$

1.233. If $a,b,c,d,e,f \in [1,3]$, then

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+e}+\frac{d-e}{e+f}+\frac{e-f}{f+a}+\frac{f-a}{a+b}\ge 0.$$

1.234. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive real numbers, then

$$\sum_{i=1}^{n}\frac{a_i}{a_{i-1}+2a_i+a_{i+1}}\le \frac{n}{4},$$

where $a_0 = a_n$ and $a_{n+1} = a_1$.

1.235. Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that

$$\frac{1}{n-2+a_1+a_2}+\frac{1}{n-2+a_2+a_3}+\cdots+\frac{1}{n-2+a_n+a_1}\le 1.$$

1.236. If $a_1, a_2, ..., a_n \ge 1$, then

$$\prod\left(a_1+\frac{1}{a_2}+n-2\right)\ge n^{n-2}(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right);$$

1.237. If $a_1, a_2, ..., a_n \ge 1$, then

$$\left(a_1+\frac{1}{a_1}\right)\left(a_2+\frac{1}{a_2}\right)\cdots\left(a_n+\frac{1}{a_n}\right)+2^n\ge 2\left(1+\frac{a_1}{a_2}\right)\left(1+\frac{a_2}{a_3}\right)\cdots\left(1+\frac{a_n}{a_1}\right).$$

1.238. Let k and n be positive integers, and let $a_1, a_2, ..., a_n$ be real numbers such that

$$a_1 \le a_2 \le \cdots \le a_n.$$

Consider the inequality

$$(a_1+a_2+\cdots+a_n)^2\ge n(a_1a_{k+1}+a_2a_{k+2}+\cdots+a_na_{n+k}),$$

where $a_{n+i} = a_i$ for any positive integer i. Prove this inequality for

(a) $n = 2k$;

(b) $n = 4k$.

1.239. If $a_1, a_2, \ldots, a_n$ are real numbers, then

$$a_1(a_1+a_2)+a_2(a_2+a_3)+\cdots+a_n(a_n+a_1) \geq \frac{2}{n}(a_1+a_2+\cdots+a_n)^2.$$

1.240. If $a_1,\ a_2,\ \ldots,\ a_n \in [1,\ 2]$, then

$$\sum_{i=1}^{n} \frac{3}{a_i+2a_{i+1}} \geq \sum_{i=1}^{n} \frac{2}{a_i+a_{i+1}},$$

where $a_{n+1}=a_1$.

1.241. Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be real numbers such that $a_1+a_2+\cdots+a_n=n$.

(a) If $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n$, then

$$a_1^3+a_2^3+\cdots+a_n^3+2n \geq 3(a_1^2+a_2^2+\cdots+a_n^2);$$

(b) If $a_1 \leq 1 \leq a_2 \leq \cdots \leq a_n$, then

$$a_1^3+a_2^3+\cdots+a_n^3+2n \leq 3(a_1^2+a_2^2+\cdots+a_n^2).$$

1.242. Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be nonnegative real numbers such that $a_1+a_2+\cdots+a_n=n$.

(a) If $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n$, then

$$a_1^4+a_2^4+\cdots+a_n^4+5n \geq 6(a_1^2+a_2^2+\cdots+a_n^2);$$

(b) If $a_1 \leq 1 \leq a_2 \leq \cdots \leq a_n$, then

$$a_1^4+a_2^4+\cdots+a_n^4+6n \leq 7(a_1^2+a_2^2+\cdots+a_n^2).$$

1.243. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n, \qquad \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}=n,$$

then

$$a_1^2+a_2^2+\cdots+a_n^2+2n \geq 3(a_1+a_2+\cdots+a_n).$$

1.244. If $a_1, a_2, \ldots, a_n$ are real numbers such that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

(a) $$\frac{a_1+1}{a_1^2+1} + \frac{a_2+1}{a_2^2+1} + \cdots + \frac{a_n+1}{a_n^2+1} \le n;$$

(b) $$\frac{1}{a_1^2+3} + \frac{1}{a_2^2+3} + \cdots + \frac{1}{a_1^2+3} \le \frac{n}{4}.$$

1.245. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{a_1^2-1}{(a_1+3)^2} + \frac{a_2^2-1}{(a_2+3)^2} + \cdots + \frac{a_n^2-1}{(a_n+3)^2} \ge 0.$$

1.246. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1}{3a_1^3+4} + \frac{1}{3a_2^3+4} + \cdots + \frac{1}{3a_n^3+4} \ge \frac{n}{7}.$$

1.247. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\sqrt{\frac{3a_1}{4-a_1}} + \sqrt{\frac{3a_2}{4-a_2}} + \cdots + \sqrt{\frac{3a_n}{4-a_n}} \le n.$$

1.248. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n,$$

then

$$\frac{1}{3-a_1} + \frac{1}{3-a_2} + \cdots + \frac{1}{3-a_n} \le \frac{n}{2}.$$

1.249. If $a_1, a_2, \ldots, a_n$ are real numbers such that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$(1 + a_1^2)(1 + a_2^2)\cdots(1 + a_n^2) \ge 2^n.$$

1.250. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1+1)^2} + \frac{1}{(a_2+1)^2} + \cdots + \frac{1}{(a_n+1)^2} \ge \frac{n}{4}.$$

1.251. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1+2)^2} + \frac{1}{(a_2+2)^2} + \cdots + \frac{1}{(a_n+2)^2} \ge \frac{n}{9}.$$

1.252. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$a_1^n + a_2^n + \cdots + a_n^n - n \ge n^2\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n\right).$$

1.253. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are real numbers such that

$$a_1 + a_2 + \cdots + a_n = n, \qquad a_1 \ge a_2 \ge 1 \ge a_3 \ge \cdots \ge a_n,$$

then

$$a_1^4 + a_2^4 + \cdots + a_n^4 - n \ge \frac{14}{3}(a_1^2 + a_2^2 + \cdots + a_n^2 - n).$$

1.254. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \quad a_1 a_2 \cdots a_n = 1.$$

Prove that

$$\frac{1-a_1}{3+a_1^2} + \frac{1-a_2}{3+a_2^2} + \cdots + \frac{1-a_n}{3+a_n^2} \ge 0.$$

1.255. Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative real numbers such that

$$a_1 \ge \cdots \ge a_k \ge 1 \ge a_{k+1} \ge \cdots \ge a_n, \qquad 1 \le k \le n-1,$$

and

$$a_1 + a_2 + \cdots + a_n = p.$$

Prove that

(a) if $p \ge k$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 \le (p-k+1)^2 + k - 1;$$

(b) if $k \le p \le n$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 \ge \frac{p^2 - 2kp + kn}{n-k};$$

(c) if $p \ge n$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 \ge \frac{p^2 - 2(n-k)p + n(n-k)}{k}.$$

1.256. Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative real numbers such that

$$a_1 \ge \cdots \ge a_k \ge 1 \ge a_{k+1} \ge \cdots \ge a_n, \qquad 1 \le k \le n-1,$$

and

$$a_1 + a_2 + \cdots + a_n = n, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = q,$$

where q is a fixed number. Prove that the product $r = a_1 a_2 \cdots a_n$ is maximal when

$$a_2 = \cdots = a_k = 1, \qquad a_{k+1} = \cdots = a_n.$$

1.257. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$(a_1 a_2 \cdots a_n)^{\frac{2}{n}} (a_1^2 + a_2^2 + \cdots + a_n^2) \le n.$$

1.258. Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be nonnegative real numbers such that

$$a_1 \geq \cdots \geq a_k \geq 1 \geq a_{k+1} \geq \cdots \geq a_n, \qquad 1 \leq k \leq n-1,$$

and

$$a_1 + a_2 + \cdots + a_n = p, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = q,$$

where p and q are fixed numbers.

(a) For $p \leq n$, the product $r = a_1 a_2 \cdots a_n$ is maximal when $a_2 = \cdots = a_k = 1$ and $a_{k+1} = \cdots = a_n$;

(b) For $p \geq n$ and $q \geq n-1+(p-n+1)^2$, the product $r = a_1 a_2 \cdots a_n$ is maximal when $a_2 = \cdots = a_k = 1$ and $a_{k+1} = \cdots = a_n$;

(c) For $p \geq n$ and $q < n-1+(p-n+1)^2$, the product $r = a_1 a_2 \cdots a_n$ is maximal when $a_2 = \cdots = a_k$ and $a_{k+1} = \cdots = a_n = 1$.

1.259. If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are nonnegative real numbers such that

$$a_1 \leq a_2 \leq 1 \leq a_3 \leq \cdots \leq a_n, \qquad a_1 + a_2 + \cdots + a_n = n-1,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 + 10a_1 a_2 \cdots a_n \leq n+1.$$

1.260. If a, b, c, d, e are nonnegative real numbers such that

$$a \leq b \leq 1 \leq c \leq d \leq e, \qquad a+b+c+d+e = 8,$$

then

$$a^2 + b^2 + c^2 + d^2 + e^2 + 3abcde \leq 38.$$

1.2 Solutions

P 1.1. *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$ab^2+bc^2+ca^2 \le 4.$$

(Canada, 1999)

First Solution. Assume that $a=\max\{a,b,c\}$. Since

$$ab^2+bc^2+ca^2 \le ab\cdot\frac{a+b}{2}+abc+ca^2=\frac{a(a+b)(b+2c)}{2},$$

it suffices to show that

$$a(a+b)(b+2c)\le 8.$$

By the AM-GM inequality, we have

$$a(a+b)(b+2c)\le\left[\frac{a+(a+b)+(b+2c)}{3}\right]^3=8\left(\frac{a+b+c}{3}\right)^3=8.$$

The equality holds for $a=2,\ b=0,\ c=1$ (or any cyclic permutation).

Second Solution. Let (x,y,z) be a permutation of (a,b,c) such that

$$x\ge y\ge z.$$

Since

$$xy\ge zx\ge yz,$$

by the rearrangement inequality, we have

$$\begin{aligned}ab^2+bc^2+ca^2&=b\cdot ab+c\cdot bc+a\cdot ca\\&\le x\cdot xy+y\cdot zx+z\cdot yz\\&=y(x^2+xz+z^2).\end{aligned}$$

Using this result and the AM-GM inequality, we get

$$\begin{aligned}ab^2+bc^2+ca^2&\le y(x+z)^2=4y\cdot\frac{x+z}{2}\cdot\frac{x+z}{2}\\&\le 4\left(\frac{y+\frac{x+z}{2}+\frac{x+z}{2}}{3}\right)^3\\&=4\left(\frac{x+y+z}{3}\right)^3=4.\end{aligned}$$

Third Solution. Without loss of generality, assume that b is between a and c; that is,

$$(b-a)(b-c)\le 0,\quad b^2+ac\le b(a+c).$$

Since

$$ab^2+bc^2+ca^2 = a(b^2+ac)+bc^2 \le ab(a+c)+bc^2 = b(a^2+ac+c^2)$$
$$\le b(a+c)^2 = b(3-b)^2,$$

it suffices to show that

$$b(3-b)^2 \le 4.$$

Indeed,

$$b(3-b)^2-4 = (b-1)^2(b-4) \le (b-1)^2(b-3) = -(b-1)^2(a+c) \le 0.$$

Fourth Solution. Write the inequality in the homogeneous form

$$4(a+b+c)^3 \ge 27(ab^2+bc^2+ca^2),$$

which is equivalent to

$$4(a^3+b^3+c^3)+12(a+b)(b+c)(c+a) \ge 27(ab^2+bc^2+ca^2),$$

$$4\sum a^3+12\left(\sum a^2b+\sum ab^2+2abc\right) \ge 27\sum ab^2,$$

$$4\sum a^3+12\sum a^2b+24abc \ge 15\sum ab^2.$$

On the other hand, the obvious inequality

$$\sum a(2a-pb-qc)^2 \ge 0$$

is equivalent to

$$4\sum a^3+(q^2-4p)\sum a^2b+6pqabc \ge (4q-p^2)\sum ab^2.$$

Setting $p=1$ and $q=4$ leads to the desired inequality; in addition,

$$4(a+b+c)^3-27(ab^2+bc^2+ca^2) = \sum a(2a-b-4c)^2 \ge 0.$$

□

P 1.2. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$(ab+bc+ca)(ab^2+bc^2+ca^2) \le 9.$$

Solution. Let (x, y, z) be a permutation of (a, b, c) such that $x \ge y \ge z$. As shown in the second solution of P 1.1,

$$ab^2 + bc^2 + ca^2 \le y(x^2 + xz + z^2).$$

Consequently, it suffices to show that

$$y(xy + yz + zx)(x^2 + xz + z^2) \le 9.$$

By the AM-GM inequality, we get

$$4(xy + yz + zx)(x^2 + xz + z^2) \le (xy + yz + zx + x^2 + xz + z^2)^2$$

$$= (x + z)^2(x + y + z)^2 = 9(x + z)^2.$$

Thus, we still have to show that

$$y(x + z)^2 \le 4.$$

This follows from the AM-GM inequality, as follows:

$$2y(x + z)^2 \le \left[\frac{2y + (x + z) + (x + z)}{3}\right]^3 = 8.$$

The equality holds for $a = b = c = 1$.

□

P 1.3. *If a, b, c are nonnegative real numbers such that $a^2 + b^2 + c^2 = 3$, then*

(a) $$ab^2 + bc^2 + ca^2 \le abc + 2;$$

(b) $$\frac{a}{b + 2} + \frac{b}{c + 2} + \frac{c}{a + 2} \le 1.$$

(Vasile C., 2005)

Solution. (a) ***First Solution.*** Without loss of generality, assume that b is between a and c; that is,

$$(b - a)(b - c) \le 0, \quad b^2 + ac \le b(a + c).$$

Since

$$ab^2 + bc^2 + ca^2 = a(b^2 + ac) + bc^2 \le ab(a + c) + bc^2 = b(a^2 + c^2) + abc,$$

it suffices to show that

$$b(a^2 + c^2) \le 2.$$

We have

$$2 - b(a^2 + c^2) = 2 - b(3 - b^2) = (b - 1)^2(b + 2) \ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=0$, $b=1$, $c=\sqrt{2}$ (or any cyclic permutation).

Second Solution. Let (x,y,z) be a permutation of (a,b,c) such that $x \ge y \ge z$. As shown in the second solution of P 1.1,

$$ab^2+bc^2+ca^2 \le y(x^2+xz+z^2).$$

Therefore, it suffices to show that

$$y(x^2+xz+z^2) \le xyz+2,$$

which can be written as

$$y(x^2+z^2) \le 2.$$

Indeed,

$$2-y(x^2+z^2)=2-y(3-y^2)=(y-1)^2(y+2) \ge 0.$$

(b) Write the inequality as follows:

$$\sum a(a+2)(c+2) \le (a+2)(b+2)(c+2),$$

$$ab^2+bc^2+ca^2+2(a^2+b^2+c^2) \le abc+8,$$

$$ab^2+bc^2+ca^2 \le abc+2.$$

The last inequality is just the inequality in (a).

□

P 1.4. *If $a,b,c \ge 1$, then*

(a) $$2(ab^2+bc^2+ca^2)+3 \ge 3(ab+bc+ca);$$

(b) $$ab^2+bc^2+ca^2+6 \ge 3(a+b+c).$$

Solution. (a) ***First Solution.*** From

$$a(b-1)^2+b(c-1)^2+c(a-1)^2 \ge 0,$$

we get

$$ab^2+bc^2+ca^2 \ge 2(ab+bc+ca)-(a+b+c).$$

Using this inequality gives

$$2(ab^2+bc^2+ca^2)+3-3(ab+bc+ca) \ge (ab+bc+ca)-2(a+b+c)+3$$

$$=(a-1)(b-1)+(b-1)(c-1)+(c-1)(a-1) \ge 0.$$

The equality holds for $a=b=c=1$.

Second Solution. From

$$\sum b(a-1)(b-1) \geq 0,$$

we get

$$ab^2 + bc^2 + ca^2 \geq a^2 + b^2 + c^2 + ab + bc + ca - (a + b + c).$$

Thus, it suffices to show that

$$2(a^2 + b^2 + c^2) + 2(ab + bc + ca) - 2(a + b + c) + 3 \geq 3(ab + bc + ca),$$

which is equivalent to

$$2(a^2 + b^2 + c^2) - 2(a + b + c) + 3 \geq ab + bc + ca,$$

$$(a-1)^2 + (b-1)^2 + (c-1)^2 + (a^2 + b^2 + c^2 - ab - bc - ca) \geq 0,$$

$$2(a-1)^2 + 2(b-1)^2 + 2(c-1)^2 + (a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0.$$

(b) The inequality in (b) follows by summing the inequality in (a) and the obvious inequality

$$3(a-1)(b-1) + 3(b-1)(c-1) + 3(c-1)(a-1) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.5. *If a, b, c are nonnegative real numbers such that*

$$a + b + c = 3, \qquad a \geq b \geq c,$$

then

(a) $$a^2b + b^2c + c^2a \geq ab + bc + ca;$$

(b) $$8(ab^2 + bc^2 + ca^2) + 3abc \leq 27;$$

(c) $$\frac{18}{a^2b + b^2c + c^2a} \leq \frac{1}{abc} + 5.$$

Solution. (a) Write the inequality in the homogeneous form

$$3(a^2b + b^2c + c^2a) \geq (a + b + c)(ab + bc + ca),$$

which is equivalent to

$$a^2b + b^2c + c^2a - 3abc \geq ab^2 + bc^2 + ca^2 - a^2b - b^2c - c^2a.$$

This inequality is true because

$$a^2b + b^2c + c^2a - 3abc \geq 0$$

(by the AM-GM inequality) and

$$ab^2+bc^2+ca^2-a^2b-b^2c-c^2a=(a-b)(b-c)(c-a)\le 0.$$

The equality holds for $a=b=c=1$, and also for $a=3$ and $b=c=0$.

(b) Write the inequality in the homogeneous form

$$(a+b+c)^3\ge 8(ab^2+bc^2+ca^2)+3abc,$$

$$\sum a^3+3abc+3\sum a^2b\ge 5\sum ab^2,$$

$$\sum a^3+3abc-\left(\sum ab^2+\sum a^2b\right)\ge 4\left(\sum ab^2-\sum a^2b\right),$$

$$\sum a^3+3abc-\sum ab(a+b)\ge 4(a-b)(b-c)(c-a).$$

The inequality is true since

$$(a-b)(b-c)(c-a)\le 0$$

and, by Schur's inequality of degree three,

$$\sum a^3+3abc-\sum ab(a+b)\ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=b=3/2$ and $c=0$.

(c) Since

$$ab^2+bc^2+ca^2-a^2b-b^2c-c^2a=(a-b)(b-c)(c-a)\le 0,$$

it suffices to prove the symmetric inequality

$$\frac{36}{(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)}\le\frac{1}{abc}+5,$$

which is equivalent to

$$\frac{36}{(a+b+c)(ab+bc+ca)-3abc}\le\frac{1}{abc}+5,$$

$$\frac{12}{ab+bc+ca-abc}\le\frac{1}{abc}+5,$$

$$\frac{12}{a(b+c)-(a-1)bc}\le\frac{1}{a\cdot bc}+5,$$

$$\frac{12}{a(3-a)-(a-1)bc}\le\frac{1}{a\cdot bc}+5.$$

Since $a-1\ge 0$ and

$$4bc\le (b+c)^2=(3-a)^2,$$

it suffices to show that

$$\frac{48}{4a(3-a)-(a-1)(3-a)^2} \le \frac{4}{a(3-a)^2}+5,$$

which is equivalent to

$$\frac{48}{(3-a)(3+a^2)} \le \frac{4}{a(3-a)^2}+5,$$

$$5a^5-30a^4+60a^3-38a^2-9a+12 \ge 0,$$

$$(a-1)^2(5a^3-20a^2+15a+12) \ge 9.$$

We need to show that $1 \le a \le 3$ involves

$$5a^3-20a^2+15a+12 \ge 0.$$

If $1 \le a \le 2$, then

$$5a^3-20a^2+15a+12 = 5a(a-2)^2+(12-5a) > 0.$$

If $2 \le a \le 3$, then

$$5a^3-20a^2+15a+12 = 5(a-2)^3+10a^2-45a+52 \ge 10a^2-45a+52 > 0$$
$$= 10\left(a-\frac{9}{4}\right)^2+\frac{11}{8} > 0.$$

The equality holds for $a=b=c=1$.

□

P 1.6. *If a,b,c are nonnegative real numbers such that*

$$a^2+b^2+c^2=3, \qquad a \ge b \ge c,$$

then

$$ab^2+bc^2+ca^2 \le \frac{3}{4}(ab+bc+ca+1).$$

Solution. Let us denote

$$p=a+b+c, \qquad q=ab+bc+ca.$$

From $a^2+b^2+c^2=3$, it follows that

$$2q=p^2-3.$$

In addition, from the known inequalities

$$(a+b+c)^2 \ge a^2+b^2+c^2$$

and

$$3(a^2+b^2+c^2) \ge (a+b+c)^2,$$

we get

$$\sqrt{3} \le p \le 3.$$

Since

$$ab^2+bc^2+ca^2-a^2b-b^2c-c^2a=(a-b)(b-c)(c-a) \le 0,$$

it suffices to show that

$$ab^2+bc^2+ca^2+(a^2b+b^2c+c^2a) \le \frac{3}{2}(ab+bc+ca+1).$$

which is equivalent to

$$pq \le 3abc+\frac{3}{2}(q+1),$$

$$6abc+3(q+1) \ge 2pq.$$

Consider two cases: $\sqrt{3} \le p \le \frac{12}{5}$ and $\frac{12}{5} \le p \le 3$.

Case 1: $\sqrt{3} \le p \le \frac{12}{5}$. Since

$$6abc+3(q+1)-2pq \ge 3(q+1)-2pq = 3-(2p-3)q = \frac{1}{2}[6-(2p-3)(p^2-3)],$$

it suffices to show that

$$(2p-3)(p^2-3) \le 6.$$

Indeed, we have

$$(2p-3)(p^2-3) \le \left(\frac{24}{5}-3\right)\left(\frac{144}{25}-3\right)=\frac{621}{125}<6.$$

Case 2: $\frac{12}{5} \le p \le 3$. According to Schur's inequality of degree three, we have

$$p^3+9abc \ge 4pq.$$

Thus, it suffices to prove that

$$2(4pq-p^3)+9(q+1) \ge 6pq,$$

which is equivalent to

$$(2p+9)q-2p^3+9 \ge 0,$$

$$(2p+9)(p^2-3)-4p^3+18 \ge 0,$$
$$-2p^3+9p^2-6p-9 \ge 0,$$
$$(3-p)(2p^2-3p-3) \ge 0.$$

This inequality is true since $3-p \ge 0$ and

$$2p^2-3p-3 \ge \frac{24}{5}p-3p-3 = \frac{9}{5}p-3 \ge \frac{9}{5}\cdot\frac{12}{5}-3 > 0.$$

The equality holds for $a=b=c=1$.

□

P 1.7. *If a,b,c are nonnegative real numbers such that $a^2+b^2+c^2=3$, then*

$$a^2b^3+b^2c^3+c^2a^3 \le 3.$$

(Vasile C., 2005)

Solution. Let (x,y,z) be a permutation of (a,b,c) such that

$$x \ge y \ge z.$$

Since

$$x^2y^2 \ge z^2x^2 \ge y^2z^2,$$

the rearrangement inequality yields

$$a^2b^3+b^2c^3+c^2a^3 = b\cdot a^2b^2+c\cdot b^2c^2+a\cdot c^2a^2 \le x\cdot x^2y^2+y\cdot z^2x^2+z\cdot y^2z^2$$
$$= y(x^3y+z^2x^2+yz^3) \le y\left(x^2\cdot\frac{x^2+y^2}{2}+z^2x^2+z^2\cdot\frac{y^2+z^2}{2}\right)$$
$$= \frac{y(x^2+z^2)(x^2+y^2+z^2)}{2} = \frac{3y(x^2+z^2)}{2}.$$

Thus, it suffices to show that

$$y(x^2+z^2) \le 2$$

for $x^2+y^2+z^2=3$. By the AM-GM inequality, we get

$$6 = 2y^2+(x^2+z^2)+(x^2+z^2) \ge 3\sqrt[3]{2y^2(x^2+z^2)^2}.$$

The equality holds for $a=b=c=1$.

□

P 1.8. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$a^4b^2+b^4c^2+c^4a^2+4 \ge a^3b^3+b^3c^3+c^3a^3.$$

Solution. Write the inequality as

$$a^2(a^2b^2 + c^4 - ab^3 - ac^3) + 4 \ge b^2c^2(bc - b^2).$$

Since

$$\begin{aligned} 2\sum(a^2b^2 + c^4 - ab^3 - ac^3) &= \sum[a^4 + b^4 + 2a^2b^2 - 2ab(a^2 + b^2)] \\ &= \sum(a^2 + b^2)(a - b)^2 \ge 0, \end{aligned}$$

we may assume (without loss of generality) that

$$a^2b^2 + c^4 - ab^3 - ac^3 \ge 0.$$

Thus, it suffices to show that

$$4 \ge b^2c^2(bc - b^2).$$

Since

$$bc - b^2 \le \frac{c^2}{4},$$

it is enough to prove that

$$16 \ge b^2c^4.$$

From

$$3 = a + b + c \ge b + \frac{c}{2} + \frac{c}{2} \ge 3\sqrt[3]{b\left(\frac{c}{2}\right)^2},$$

the conclusion follows. The equality holds for $a = 0$, $b = 1$, $c = 2$ (or any cyclic permutation).

□

P 1.9. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

(a) $$ab^2 + bc^2 + ca^2 + abc \le 4;$$

(b) $$\frac{a}{4-b} + \frac{b}{4-c} + \frac{c}{4-a} \le 1;$$

(c) $$ab^3 + bc^3 + ca^3 + (ab + bc + ca)^2 \le 12;$$

(d) $$\frac{ab^2}{1+a+b} + \frac{bc^2}{1+b+c} + \frac{ca^2}{1+c+a} \le 1.$$

Solution. (a) ***First Solution.*** Let (x, y, z) be a permutation of (a, b, c) such that

$$x \geq y \geq z.$$

As shown in the second solution of P 1.1,

$$ab^2 + bc^2 + ca^2 \leq y(x^2 + xz + z^2);$$

hence

$$ab^2 + bc^2 + ca^2 + abc \leq y(x + z)^2.$$

Thus, it suffices to show that $x + y + z = 3$ involves

$$y(x + z)^2 \leq 4.$$

According to the AM-GM inequality, we have

$$\frac{1}{4}y(x+z)^2 = y \cdot \frac{x+z}{2} \cdot \frac{x+z}{2} \leq \left(\frac{y + \frac{x+z}{2} + \frac{x+z}{2}}{3} \right)^3 = 1.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$, $b = 1$, $c = 2$ (or any cyclic permutation).

Second Solution. Without loss of generality, assume that b is between a and c; that is,

$$(b - a)(b - c) \leq 0, \quad b^2 + ca \leq b(c + a).$$

Therefore,

$$ab^2 + bc^2 + ca^2 + abc = a(b^2 + ca) + bc^2 + abc \leq ab(c + a) + bc^2 + abc$$

$$= b(a + c)^2 = b(3 - b)^2 = 4 + (b^3 - 6b^2 + 9b - 4) = 4 - (1 - b)^2(4 - b) \leq 4.$$

Third Solution. Write the inequality in the homogeneous form

$$4(a + b + c)^3 \geq 27(ab^2 + bc^2 + ca^2 + abc).$$

Without loss of generality, suppose that $a = min\{a, b, c\}$. Putting $b = a + x$ and $c = a + y$, where $x, y \geq 0$, the inequality can be restated as

$$9(x^2 - xy + y^2)a + (2x - y)^2(x + 4y) \geq 0,$$

which is obviously true.

(b) ***First Solution.*** Write the inequality in the homogeneous form

$$\sum \frac{a}{4a + b + 4c} \leq \frac{1}{3}.$$

Multiplying by $a+b+c$, the inequality becomes as follows:

$$\sum \frac{a^2+ab+ac}{4a+b+4c} \le \frac{a+b+c}{3},$$

$$\sum \left(\frac{a^2+ab+ac}{4a+b+4c} - \frac{a}{4} \right) \le \frac{a+b+c}{12},$$

$$\sum \frac{9ab}{4a+b+4c} \le a+b+c.$$

Since

$$\frac{9}{4a+b+4c} = \frac{9}{(2a+c)+(2a+c)+(2c+b)} \le \frac{1}{2a+c} + \frac{1}{2a+c} + \frac{1}{2c+b}$$
$$= \frac{2}{2a+c} + \frac{1}{2c+b},$$

we have

$$\sum \frac{9ab}{4a+b+4c} \le \sum \frac{2ab}{2a+c} + \sum \frac{ab}{2c+b} = \sum \frac{2ab}{2a+c} + \sum \frac{bc}{2a+c}$$
$$= \sum \frac{2ab+bc}{2a+c} = \sum b = a+b+c.$$

The equality holds for $a=b=c=1$, and also for $a=0$, $b=1$, $c=2$ (or any cyclic permutation).

Second Solution. Write the inequality as follows:

$$\sum a(4-a)(4-c) \le (4-a)(4-b)(4-c),$$

$$32 + \sum ab^2 + abc \le 4\left(\sum a^2 + 2\sum ab\right),$$

$$32 + \sum ab^2 + abc \le 4\left(\sum a\right)^2,$$

$$ab^2 + bc^2 + ca^2 + abc \le 4.$$

The last inequality is just the inequality in (a).

(c) Using the inequality in (a), we get

$$(a+b+c)(ab^2+bc^2+ca^2+abc) \le 12,$$

which is equivalent to the desired inequality

$$ab^3 + bc^3 + ca^3 + (ab+bc+ca)^2 \le 12.$$

(d) Let $q = ab+bc+ca$. Since

$$\sum ab^2(1+b+c)(1+c+a) = \sum ab^2(4+q+c+c^2) = (4+q)\sum ab^2 + (3+q)abc$$

and

$$\prod(1+a+b) = 1+\sum(a+b)+\sum(b+c)(c+a)+\prod(a+b)$$
$$= 7+3q+\sum c^2+(3q-abc) = 16+4q-abc,$$

the inequality is equivalent to

$$(4+q)\sum ab^2+(3+q)abc \le 16+4q-abc,$$

$$(4+q)\left(\sum ab^2+abc-4\right) \le 0.$$

According to (a), the desired inequality is clearly true.

Remark. The following statement is also valid:

- *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$ab^2+bc^2+ca^2+abc+(a-1)^2(b-1)^2(c-1)^2 \le 4,$$

with equality for $a=b=c=1$, and also for $a=0$, $b=1$, $c=2$ (or any cyclic permutation).

Having in view the second solution of (a), it is enough to show that

$$(a-1)^2(b-1)^2(c-1)^2 \le (4-b)(1-b)^2,$$

where b is between a and c. This is true if

$$|(a-1)(c-1)| \le \sqrt{4-b}.$$

Assuming that $a \le c$ (hence $a \le b \le c$, $a \le 1$, $c \ge 1$), the inequality can be written as follows:

$$(1-a)(c-1) \le \sqrt{4-b},$$

$$a+c-1 \le ac+\sqrt{4-b},$$

$$2-b \le ac+\sqrt{4-b}.$$

This is true if

$$2-b \le \sqrt{4-b}.$$

Indeed,

$$\sqrt{4-b}-(2-b) = \frac{4-b-(2-b)^2}{\sqrt{4-b}+2-b} = \frac{b(3-b)}{\sqrt{4-b}+2-b}$$
$$= \frac{b(a+c)}{\sqrt{4-b}+2-b}) \ge 0.$$

□

P 1.10. *If a, b, c are positive real numbers, then*

$$\frac{1}{a(a+2b)}+\frac{1}{b(b+2c)}+\frac{1}{c(c+2a)}\ge\frac{3}{ab+bc+ca}.$$

First Solution. Write the inequality as

$$\sum\frac{a(b+c)+bc}{a(a+2b)}\ge 3,$$

$$\sum\frac{b+c}{a+2b}+\sum\frac{bc}{a(a+2b)}\ge 3.$$

It suffices to show that

$$\sum\frac{b+c}{a+2b}\ge 2$$

and

$$\sum\frac{bc}{a(a+2b)}\ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{b+c}{a+2b}\ge\frac{\left[\sum(b+c)\right]^2}{\sum(b+c)(a+2b)}=\frac{4\left(\sum a\right)^2}{2\sum a^2+4\sum ab}=2$$

and

$$\sum\frac{bc}{a(a+2b)}\ge\frac{\left(\sum bc\right)^2}{abc\sum(a+2b)}=\frac{\left(\sum bc\right)^2}{3abc\sum a}=1+\frac{\sum a^2(b-c)^2}{6abc\sum a}\ge 1.$$

The equality holds for $a=b=c$.

Second Solution. We apply the Cauchy-Schwarz inequality in the following way

$$\sum\frac{1}{a(a+2b)}\ge\frac{\left(\sum c\right)^2}{\sum ac^2(a+2b)}=\frac{\left(\sum a\right)^2}{\sum a^2b^2+2abc\sum a}.$$

Thus, it suffices to show that

$$\frac{\left(\sum a\right)^2}{\sum a^2b^2+2abc\sum a}\ge\frac{3}{\sum ab},$$

which is equivalent to

$$\left(\sum ab\right)\left(\sum a^2+2\sum ab\right)\ge 3\sum a^2b^2+6abc\sum a,$$

$$\sum ab(a^2+b^2)\ge\sum a^2b^2+abc\sum a.$$

The last inequality follows by summing the obvious inequalities

$$\sum ab(a^2+b^2) \ge 2\sum a^2b^2$$

and

$$\sum a^2b^2 \ge abc\sum a.$$

□

P 1.11. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a}{b^2+2c}+\frac{b}{c^2+2a}+\frac{c}{a^2+2b} \ge 1.$$

Solution. Using the Cauchy-Schwarz inequality, we get

$$\sum \frac{a}{b^2+2c} \ge \frac{\left(\sum a\right)^2}{\sum a(b^2+2c)} = 1+\frac{\sum a^2-\sum ab^2}{\sum ab^2+2\sum ab}.$$

Thus, it suffices to show that

$$\sum a^2 - \sum ab^2 \ge 0.$$

Write this inequality in the homogeneous form

$$(a+b+c)(a^2+b^2+c^2) \ge 3(ab^2+bc^2+ca^2),$$

which is equivalent to the obvious inequality

$$a(a-c)^2+b(b-a)^2+c(c-b)^2 \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.12. *If a,b,c are positive real numbers such that $a+b+c \ge 3$, then*

$$\frac{a-1}{b+1}+\frac{b-1}{c+1}+\frac{c-1}{a+1} \ge 0.$$

Solution. Write the inequality as

$$(a^2-1)(c+1)+(b^2-1)(a+1)+(c^2-1)(b+1)\geq 0,$$

$$ab^2+bc^2+ca^2+a^2+b^2+c^2\geq a+b+c+3.$$

From

$$a(b-1)^2+b(c-1)^2+c(a-1)^2\geq 0,$$

we get

$$ab^2+bc^2+ca^2\geq 2(ab+bc+ca)-(a+b+c).$$

Using this inequality yields

$$ab^2+bc^2+ca^2+a^2+b^2+c^2-a-b-c-3\geq (a+b+c)^2-2(a+b+c)-3$$

$$=(a+b+c-3)(a+b+c+1)\geq 0.$$

The equality holds for $a=b=c=1$.

□

P 1.13. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

(a) $$\frac{1}{2ab^2+1}+\frac{1}{2bc^2+1}+\frac{1}{2ca^2+1}\geq 1;$$

(b) $$\frac{1}{ab^2+2}+\frac{1}{bc^2+2}+\frac{1}{ca^2+2}\geq 1.$$

Solution. By the AM-GM inequality, we have

$$1=\left(\frac{a+b+c}{3}\right)^3\geq abc.$$

(a) Since

$$2ab^2+1\leq \frac{2b}{c}+1=\frac{2b+c}{c},$$

it suffices to show that

$$\frac{c}{2b+c}+\frac{a}{2c+a}+\frac{b}{2a+b}\geq 1.$$

Using the Cauchy-Schwarz inequality, we get

$$\sum\frac{c}{2b+c}\geq\frac{\left(\sum c\right)^2}{\sum c(2b+c)}=\frac{(a+b+c)^2}{(a+b+c)^2}=1.$$

The equality holds for $a=b=c=1$.

(b) By expanding, the inequality can be restated as

$$a^3b^3c^3 + abc(a^2b + b^2c + c^2a) \le 4.$$

Applying the AM-GM inequality gives

$$\begin{aligned}(a+b+c)^3 &= \sum a^3 + 6abc + 3\sum ab^2 + 3\sum a^2b \\ &\ge 3abc + 6abc + 9abc + 3\sum a^2b,\end{aligned}$$

i.e.

$$a^2b + b^2c + c^2a \le 9 - 6abc.$$

Therefore, it suffices to show that

$$a^3b^3c^3 + abc(9 - 6abc) \le 4,$$

which is equivalent to the obvious inequality

$$(abc-1)^2(abc-4) \le 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.14. *If a, b, c are positive real numbers such that $a + b + c = 3$, then*

$$\frac{ab}{9-4bc} + \frac{bc}{9-4ca} + \frac{ca}{9-4ab} \le \frac{3}{5}.$$

Solution. We have

$$\begin{aligned}\sum \frac{ab}{9-4bc} &\le \sum \frac{ab}{9-(b+c)^2} = \sum \frac{b}{3+b+c} = \sum \frac{b}{a+2b+2c} \\ &= \frac{1}{2}\sum\left[1 - \frac{a+2c}{a+2b+2c}\right] = \frac{3}{2} - \frac{1}{2}\sum \frac{a+2c}{a+2b+2c}.\end{aligned}$$

Thus, it suffices to show that

$$\sum \frac{a+2c}{a+2b+2c} \ge \frac{9}{5}.$$

Using the Cauchy-Schwarz inequality, we get

$$\sum \frac{a+2c}{a+2b+2c} \ge \frac{[\sum(a+2c)]^2}{\sum(a+2c)(a+2b+2c)} = \frac{9(a+b+c)^2}{5(a+b+c)^2} = \frac{9}{5}.$$

The equality holds for $a = b = c = 1$.

□

P 1.15. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

(a) $$\frac{a^2}{2a+b^2}+\frac{b^2}{2b+c^2}+\frac{c^2}{2c+a^2}\ge 1;$$

(b) $$\frac{a^2}{a+2b^2}+\frac{b^2}{b+2c^2}+\frac{c^2}{c+2a^2}\ge 1.$$

Solution. (a) By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^2}{2a+b^2}\ge\frac{\left(\sum a^2\right)^2}{\sum a^2(2a+b^2)}=\frac{\sum a^4+2\sum a^2b^2}{2\sum a^3+\sum a^2b^2}.$$

Thus, it suffices to prove that

$$\sum a^4+\sum a^2b^2\ge 2\sum a^3,$$

which is equivalent to the homogeneous inequalities

$$3\sum a^4+3\sum a^2b^2\ge 2\left(\sum a\right)\left(\sum a^3\right),$$

$$\sum a^4+3\sum a^2b^2-2\sum ab(a^2+b^2)\ge 0,$$

$$\sum(a-b)^4\ge 0.$$

The equality holds for $a=b=c=1$.

(b) By the Cauchy-Schwarz inequality, we get

$$\sum\frac{a^2}{a+2b^2}\ge\frac{\left(\sum a^2\right)^2}{\sum a^2(a+2b^2)}=\frac{\sum a^4+2\sum a^2b^2}{\sum a^3+2\sum a^2b^2}.$$

Thus, it suffices to prove that

$$\sum a^4\ge\sum a^3.$$

We have

$$\sum a^4-\sum a^3=\sum(a^4-a^3-a+1)=\sum(a-1)(a^3-1)\ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.16. *Let a,b,c be positive real numbers such that $a+b+c=3$. Then,*

$$\frac{1}{a+b^2+c^3}+\frac{1}{b+c^2+a^3}+\frac{1}{c+a^2+b^3}\le 1.$$

(Vasile C., 2009)

Solution *(by Vo Quoc Ba Can).* By the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{a+b^2+c^3} \le \sum \frac{a^3+b^2+c}{(a^2+b^2+c^2)^2} = \frac{\sum a^3 + \sum a^2 + 3}{(a^2+b^2+c^2)^2}.$$

Therefore, it suffices to show that

$$(a^2+b^2+c^2)^2 \ge a^3+b^3+c^3+(a^2+b^2+c^2)+3,$$

or, equivalently,

$$(a^2+b^2+c^2)^2 + \sum a^2(3-a) \ge 4(a^2+b^2+c^2)+3.$$

Let us denote $t = a^2+b^2+c^2$. Applying again the Cauchy-Schwarz inequality, we get

$$\sum a^2(3-a) \ge \frac{[\sum a(3-a)]^2}{\sum(3-a)} = \frac{(9-a^2-b^2-c^2)^2}{6}.$$

Thus, it is enough to show that

$$t^2 + \frac{(9-t)^2}{6} \ge 4t+3.$$

This inequality reduces to $(t-3)^2 \ge 0$. The equality occurs for $a=b=c=1$.

□

P 1.17. *If a,b,c are positive real numbers, then*

$$\frac{1+a^2}{1+b+c^2} + \frac{1+b^2}{1+c+a^2} + \frac{1+c^2}{1+a+b^2} \ge 2.$$

Solution. From

$$1+b+c^2 \le 1+\frac{1+b^2}{2}+c^2,$$

we have

$$\frac{1+a^2}{1+b+c^2} \ge \frac{2(1+a^2)}{1+b^2+2(1+c^2)}.$$

Thus, it suffices to show that

$$\frac{x}{y+2z} + \frac{y}{z+2x} + \frac{z}{x+2y} \ge 1,$$

where

$$x = 1+a^2, \quad y = 1+b^2, \quad z = 1+c^2.$$

Using the Cauchy-Schwarz inequality gives

$$\frac{x}{y+2z}+\frac{y}{z+2x}+\frac{z}{x+2y} \ge \frac{(x+y+z)^2}{x(y+2z)+y(z+2x)+z(x+2y)}$$
$$=\frac{(x+y+z)^2}{3(xy+yz+zx)} \ge 1.$$

The equality occurs for $a=b=c=1$.

□

P 1.18. *If a,b,c are nonnegative real numbers, then*

$$\frac{a}{4a+4b+c}+\frac{b}{4b+4c+a}+\frac{c}{4c+4a+b} \le \frac{1}{3}.$$

(Pham Kim Hung, 2007)

Solution. If two of a,b,c are zero, then the inequality is trivial. Otherwise, multiplying by $4(a+b+c)$, the inequality becomes as follows:

$$\sum \frac{4a(a+b+c)}{4a+4b+c} \le \frac{4}{3}(a+b+c),$$

$$\sum \left[\frac{4a(a+b+c)}{4a+4b+c}-a\right] \le \frac{1}{3}(a+b+c),$$

$$\sum \frac{ca}{4a+4b+c} \le \frac{1}{9}(a+b+c).$$

By the Cauchy-Schwarz inequality, we get

$$\frac{9}{4a+4b+c}=\frac{9}{(2b+c)+2(2a+b)} \le \frac{1}{2b+c}+\frac{2}{2a+b}.$$

Therefore,

$$\sum \frac{ca}{4a+4b+c} \le \frac{1}{9}\sum ca\left(\frac{1}{2b+c}+\frac{2}{2a+b}\right)$$
$$=\frac{1}{9}\left(\sum \frac{ca}{2b+c}+\sum \frac{2ab}{2b+c}\right)=\frac{1}{9}\sum a,$$

as desired. The equality occurs for $a=b=c$, and also for $a=2b$ and $c=0$ (or any cyclic permutation).

□

P 1.19. *If a,b,c are positive real numbers, then*

$$\frac{a+b}{a+7b+c}+\frac{b+c}{b+7c+a}+\frac{c+a}{c+7a+b} \ge \frac{2}{3}.$$

Solution. Write the inequality as

$$\sum\left(\frac{a+b}{a+7b+c}-\frac{1}{k}\right)\geq\frac{2}{3}-\frac{3}{k},\quad k>0,$$

$$\sum\frac{(k-1)a+(k-7)b-c}{a+7b+c}\geq\frac{2k-9}{3}.$$

Consider that all fractions in the left hand side are nonnegative and apply the Cauchy-Schwarz inequality, as follows:

$$\sum\frac{(k-1)a+(k-7)b-c}{a+7b+c}\geq\frac{[(k-1)\sum a+(k-7)\sum b-\sum c]^2}{\sum(a+7b+c)[(k-1)a+(k-7)b-c]}$$

$$=\frac{(2k-9)^2\left(\sum a\right)^2}{(8k-51)\sum a^2+2(5k-15)\sum ab}.$$

We choose $k=12$ to have $8k-51=5k-15$, hence

$$(8k-51)\sum a^2+2(5k-15)\sum ab=45\left(\sum a\right)^2.$$

For this value of k, the desired inequality

$$\sum\frac{(k-1)a+(k-7)b-c}{a+7b+c}\geq\frac{2k-9}{3}$$

can be restated as

$$\sum\frac{11a+5b-c}{a+7b+c}\geq 5.$$

Without loss of generality, assume that $a=\max\{a,b,c\}$. Consider further two cases.

Case 1: $11b+5c-a\geq 0$. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{11a+5b-c}{a+7b+c}\geq\frac{[\sum(11a+5b-c)]^2}{\sum(a+7b+c)(11a+5b-c)}=\frac{225\left(\sum a\right)^2}{45\left(\sum a\right)^2}=5.$$

Case 2: $11b+5c-a<0$. We have

$$\sum\frac{a+b}{a+7b+c}>\frac{a+b}{a+7b+c}=\frac{2}{3}+\frac{a-11b-2c}{3(a+7b+c)}>\frac{2}{3}.$$

Thus, the proof is completed. The equality holds for $a=b=c$.

□

P 1.20. *If a,b,c are positive real numbers, then*

$$\frac{a+b}{a+3b+c}+\frac{b+c}{b+3c+a}+\frac{c+a}{c+3a+b}\geq\frac{6}{5}.$$

(Vasile C., 2007)

Solution. Due to homogeneity, we may assume that

$$a+b+c=1,$$

when the inequality becomes

$$\sum \frac{1-c}{1+2b} \ge \frac{6}{5},$$

$$5\sum(1-c)(1+2c)(1+2a) \ge 6(2a+1)(2b+1)(2c+1),$$

$$5\left(4+6\sum ab-4\sum a^2b\right)=6\left(3+4\sum ab+8abc\right),$$

$$1+3\sum ab \ge 10\sum a^2b+24abc,$$

$$(a+b+c)^3+3(a+b+c)(ab+bc+ca) \ge 10(a^2b+b^2c+c^a)+24abc,$$

$$\sum a^3+6\sum ab^2 \ge 4\sum a^2b+9abc,$$

$$\left[2\sum a^3-\sum ab(a+b)\right]+3\left[\sum ab(a+b)-6abc\right]+10\left(\sum ab^2-\sum a^2b\right) \ge 0,$$

$$\sum(a+b)(a-b)^2+3\sum c(a-b)^2+10\left(\sum ab^2-\sum a^2b\right) \ge 0,$$

$$\sum(a+b+3c)(a-b)^2+10(a-b)(b-c)(c-a) \ge 0.$$

Assume that

$$a=\min\{a,b,c\},$$

and use the substitution

$$b=a+x, \quad c=a+y, \qquad x,y \ge 0.$$

The inequality becomes

$$(5a+x+3y)x^2+(5a+x+y)(x-y)^2+(5a+3x+y)y^2-10xy(x-y) \ge 0.$$

Clearly, it suffices to consider the case $a=0$, when the inequality becomes

$$x^3-4x^2y+6xy^2+y^3 \ge 0.$$

Indeed, we have

$$x^3-4x^2y+6xy^2+y^3=x(x-2y)^2+2xy^2+y^3 \ge 0.$$

The equality holds for $a=b=c$.

□

P 1.21. *If a, b, c are positive real numbers, then*

$$\frac{2a+b}{2a+c}+\frac{2b+c}{2b+a}+\frac{2c+a}{2c+b} \ge 3.$$

(Pham Kim Hung, 2007)

Solution. Without loss of generality, assume that $a = \max\{a, b, c\}$. There are two cases to consider.

Case 1: $a \le 2b + 2c$. Write the inequality as

$$\sum \left(\frac{2a+b}{2a+c} - \frac{1}{2} \right) \ge \frac{3}{2},$$

$$\sum \frac{2a+2b-c}{2a+c} \ge 3.$$

Since

$$2a+2b-c > 0, \quad 2b+2c-a \ge 0, \quad 2c+2a-b > 0,$$

we may apply the Cauchy-Schwarz inequality to get

$$\sum \frac{2a+2b-c}{2a+c} \ge \frac{\left[\sum(2a+2b-c)\right]^2}{\sum(2a+2b-c)(2a+c)} = \frac{9\left(\sum a\right)^2}{3\left(\sum a\right)^2} = 3.$$

Case 2: $a > 2b + 2c$. Since

$$2a + c - (2b + a) = (a - 2b - 2c) + 3c > 0,$$

we have

$$\frac{2a+b}{2a+c}+\frac{2b+c}{2b+a} > \frac{2a+b}{2a+c}+\frac{2b+c}{2a+c} = 1+\frac{3b}{2a+c} > 1.$$

Therefore, it suffices to show that

$$\frac{2c+a}{2c+b} \ge 2.$$

Indeed,

$$\frac{2c+a}{2c+b} > \frac{2c+2b+2c}{2c+b} = 2.$$

Thus, the proof is completed. The equality holds for $a = b = c$.

□

P 1.22. *If a, b, c are positive real numbers, then*

$$\frac{a(a+b)}{a+c}+\frac{b(b+c)}{b+a}+\frac{c(c+a)}{c+b} \le \frac{3(a^2+b^2+c^2)}{a+b+c}.$$

(Pham Huu Duc, 2007)

Solution. Write the inequality as

$$\sum \frac{a(a+b)(a+b+c)}{a+c} \le 3(a^2+b^2+c^2),$$

$$\sum \frac{ab(a+b)+a(a+b)(a+c)}{a+c} \le 3(a^2+b^2+c^2),$$

$$\sum \frac{ab(a+b)}{a+c} \le 2(a^2+b^2+c^2)-(ab+bc+ca).$$

Let (x,y,z) be a permutation of (a,b,c) such that $x \ge y \ge z$. Since

$$x+y \ge z+x \ge y+z$$

and

$$xy(x+y) \ge zx(z+x) \ge yz(y+z),$$

by the rearrangement inequality, we have

$$\sum \frac{ab(a+b)}{a+c} \le \frac{xy(x+y)}{y+z} + \frac{zx(z+x)}{z+x} + \frac{yz(y+z)}{x+y}.$$

Consequently, it suffices to show that

$$\frac{xy(x+y)}{y+z} + \frac{yz(y+z)}{x+y} \le 2(x^2+y^2+z^2) - xy - yz - 2zx.$$

Write this inequality as follows:

$$xy\left(\frac{x+y}{y+z}-1\right) + yz\left(\frac{y+z}{x+y}-1\right) \le 2(x^2+y^2+z^2-xy-yz-zx),$$

$$\frac{xy(x-z)}{y+z} + \frac{yz(z-x)}{x+y} \le (x-y)^2+(y-z)^2+(z-x)^2,$$

$$\frac{y(x+y+z)(z-x)^2}{(x+y)(y+z)} \le (x-y)^2+(y-z)^2+(z-x)^2.$$

Since

$$y(x+y+z) < (x+y)(y+z),$$

the last inequality is clearly true. The equality holds for $a=b=c$.

□

P 1.23. *If a,b,c are real numbers, then*

$$\frac{a^2-bc}{4a^2+b^2+4c^2} + \frac{b^2-ca}{4b^2+c^2+4a^2} + \frac{c^2-ab}{4c^2+a^2+4b^2} \ge 0.$$

(Vasile C., 2006)

Solution. Since
$$\frac{4(a^2-bc)}{4a^2+b^2+4c^2}=1-\frac{(b+2c)^2}{4a^2+b^2+4c^2},$$
we may rewrite the inequality as
$$\frac{(b+2c)^2}{4a^2+b^2+4c^2}+\frac{(c+2a)^2}{4b^2+c^2+4a^2}+\frac{(a+2b)^2}{4c^2+a^2+4b^2}\le 3.$$
Using the Cauchy-Schwarz inequality gives
$$\frac{(b+2c)^2}{4a^2+b^2+4c^2}=\frac{(b+2c)^2}{(2a^2+b^2)+2(2c^2+a^2)}\le\frac{b^2}{2a^2+b^2}+\frac{2c^2}{2c^2+a^2}.$$
Therefore,
$$\sum\frac{(b+2c)^2}{4a^2+b^2+4c^2}\le\sum\frac{b^2}{2a^2+b^2}+\sum\frac{2c^2}{2c^2+a^2}=\sum\frac{b^2}{2a^2+b^2}+\sum\frac{2a^2}{2a^2+b^2}=3.$$
The equality occurs when
$$a(2b^2+c^2)=b(2c^2+a^2)=c(2a^2+b^2);$$
that is, when $a=b=c$, and also when $a=2b=4c$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

- *Let a,b,c be real numbers. If $k>0$, then*
$$\frac{a^2-bc}{2ka^2+b^2+k^2c^2}+\frac{b^2-ca}{2kb^2+c^2+k^2a^2}+\frac{c^2-ab}{2kc^2+a^2+k^2b^2}\ge 0,$$
with equality for $a=b=c$, and also for $a=kb=k^2c$ (or any cyclic permutation).

□

P 1.24. *If a,b,c are real numbers, then*

(a) $$a(a+b)^3+b(b+c)^3+c(c+a)^3\ge 0;$$

(b) $$a(a+b)^5+b(b+c)^5+c(c+a)^5\ge 0.$$

(Vasile C., 1989)

Solution. (a) Using the substitution
$$b+c=2x,\quad c+a=2y,\quad a+b=2z,$$
which are equivalent to
$$a=y+z-x,\quad b=z+x-y,\quad c=x+y-z,$$

the inequality becomes in succession

$$x^4+y^4+z^4+xy^3+yz^3+zx^3 \ge x^3y+y^3z+z^3x,$$

$$\sum(x^4+2xy^3-2x^3y+y^4)\ge 0,$$

$$\sum(x^2-xy-y^2)^2+\sum x^2y^2\ge 0,$$

the last being clearly true. The equality occurs for $a=b=c=0$.

(b) Using the same substitution, the inequality turns into

$$x^6+y^6+z^6+xy^5+yz^5+zx^5 \ge x^5y+y^5z+z^5x,$$

which is equivalent to

$$\sum[x^6+y^6-2xy(x^4-y^4)]\ge 0,$$

$$\sum[(x^2+y^2)(x^4-x^2y^2+y^4)-2xy(x^2+y^2)(x^2-y^2)]\ge 0,$$

$$\sum(x^2+y^2)(x^2-xy-y^2)^2\ge 0.$$

The equality occurs for $a=b=c=0$.

□

P 1.25. *If a,b,c are real numbers, then*

$$3(a^4+b^4+c^4)+4(a^3b+b^3c+c^3a)\ge 0.$$

(Vasile C., 2005)

Solution. If a,b,c are nonnegative, then the inequality is trivial. Since the inequality remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, it suffices to consider the case when only one of a,b,c is negative; let $c<0$. Replacing now c with $-c$, the inequality can be restated as

$$3(a^4+b^4+c^4)+4a^3b\ge 4(b^3c+c^3a),$$

where $a,b,c\ge 0$. It is enough to prove that

$$3(a^4+b^4+c^4+a^3b)\ge 4(b^3c+c^3a).$$

Case 1: $a\le b$. Since $a^3b\ge a^4$, it suffices to show that

$$6a^4+3b^4+3c^4\ge 4(b^3c+ac^3).$$

Using the AM-GM inequality yields

$$3b^4+c^4\ge 4\sqrt[4]{b^{12}c^4}=4b^3c.$$

Therefore, it suffices to show that

$$6a^4 + 2c^4 \ge 4ac^3.$$

Indeed, we have

$$3a^4 + c^4 = 3a^4 + \frac{1}{3}c^4 + \frac{1}{3}c^4 + \frac{1}{3}c^4 \ge 4\sqrt[4]{\frac{a^4c^{12}}{9}} = \frac{4}{\sqrt{3}}ac^3 \ge 2ac^3.$$

Case 2: $a \ge b$. Since $3a^3b \ge 3b^4$, it suffices to show that

$$3a^4 + 6b^4 + 3c^4 \ge 4(b^3c + ac^3).$$

By the AM-GM inequality, we get

$$6b^4 + \frac{c^4}{8} = 2b^4 + 2b^4 + 2b^4 + \frac{c^4}{8} \ge 4\sqrt[4]{b^{12}c^4} = 4b^3c.$$

Thus, we still have to show that

$$3a^4 + \frac{23}{8}c^4 \ge 4ac^3.$$

We will prove the sharper inequality

$$3a^4 + \frac{5}{2}c^4 \ge 4ac^3.$$

Indeed, we have

$$3a^4 + \frac{5}{2}c^4 = 3a^4 + \frac{5}{6}c^4 + \frac{5}{6}c^4 + \frac{5}{6}c^4 \ge 4\sqrt[4]{\frac{125a^4c^{12}}{72}} \ge 4ac^3.$$

The equality occurs for $a = b = c = 0$.

□

P 1.26. *If a, b, c are positive real numbers, then*

$$\frac{(a-b)(2a+b)}{(a+b)^2} + \frac{(b-c)(2b+c)}{(b+c)^2} + \frac{(c-a)(2c+a)}{(c+a)^2} \ge 0.$$

(Vasile C., 2006)

Solution. Since

$$\frac{(a-b)(2a+b)}{(a+b)^2} = \frac{2a^2 - b(a+b)}{(a+b)^2} = \frac{2a^2}{(a+b)^2} - \frac{b}{a+b},$$

we can write the inequality as

$$2\sum\left(\frac{a}{a+b}\right)^2-\sum\frac{b}{a+b}\ge 0.$$

According to P 1.1 in Volume 2, we have

$$\begin{aligned}2\sum\left(\frac{a}{a+b}\right)^2&=\sum\left(\frac{a}{a+b}\right)^2+\sum\left(\frac{b}{b+c}\right)^2\\&=\sum\left[\frac{1}{(1+b/a)^2}+\frac{1}{(1+c/b)^2}\right]\\&\ge\sum\frac{1}{1+c/a}=\sum\frac{a}{a+c}=\sum\frac{b}{b+a}.\end{aligned}$$

Therefore,

$$2\sum\left(\frac{a}{a+b}\right)^2-\sum\frac{b}{a+b}\ge\sum\frac{b}{b+a}-\sum\frac{b}{a+b}=0.$$

The equality holds for $a=b=c$.

□

P 1.27. *If a,b,c are positive real numbers, then*

$$\frac{(a-b)(2a+b)}{a^2+ab+b^2}+\frac{(b-c)(2b+c)}{b^2+bc+c^2}+\frac{(c-a)(2c+a)}{c^2+ca+a^2}\ge 0.$$

(Vasile C., 2006)

Solution. Since

$$\frac{(a-b)(2a+b)}{a^2+ab+b^2}=\frac{3a^2-(a^2+ab+b^2)}{a^2+ab+b^2}=\frac{3a^2}{a^2+ab+b^2}-1,$$

we can write the inequality as

$$\sum\frac{a^2}{a^2+ab+b^2}\ge 1,$$

$$\sum\frac{1}{1+b/a+(b/a)^2}\ge 1.$$

Clearly, this inequality follows immediately from P 1.45 in Volume 2. The equality holds for $a=b=c$.

□

P 1.28. *If a, b, c are positive real numbers, then*

$$\frac{(a-b)(3a+b)}{a^2+b^2}+\frac{(b-c)(3b+c)}{b^2+c^2}+\frac{(c-a)(3c+a)}{c^2+a^2}\ge 0.$$

(Vasile C., 2006)

Solution. Since

$$(a-b)(3a+b)=(a-b)^2+2(a^2-b^2),$$

we can write the inequality as

$$\sum\frac{(a-b)^2}{a^2+b^2}+2\sum\frac{a^2-b^2}{a^2+b^2}\ge 0.$$

Using the identity

$$\sum\frac{a^2-b^2}{a^2+b^2}+\prod\frac{a^2-b^2}{a^2+b^2}=0,$$

the inequality becomes

$$\sum\frac{(a-b)^2}{a^2+b^2}\ge 2\prod\frac{a^2-b^2}{a^2+b^2}.$$

By the AM-GM inequality, we have

$$\sum\frac{(a-b)^2}{a^2+b^2}\ge 3\sqrt[3]{\prod\frac{(a-b)^2}{a^2+b^2}}.$$

Thus, it suffices to show that

$$3\sqrt[3]{\prod\frac{(a-b)^2}{a^2+b^2}}\ge 2\prod\frac{a^2-b^2}{a^2+b^2},$$

which is equivalent to

$$27\prod\frac{(a-b)^2}{a^2+b^2}\ge 8\prod\frac{(a^2-b^2)^3}{(a^2+b^2)^3}.$$

This inequality is true if

$$27\prod(a^2+b^2)^2\ge\prod(a-b)(a+b)^3.$$

Assume that $a=\max\{a,b,c\}$. For the nontrivial case $a>c>b$, we can get this inequality by multiplying the inequalities

$$3(a^2+b^2)^2\ge 2(a-b)(a+b)^3,$$

$$3(c^2+b^2)^2\ge 2(c-b)(c+b)^3,$$

$$3(a^2+c^2)^2\ge 2(a-c)(a+c)^3.$$

These inequalities are true because

$$3(a^2+b^2)^2-2(a-b)(a+b)^3=a^2(a-2b)^2+b^2(2a^2+4ab+5b^2)>0.$$

The equality holds for $a=b=c$.

□

P 1.29. *Let a,b,c be positive real numbers such that $abc=1$. Then,*

$$\frac{1}{1+a+b^2}+\frac{1}{1+b+c^2}+\frac{1}{1+c+a^2}\le 1.$$

(Vasile C., 2005)

Solution. Using the substitution

$$a=x^3,\quad b=y^3,\quad c=z^3,$$

we have to show that $xyz=1$ involves

$$\frac{1}{1+x^3+y^6}+\frac{1}{1+y^3+z^6}+\frac{1}{1+z^3+x^6}\le 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{1}{1+x^3+y^6}\le\sum\frac{z^4+x+y^{-2}}{(z^2+x^2+y^2)^2}=\frac{\sum(z^4+x^2yz+x^2z^2)}{(x^2+y^2+z^2)^2}.$$

So, it remains to show that

$$(x^2+y^2+z^2)^2\ge\sum x^4+xyz\sum x+\sum x^2y^2,$$

which is equivalent to the known inequality

$$\sum x^2y^2\ge xyz\sum x.$$

The equality occurs for $a=b=c=1$.

Remark. Actually, the following generalization holds:

- *Let a,b,c be positive real numbers such that $abc=1$. If $k\ge 0$, then*

$$\frac{1}{1+a+b^k}+\frac{1}{1+b+c^k}+\frac{1}{1+c+a^k}\le 1.$$

□

P 1.30. *Let a,b,c be positive real numbers such that $abc=1$. Then,*

$$\frac{a}{(a+1)(b+2)}+\frac{b}{(b+1)(c+2)}+\frac{c}{(c+1)(a+2)}\ge\frac{1}{2}.$$

Solution. Using the substitution

$$a = \frac{x}{y}, \quad b = \frac{y}{z}, \quad c = \frac{z}{x},$$

where x, y, z are positive real numbers, the inequality can be restated as

$$\frac{zx}{(x+y)(y+2z)} + \frac{xy}{(y+z)(z+2x)} + \frac{yz}{(z+x)(x+2y)} \geq \frac{1}{2}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{zx}{(x+y)(y+2z)} \geq \frac{\left(\sum zx\right)^2}{\sum zx(x+y)(y+2z)} = \frac{1}{2}.$$

The equality occurs for $a = b = c = 1$.

□

P 1.31. *If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then*

$$(a+2b)(b+2c)(c+2a) \geq 27.$$

(Michael Rozenberg, 2007)

Solution. Write the inequality in the homogeneous form

$$A + B \geq 0,$$

where

$$\begin{aligned} A &= (a+2b)(b+2c)(c+2a) - 3(a+b+c)(ab+bc+ca) \\ &= (a-b)(b-c)(c-a) \end{aligned}$$

and

$$B = 3(ab+bc+ca)[a+b+c-\sqrt{3(ab+bc+ca)}].$$

Since

$$\begin{aligned} B &= \frac{3(ab+bc+ca)[(a-b)^2+(b-c)^2+(c-a)^2]}{2(a+b+c+\sqrt{3(ab+bc+ca)})]} \\ &\geq \frac{3(ab+bc+ca)[(a-b)^2+(b-c)^2+(c-a)^2]}{4(a+b+c)}, \end{aligned}$$

it suffices to show that

$$4(a+b+c)(a-b)(b-c)(c-a) + 3(ab+bc+ca)[(a-b)^2+(b-c)^2+(c-a)^2] \geq 0.$$

Consider $c = \min\{a, b, c\}$, and use the substitution

$$a = c + x, \quad b = c + y, \qquad x, y \ge 0.$$

The inequality becomes

$$-4xy(x-y)(3c+x+y)+6(x^2-xy+y^2)[3c^2+2(x+y)c+xy] \ge 0,$$

which is equivalent to

$$9(x^2-xy+y^2)c^2+6Cc+D \ge 0,$$

where

$$C = x^3-x^2y+xy^2+y^3 \ge x(x^2-xy+y^2),$$
$$D = xy(x^2+5y^2-3xy) \ge (2\sqrt{5}-3)x^2y^2.$$

Since $C \ge 0$ and $D \ge 0$, the inequality is obvious. The equality holds for $a = b = c = 1$.

□

P 1.32. *If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then*

$$\frac{a}{a+a^3+b} + \frac{b}{b+b^3+c} + \frac{c}{c+c^3+a} \le 1.$$

(Andrei Ciupan, 2005)

Solution. Write the inequality as

$$\frac{1}{1+a^2+b/a} + \frac{1}{1+b^2+c/b} + \frac{1}{1+c^2+a/c} \le 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{1}{1+a^2+b/a} \le \sum \frac{c^2+1+ab}{(c+a+b)^2} = 1.$$

The equality holds for $a = b = c = 1$.

□

P 1.33. *If a, b, c are positive real numbers such that $a \ge b \ge c$ and $ab+bc+ca = 3$, then*

$$\frac{1}{a+2b} + \frac{1}{b+2c} + \frac{1}{c+2a} \ge 1.$$

Solution. According to the well known inequality

$$x + y + z \geq \sqrt{3(xy + yz + zx)},$$

where x, y, z are positive real numbers, it suffices to prove that

$$\frac{1}{(a+2b)(b+2c)} + \frac{1}{(b+2c)(c+2a)} + \frac{1}{(c+2a)(a+2b)} \geq \frac{1}{3}.$$

This is equivalent to the following inequalities

$$9(a+b+c) \geq (a+2b)(b+2c)(c+2a),$$

$$3(a+b+c)(ab+bc+ca) \geq (a+2b)(b+2c)(c+2a),$$

$$a^2b + b^2c + c^2a \geq ab^2 + bc^2 + ca^2,$$

$$(a-b)(b-c)(a-c) \geq 0.$$

The last inequality is clearly true for $a \geq b \geq c$. The equality occurs for $a = b = c = 1$.

□

P 1.34. *If $a, b, c \in [0, 1]$, then*

$$\frac{a}{4b^2+5} + \frac{b}{4c^2+5} + \frac{c}{4a^2+5} \leq \frac{1}{3}.$$

Solution. Let

$$E(a,b,c) = \frac{a}{4b^2+5} + \frac{b}{4c^2+5} + \frac{c}{4a^2+5}.$$

We have

$$\begin{aligned} E(a,b,c) - E(1,b,c) &= \frac{a-1}{4b^2+5} + c\left(\frac{1}{4a^2+5} - \frac{1}{9}\right) \\ &= (1-a)\left[\frac{4c(1+a)}{9(4a^2+5)} - \frac{1}{4b^2+5}\right] \\ &\leq (1-a)\left[\frac{4(1+a)}{9(4a^2+5)} - \frac{1}{9}\right] \\ &= \frac{-(1-a)(1-2a)^2}{9(4a^2+5)} \leq 0, \end{aligned}$$

and, similarly,

$$E(a,b,c) - E(a,1,c) \leq 0, \quad E(a,b,c) - E(a,b,1) \leq 0.$$

Therefore,

$$E(a,b,c) \le E(1,b,c) \le E(1,1,c) \le E(1,1,1) = \frac{1}{3}.$$

The equality occurs for $a = b = c = 1$, and also for $a = \frac{1}{2}$ and $b = c = 1$ (or any cyclic permutation).

□

P 1.35. *If $a,b,c \in \left[\frac{1}{3}, 3\right]$, then*

$$\frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a} \ge \frac{7}{5}.$$

Solution. Assume that $a = \max\{a,b,c\}$ and show that

$$E(a,b,c) \ge E(a,b,\sqrt{ab}\,) \ge \frac{7}{5},$$

where

$$E(a,b,c) = \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{c+a}.$$

We have

$$\begin{aligned} E(a,b,c) - E(a,b,\sqrt{ab}\,) &= \frac{b}{b+c} + \frac{c}{c+a} - \frac{2\sqrt{b}}{\sqrt{a}+\sqrt{b}} \\ &= \frac{\left(\sqrt{a}-\sqrt{b}\right)\left(\sqrt{ab}-c\right)^2}{(b+c)(c+a)\left(\sqrt{a}+\sqrt{b}\right)} \ge 0. \end{aligned}$$

Substituting $x = \sqrt{\frac{a}{b}}$, the hypothesis $a,b,c \in \left[\frac{1}{3}, 3\right]$ involves $x \in \left[\frac{1}{3}, 3\right]$. Then,

$$\begin{aligned} E(a,b,\sqrt{ab}) - \frac{7}{5} &= \frac{a}{a+b} + \frac{2\sqrt{b}}{\sqrt{a}+\sqrt{b}} - \frac{7}{5} \\ &= \frac{x^2}{x^2+1} + \frac{2}{x+1} - \frac{7}{5} \\ &= \frac{3 - 7x + 8x^2 - 2x^3}{5(x+1)(x^2+1)} \\ &= \frac{(3-x)[x^2 + (1-x)^2]}{5(x+1)(x^2+1)} \ge 0. \end{aligned}$$

The equality holds for $a = 3$, $b = \frac{1}{3}$ and $c = 1$ (or any cyclic permutation).

□

P 1.36. *If* $a, b, c \in \left[\frac{1}{\sqrt{2}}, \sqrt{2}\right]$, *then*

$$\frac{3}{a+2b}+\frac{3}{b+2c}+\frac{3}{c+2a} \ge \frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}.$$

Solution. Write the inequality as

$$\sum \left(\frac{3}{a+2b} - \frac{2}{a+b} + \frac{1}{ka} - \frac{1}{kb} \right) \ge 0, \quad k > 0,$$

$$\sum \frac{-(a-b)[a^2-(k-3)ab+2b^2]}{kab(a+2b)(a+b)} \ge 0.$$

Choosing $k = 6$, the inequality becomes

$$\sum \frac{(a-b)^2(2b-a)}{6ab(a+2b)(a+b)} \ge 0.$$

Since

$$2b - a \ge \frac{2}{\sqrt{2}} - \sqrt{2} = 0,$$

the conclusion follows. The equality holds for $a = b = c$.

□

P 1.37. *If* a, b, c *are nonnegative real numbers, no two of which are zero, then*

$$\frac{4abc}{ab^2+bc^2+ca^2+abc} + \frac{a^2+b^2+c^2}{ab+bc+ca} \ge 2.$$

(Vo Quoc Ba Can, 2009)

First Solution. Without loss of generality, assume that b is between a and c; that is,

$$b^2 + ca \le b(c+a).$$

Then,

$$\begin{aligned} ab^2+bc^2+ca^2+abc &= a(b^2+ca)+bc^2+abc \\ &\le ab(c+a)+bc^2+abc \\ &= b(a+c)^2, \end{aligned}$$

and it suffices to prove that

$$\frac{4ac}{(a+c)^2} + \frac{a^2+b^2+c^2}{ab+bc+ca} \ge 2.$$

This inequality is equivalent to

$$[a^2+c^2-b(a+c)]^2 \ge 0.$$

The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Let (x,y,z) be a permutation of (a,b,c) such that $x \ge y \ge z$. As we have shown in the second solution of P 1.1,

$$ab^2+bc^2+ca^2 \le y(x^2+xz+z^2);$$

hence

$$ab^2+bc^2+ca^2+abc \le y(x+z)^2.$$

Thus, it suffices to prove that

$$\frac{4xyz}{y(x+z)^2}+\frac{x^2+y^2+z^2}{xy+yz+zx} \ge 2,$$

which is equivalent to

$$\frac{x^2+y^2+z^2}{xy+yz+zx} \ge \frac{2(x^2+z^2)}{x+z)^2},$$

$$(x^2+z^2)^2-2y(x+z)(x^2+z^2)+y^2(x+z)^2 \ge 0,$$

$$(x^2+z^2-xy-yz)^2 \ge 0.$$

□

P 1.38. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$\frac{1}{ab^2+8}+\frac{1}{bc^2+8}+\frac{1}{ca^2+8} \ge \frac{1}{3}.$$

(Vasile C., 2007)

Solution. By expanding, we can write the inequality as

$$64 \ge r^3+16A+5rB,$$

$$64 \ge r^3+(16-5r)A+5r(A+B),$$

where

$$r=abc, \quad A=ab^2+bc^2+ca^2, \quad B=a^2b+b^2c+c^2a.$$

By the AM-GM inequality, we have

$$r \le \left(\frac{a+b+c}{3}\right)^3=1.$$

On the other hand, by the inequality (a) in P 1.9, we get

$$A \le 4 - r,$$

and by Schur's inequality, we have

$$(a+b+c)^3 + 9abc \ge 4(a+b+c)(ab+bc+ca),$$

which is equivalent to

$$A + B \le \frac{27-3r}{4}.$$

Therefore, it suffices to prove that

$$64 \ge r^3 + (16-5r)(4-r) + \frac{5r(27-3r)}{4}.$$

We can write this inequality in the obvious form

$$r(1-r)(9+4r) \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$, $b = 1$, $c = 2$ (or any cyclic permutation).

□

P 1.39. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

$$\frac{ab}{bc+3} + \frac{bc}{ca+3} + \frac{ca}{ab+3} \le \frac{3}{4}.$$

(Vasile C., 2008)

Solution. Using the inequality (a) in P 1.9, namely

$$a^2b + b^2c + c^2a \le 4 - abc,$$

we have

$$\begin{aligned}\sum ab(ca+3)(ab+3) &= abc\sum a^2b + 9abc + 3\sum a^2b^2 + 9\sum ab \\ &\le 13abc - a^2b^2c^2 + 3\sum a^2b^2 + 9\sum ab.\end{aligned}$$

On the other hand,

$$(ab+3)(bc+3)(ca+3) = a^2b^2c^2 + 9abc + 9\sum ab + 27.$$

Therefore, it suffices to prove that

$$4\left(13abc - a^2b^2c^2 + 3\sum a^2b^2 + 9\sum ab\right) \le 3\left(a^2b^2c^2 + 9abc + 9\sum ab + 27\right),$$

which is equivalent to

$$7a^2b^2c^2+81 \ge 25abc+12\sum a^2b^2+9\sum ab,$$

$$7r^2+47r \ge 3(q+3)(4q-9),$$

where

$$q=ab+bc+ca, \quad r=abc, \quad q \le 3, \;\; r \le 1.$$

Since

$$7r^2+47r \ge 9r^2+45r,$$

it suffices to show that

$$3r^2+15r \ge (q+3)(4q-9).$$

Consider the non-trivial case

$$\frac{9}{4} < q \le 3,$$

and apply the fourth degree Schur's inequality

$$r \ge \frac{(p^2-q)(4q-p^2)}{6p} = \frac{(9-q)(4q-9)}{18}.$$

It remains to show that

$$\frac{(9-q)^2(4q-9)^2}{108}+\frac{5(9-q)(4q-9)}{6} \ge (q+3)(4q-9),$$

which is equivalent to

$$(4q-9)(3-q)(69q-4q^2-81) \ge 0.$$

This is true because

$$69q-4q^2-81=(3-q)(4q-9)+6(8q-9)>0.$$

The equality holds for $a=b=c=1$, and also for $a=0$ and $b=c=\frac{3}{2}$ (or any cyclic permutation).

□

P 1.40. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

(a)
$$\frac{a}{b^2+3}+\frac{b}{c^2+3}+\frac{c}{a^2+3} \ge \frac{3}{4};$$

(b)
$$\frac{a}{b^3+1}+\frac{b}{c^3+1}+\frac{c}{a^3+1} \ge \frac{3}{2}.$$

(Vasile Cîrtoaje and Bin Zhao, 2005)

Solution. (a) By the AM-GM inequality, we have

$$b^2+3=b^2+1+1+1\ge 4\sqrt[4]{b^2\cdot 1^3}=4\sqrt{b}.$$

Therefore,

$$\frac{3a}{b^2+3}=a-\frac{ab^2}{b^2+3}\ge a-\frac{ab^2}{4\sqrt{b}}=a-\frac{1}{4}ab\sqrt{b}.$$

Taking account of this inequality and the similar ones, it suffices to prove that

$$ab\sqrt{b}+bc\sqrt{c}+ca\sqrt{a}\le 3.$$

This inequality follows immediately by replacing a,b,c with $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ in the inequality in P 1.7. The equality holds for $a=b=c=1$.

(b) Using the AM-GM Inequality gives

$$\frac{a}{b^3+1}=a-\frac{ab^3}{b^3+1}\ge a-\frac{ab^3}{2b\sqrt{b}}=a-\frac{1}{2}ab\sqrt{b},$$

and, similarly,

$$\frac{b}{c^3+1}\ge b-\frac{1}{2}bc\sqrt{c},\quad \frac{c}{a^3+1}\ge c-\frac{1}{2}ca\sqrt{a}.$$

Thus, it suffices to show that

$$ab\sqrt{b}+bc\sqrt{c}+ca\sqrt{a}\le 3,$$

which follows from the inequality in P 1.7. The equality holds for $a=b=c=1$.

Conjecture. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. If*

$$0<k\le 3+2\sqrt{3},$$

then

$$\frac{a}{b^2+k}+\frac{b}{c^2+k}+\frac{c}{a^2+k}\ge\frac{3}{1+k}.$$

For $k=3+2\sqrt{3}$, the equality occurs when $a=b=c=1$, and again when $a=0$, $b=3-\sqrt{3}$ and $c=\sqrt{3}$ (or any cyclic permutation thereof).

□

P 1.41. *Let a,b,c be positive real numbers, and let*

$$x=a+\frac{1}{b}-1,\quad y=b+\frac{1}{c}-1,\quad z=c+\frac{1}{a}-1.$$

Prove that

$$xy+yz+zx\ge 3.$$

(Vasile C., 1991)

First Solution. Among x, y, z, there are two numbers either less than or equal to 1, or greater than or equal to 1. Let y and z be these numbers; that is,

$$(y-1)(z-1) \geq 0.$$

Since

$$xy+yz+zx-3=(y-1)(z-1)+(x+1)(y+z)-4,$$

it suffices to show that

$$(x+1)(y+z) \geq 4.$$

Since

$$y+z=b+\frac{1}{a}+c+\frac{1}{c}-2 \geq b+\frac{1}{a},$$

we have

$$(x+1)(y+z)-4 \geq (x+1)\left(b+\frac{1}{a}\right)-4=ab+\frac{1}{ab}-2 \geq 0.$$

The equality holds for $a=b=c=1$.

Second Solution. Without loss of generality, assume that $x=\max\{x, y, z\}$. Then,

$$x \geq \frac{1}{3}(x+y+z)=\frac{1}{3}\left[\left(a+\frac{1}{a}\right)+\left(b+\frac{1}{b}\right)+\left(c+\frac{1}{c}\right)-3\right]$$
$$\geq \frac{1}{3}(2+2+2-3)=1.$$

On the other hand, from

$$(x+1)(y+1)(z+1)=abc+\frac{1}{abc}+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$
$$\geq 2+a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$
$$=5+x+y+z,$$

we get

$$xyz+xy+yz+zx \geq 4.$$

Since

$$y+z=\frac{1}{a}+b+\frac{(c-1)^2}{c}>0,$$

two cases are possible: $yz \leq 0$ and $y, z>0$.

Case 1: $yz \leq 0$. Since $xyz \leq 0$, it follows that

$$xy+yz+zx \geq 4-xyz \geq 4>3.$$

Case 2: $y, z>0$. We need to show that $d \geq 1$, where

$$d=\sqrt{\frac{xy+yz+zx}{3}}.$$

By the AM-GM inequality, we have $d^3 \ge xyz$. Thus, from $xyz + xy + yz + zx \ge 4$, we get

$$d^3 + 3d^2 \ge 4,$$

$$(d-1)(d+2)^2 \ge 0,$$

hence $d \ge 1$.

□

P 1.42. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\left(a - \frac{1}{b} - \sqrt{2}\right)^2 + \left(b - \frac{1}{c} - \sqrt{2}\right)^2 + \left(c - \frac{1}{a} - \sqrt{2}\right)^2 \ge 6.$$

Solution *(by Nguyen Van Quy).* Using the substitution

$$a = \frac{y}{x}, \quad b = \frac{x}{z}, \quad c = \frac{z}{y}, \qquad x, y, z > 0,$$

the inequality becomes as follows:

$$\left(\frac{y-z}{x} - \sqrt{2}\right)^2 + \left(\frac{z-x}{y} - \sqrt{2}\right)^2 + \left(\frac{x-y}{z} - \sqrt{2}\right)^2 \ge 6,$$

$$\left(\frac{y-z}{x}\right)^2 + \left(\frac{z-x}{y}\right)^2 + \left(\frac{x-y}{z}\right)^2 - 2\sqrt{2}\left(\frac{y-z}{x} + \frac{z-x}{y} + \frac{x-y}{z}\right) \ge 0,$$

$$\left(\frac{y-z}{x}\right)^2 + \left(\frac{z-x}{y}\right)^2 + \left(\frac{x-y}{z}\right)^2 + \frac{2\sqrt{2}(y-z)(z-x)(x-y)}{xyz} \ge 0.$$

Assume that $x = \max\{x, y, z\}$. For $x \ge z \ge y$, the inequality is clearly true. Consider further that $x \ge y \ge z$ and write the desired inequality as

$$u^2 + v^2 + w^2 \ge 2\sqrt{2}\, uvw,$$

where

$$u = \frac{y-z}{x} \ge 0, \quad v = \frac{x-z}{y} \ge 0, \quad w = \frac{x-y}{z} \ge 0.$$

In addition, we have

$$uv = \left(1 - \frac{z}{y}\right)\left(1 - \frac{z}{x}\right) < 1 \cdot 1 = 1.$$

According to the AM-GM inequality, we get

$$u^2 + v^2 + w^2 \ge 2uv + w^2 \ge 2u^2v^2 + w^2 \ge 2\sqrt{2}\, uvw.$$

This completes the proof. The equality holds for $a = b = c$.

□

P 1.43. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\left|1+a-\frac{1}{b}\right|+\left|1+b-\frac{1}{c}\right|+\left|1+c-\frac{1}{a}\right|>2.$$

Solution. Using the substitution

$$a=\frac{y}{x},\quad b=\frac{x}{z},\quad c=\frac{z}{y},\qquad x,y,z>0,$$

the inequality can be restated as

$$\left|1+\frac{y-z}{x}\right|+\left|1+\frac{x-y}{z}\right|+\left|1+\frac{z-x}{y}\right|>2.$$

Without loss of generality, assume that $x=\max\{x,y,z\}$. We have

$$\left|1+\frac{y-z}{x}\right|+\left|1+\frac{x-y}{z}\right|+\left|1+\frac{z-x}{y}\right|-2\geq\left|1+\frac{y-z}{x}\right|+\left|1+\frac{x-y}{z}\right|-2$$

$$=\frac{x+y-z}{x}+\frac{z+x-y}{z}-2=\frac{y-z}{x}+\frac{x-y}{z}\geq\frac{y-z}{x}+\frac{x-y}{x}=\frac{x-z}{x}\geq 0.$$

□

P 1.44. *If a, b, c are different positive real numbers, then*

$$\left|1+\frac{a}{b-c}\right|+\left|1+\frac{b}{c-a}\right|+\left|1+\frac{c}{a-b}\right|>2.$$

(Vasile C., 2012)

Solution. Without loss of generality, assume that $a=\max\{a,b,c\}$. It suffices to show that

$$\left|1+\frac{a}{b-c}\right|+\left|1+\frac{c}{a-b}\right|>2,$$

which is equivalent to

$$\frac{a+b-c}{|b-c|}+\frac{a-b+c}{a-b}>2.$$

For $b>c$, this inequality is true since

$$\frac{a+b-c}{|b-c|}+\frac{a-b+c}{a-b}>\frac{a+b-c}{|b-c|}=\frac{a}{b-c}+1>1+1=2.$$

Also, for $b<c$, we have

$$\frac{a+b-c}{|b-c|}+\frac{a-b+c}{a-b}=\frac{a+b-c}{c-b}+\frac{a-b+c}{a-b}$$

$$=\frac{a}{c-b}+\frac{c}{a-b}>\frac{a}{c-b}+\frac{c-b}{a-b}\geq 2\sqrt{\frac{a}{a-b}}>2.$$

□

P 1.45. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

$$\left(2a - \frac{1}{b} - \frac{1}{2}\right)^2 + \left(2b - \frac{1}{c} - \frac{1}{2}\right)^2 + \left(2c - \frac{1}{a} - \frac{1}{2}\right)^2 \geq \frac{3}{4}.$$

(Vasile C., 2012)

Solution. Using the substitution

$$x = 2a - \frac{1}{b}, \quad y = 2b - \frac{1}{c}, \quad z = 2c - \frac{1}{a},$$

we can write the inequality as

$$x^2 + y^2 + z^2 \geq x + y + z.$$

From

$$x + y + z = 2\sum a - \sum \frac{1}{a}$$

and

$$xyz = 7 - 4\sum a + 2\sum \frac{1}{a},$$

it follows that

$$2(x + y + z) + xyz = 7.$$

In addition, from

$$2(|x| + |y| + |z|) + \left(\frac{|x| + |y| + |z|}{3}\right)^3 \geq 2(|x| + |y| + |z|) + |xyz|$$

$$\geq 2(x + y + z) + xyz = 7,$$

we get

$$|x| + |y| + |z| \geq 3.$$

Therefore, we have

$$x^2 + y^2 + z^2 \geq \frac{1}{3}(|x| + |y| + |z|)^2 \geq |x| + |y| + |z| \geq x + y + z.$$

The equality holds for $a = b = c = 1$.

□

P 1.46. *Let*

$$x = a + \frac{1}{b} - \frac{5}{4}, \quad y = b + \frac{1}{c} - \frac{5}{4}, \quad z = c + \frac{1}{a} - \frac{5}{4},$$

where $a \geq b \geq c > 0$. Prove that

$$xy + yz + zx \geq \frac{27}{16}.$$

(Vasile C., 2011)

Solution. Write the inequality as

$$\sum\left(ab+\frac{1}{ab}\right)+\sum\frac{b}{a}-\frac{5}{2}\sum\left(a+\frac{1}{a}\right)+6\geq 0.$$

Since

$$\sum\frac{b}{a}-\sum\frac{a}{b}=\frac{(a-b)(b-c)(a-c)}{abc}\geq 0,$$

we have

$$2\sum\frac{b}{a}\geq\sum\frac{b}{a}+\sum\frac{a}{b}=\left(\sum a\right)\left(\sum\frac{1}{a}\right)-3.$$

Thus, it suffices to prove the symmetric inequality

$$2\sum\left(ab+\frac{1}{ab}\right)+\left(\sum a\right)\left(\sum\frac{1}{a}\right)-5\sum\left(a+\frac{1}{a}\right)+9\geq 0.$$

Setting

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc,$$

we need to show that

$$(2q-5p+9)r+pq-5q+2p\geq 0$$

for all $a,b,c>0$. For fixed p and q, the linear function

$$f(r)=(2q-5p+9)r+pq-5q+2p$$

is minimal when r is either minimal or maximal. Thus, according to P 3.57 in Volume 1, it suffices to prove that $f(r)\geq 0$ for $a=0$ and for $b=c$.

For $a=0$, we need to show that

$$(b+c)bc-5bc+2(b+c)\geq 0.$$

Indeed, putting $x=\sqrt{bc}$, we have

$$(b+c)bc-5bc+2(b+c)\geq 2x^3-5x^2+4x>0.$$

For $b=c$, since

$$p=a+2b,\quad q=2ab+b^2,\quad r=ab^2,$$

the inequality $f(r)\geq 0$ becomes

$$(4ab+2b^2-5a-10b+9)ab^2+(a+2b)(2ab+b^2)-10ab-5b^2+2a+4b\geq 0;$$

that is,

$$Aa^2+2Ba+C\geq 0,$$

where

$$A = b(4b^2 - 5b + 2) > 0, \quad B = b^4 - 5b^3 + 7b^2 - 5b + 1, \quad C = b(2b^2 - 5b + 4) > 0.$$

Let

$$x = b + \frac{1}{b}, \quad x \geq 2.$$

The inequality $B \geq 0$ is equivalent to

$$b^2 + \frac{1}{b^2} - 5\left(b + \frac{1}{b}\right) + 7 \geq 0,$$

$$x^2 - 5x + 5 \geq 0,$$

$$x \geq \frac{5 + \sqrt{5}}{2}.$$

Consider two cases.

Case 1: $x \geq \dfrac{5 + \sqrt{5}}{2}$. Since $A > 0$, $B \geq 0$, $C > 0$, we have $Aa^2 + 2Ba + C > 0$.

Case 2: $2 \leq x < \dfrac{5 + \sqrt{5}}{2}$. Since $A > 0$, $B < 0$, $C > 0$ and

$$Aa^2 + 2Ba + C = (Aa^2 + C) + 2Ba \geq 2a(\sqrt{AC} + B),$$

we need to show that $AC \geq B^2$, which is equivalent to

$$8\left(b^2 + \frac{1}{b^2}\right) - 30\left(b + \frac{1}{b}\right) + 45 \geq \left[b^2 + \frac{1}{b^2} - 5\left(b + \frac{1}{b}\right) + 7\right]^2,$$

$$8x^2 - 30x + 29 \geq (x^2 - 5x + 5)^2,$$

$$(x - 2)^2(x^2 - 6x - 1) \leq 0.$$

This inequality is true for $x \leq 3 + \sqrt{10}$, therefore for $x < (5 + \sqrt{5})/2$. Thus, the proof is completed. The equality holds for $a = b = c = 1$.

□

P 1.47. *Let a, b, c be positive real numbers, and let*

$$E = \left(a + \frac{1}{a} - \sqrt{3}\right)\left(b + \frac{1}{b} - \sqrt{3}\right)\left(c + \frac{1}{c} - \sqrt{3}\right);$$

$$F = \left(a + \frac{1}{b} - \sqrt{3}\right)\left(b + \frac{1}{c} - \sqrt{3}\right)\left(c + \frac{1}{a} - \sqrt{3}\right).$$

Prove that $E \geq F$.

(Vasile C., 2011)

Solution. By expanding, the inequality becomes

$$\sum(a^2-bc)+\sum bc(bc-a^2)\ge\sqrt{3}\sum ab(b-c).$$

Since

$$\sum(a^2-bc)=\sum a^2-\sum ab\ge 0$$

and

$$\sum bc(bc-a^2)=\sum a^2b^2-abc\sum a\ge 0,$$

by the AM-GM inequality, we have

$$\sum(a^2-bc)+\sum bc(bc-a^2)\ge 2\sqrt{\left[\sum(a^2-bc)\right]\left[\sum bc(bc-a^2)\right]}.$$

Thus, it suffices to show that

$$2\sqrt{\left[\sum(a^2-bc)\right]\left[\sum bc(bc-a^2)\right]}\ge\sqrt{3}\sum ab(b-c),$$

which is equivalent to

$$2\sqrt{\left[\sum(a^2-bc)\right]\left[\sum\left(\frac{1}{a^2}-\frac{1}{bc}\right)\right]}\ge\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right),$$

$$\sqrt{[(a+c-2b)^2+3(c-a)^2]\left[3\left(\frac{1}{b}-\frac{1}{c}\right)^2+\left(\frac{2}{a}-\frac{1}{b}-\frac{1}{c}\right)^2\right]}\ge$$

$$\ge 2\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right).$$

Applying the Cauchy-Schwarz inequality, it suffices to show that

$$(a+c-2b)\left(\frac{1}{b}-\frac{1}{c}\right)+(c-a)\left(\frac{2}{a}-\frac{1}{b}-\frac{1}{c}\right)\ge 2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right),$$

which is an identity. Thus, the proof is completed. The equality holds when the following two equations are satisfied:

$$a^2+b^2+c^2-ab-bc-ca=a^2b^2+b^2c^2+c^2a^2-abc(a+b+c)$$

and

$$3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=2\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

□

P 1.48. *If a,b,c are positive real numbers such that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}=5$, then*

$$\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\ge\frac{17}{4}.$$

(Vasile C., 2007)

Solution. Making the substitution

$$x=\frac{a}{b},\quad y=\frac{b}{c},\quad z=\frac{c}{a},$$

we need to show that if x,y,z are positive real numbers satisfying

$$xyz=1,\qquad x+y+z=5,$$

then

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\ge\frac{17}{4}.$$

From $(y+z)^2\ge 4yz$, we get

$$(5-x)^2\ge\frac{4}{x};$$

therefore,

$$(5-x)+(5-x)+\frac{x}{4}\ge 3\sqrt[3]{(5-x)^2\frac{x}{4}}\ge 3,$$

which involves $x\le 4$. We have

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{17}{4}=\frac{1}{x}+\frac{y+z}{yz}-\frac{17}{4}=\frac{1}{x}+x(5-x)-\frac{17}{4}$$

$$=\frac{4-17x+20x^2-4x^3}{4x}=\frac{(4-x)(1-2x)^2}{4x}\ge 4.$$

The equality holds when one of x,y,z is 4 and the others are $\frac{1}{2}$; that is, when

$$a=4b=2c$$

(or any cyclic permutation).

□

P 1.49. *If a,b,c are positive real numbers, then*

(a) $$1+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 2\sqrt{1+\frac{b}{a}+\frac{c}{b}+\frac{a}{c}}\,;$$

(b) $$1+2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\ge\sqrt{1+16\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)}\,;$$

(c) $$3+\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 2\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}\,.$$

(Vasile C., 2007)

Solution. Let

$$x = \frac{a}{b}, \quad y = \frac{b}{c}, \quad z = \frac{c}{a}$$

and

$$p = x + y + z, \quad q = xy + yz + zx.$$

By the AM-GM inequality, we have

$$p \ge 3\sqrt[3]{xyz} = 3.$$

(a) We need to show that $xyz = 1$ involves

$$1 + x + y + z \ge 2\sqrt{1 + xy + yz + zx},$$

which is equivalent to

$$(1+p)^2 \ge 4 + 4q$$

or

$$p + 3 \ge 2\sqrt{p + q + 3}.$$

First Solution. By Schur's inequality of degree three, we have

$$p^3 + 9 \ge 4pq.$$

Thus,

$$(1+p)^2 - 4 - 4q \ge 1 + p)^2 - 4 - \left(p^2 + \frac{9}{p}\right) = \frac{(p-3)(2p+3)}{p} \ge 0.$$

The equality holds for $a = b = c$.

Second Solution. Without loss of generality, assume that b is between a and c. By the AM-GM inequality, we have

$$2\sqrt{p+q+3} = 2\sqrt{(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)} \le \frac{a+b+c}{b} + b\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Therefore,

$$\begin{aligned} p + 3 - 2\sqrt{p+q+3} &\ge \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 3 - \frac{a+b+c}{b} - b\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \\ &= \frac{(a-b)(b-c)}{ab} \ge 0. \end{aligned}$$

(b) We have to show that $xyz = 1$ involves

$$1 + 2(x + y + z) \ge \sqrt{1 + 16(xy + yz + zx)},$$

which is equivalent to

$$p^2 + p \ge 4q.$$

By Schur's inequality of degree three, we have

$$p^3 + 9 \geq 4pq.$$

Thus,

$$p^2 + p - 4q \geq p^2 + p - \left(p^2 + \frac{9}{p}\right) = \frac{(p-3)(p+3)}{9} \geq 0.$$

The equality holds for $a = b = c$.

(c) Write the inequality as follows:

$$(3 + x + y + z)^2 \geq 4(3 + x + y + z + xy + yz + zx),$$

$$(x + y + z)^2 + 2(x + y + z) \geq 3 + 4(xy + yz + zx),$$

$$(1 + x + y + z)^2 \geq 4(1 + xy + yz + zx),$$

$$1 + x + y + z \geq 2\sqrt{1 + xy + yz + zx},$$

$$1 + \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 2\sqrt{1 + \frac{b}{a} + \frac{c}{b} + \frac{a}{c}}.$$

Thus, the inequality is equivalent to the inequality in (a).

□

P 1.50. *If a, b, c are positive real numbers, then*

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + 15\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \geq 16\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right).$$

Solution. Making the substitution

$$x = \frac{a}{b}, \quad y = \frac{b}{c}, \quad z = \frac{c}{a},$$

we have to show that $xyz = 1$ involves

$$x^2 + y^2 + z^2 + 15(xy + yz + zx) \geq 16(x + y + z),$$

which is equivalent to

$$(x + y + z)^2 - 16(x + y + z) + 13(xy + yz + zx) \geq 0.$$

According to P 3.58 in Volume 1, for fixed $x + y + z$ and $xyz = 1$, the expression

$$xy + yz + zx$$

is minimal when two of x, y, z are equal. Therefore, due to symmetry, it suffices to consider that $x = y$. We need to show that

$$(2x + z)^2 - 16(2x + z) + 13(x^2 + 2xz) \geq 0$$

for $x^2z=1$. Write this inequality as

$$17x^6-32x^5+30x^3-16x^2+1\geq 0,$$

or

$$(x-1)^2g(x)\geq 0,\quad g(x)=17x^4+2x^3-13x^2+2x+1.$$

Since

$$g(x)=(2x-1)^4+x(x^3+34x^2-37x+10),$$

it suffices to show that

$$x^3+34x^2-37x+10\geq 0.$$

There are two cases to consider.

Case 1: $x\in\left(0,\frac{1}{2}\right]\cup\left[\frac{10}{17},\infty\right)$. We have

$$x^3+34x^2-37x+10>34x^2-37x+10=(2x-1)(17x-10)\geq 0.$$

Case 2: $x\in\left(\frac{1}{2},\frac{10}{17}\right)$. We have

$$2(x^3+34x^2-37x+10)>2\left(\frac{1}{2}x^2+34x^2-37x+10\right)=69x^2-74x+20.$$

Since $69x^2-74x+20>0$ for all real x, the proof is completed. The equality holds for $a=b=c$.

□

P 1.51. *If a,b,c are positive real numbers such that $abc=1$, then*

(a)
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq a+b+c;$$

(b)
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{3}{2}(a+b+c-1);$$

(c)
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+2\geq\frac{5}{3}(a+b+c).$$

Solution. (a) We write the inequality as

$$\left(2\frac{a}{b}+\frac{b}{c}\right)+\left(2\frac{b}{c}+\frac{c}{a}\right)+\left(2\frac{c}{a}+\frac{a}{b}\right)\geq 3(a+b+c).$$

In virtue of the AM-GM inequality, we get

$$\left(2\frac{a}{b}+\frac{b}{c}\right)+\left(2\frac{b}{c}+\frac{c}{a}\right)+\left(2\frac{c}{a}+\frac{a}{b}\right)\geq 3\sqrt[3]{\frac{a^2}{bc}}+3\sqrt[3]{\frac{b^2}{ca}}+3\sqrt[3]{\frac{c^2}{ab}}=3(a+b+c).$$

The equality holds for $a = b = c = 1$.

(b) Using the substitution

$$a = \frac{y}{x}, \quad b = \frac{z}{y}, \quad c = \frac{x}{z},$$

where $x, y, z > 0$, the inequality can be restated as

$$2(x^3 + y^3 + z^3) + 3xyz \ge 3(x^2y + y^2z + z^2x).$$

First Solution. We get the desired inequality by summing Schur's inequality of degree three

$$x^3 + y^3 + z^3 + 3xyz \ge (x^2y + y^2z + z^2x) + (xy^2 + yz^2 + zx^2)$$

and

$$x^3 + y^3 + z^3 + xy^2 + yz^2 + zx^2 \ge 2(x^2y + y^2z + z^2x).$$

The last inequality is equivalent to

$$x(x - y)^2 + y(y - z)^2 + z(z - x)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. Multiplying by $x + y + z$, the desired inequality in x, y, z turns into

$$2\sum x^4 - \sum x^3y - 3\sum x^2y^2 + 2\sum xy^3 \ge 0.$$

Write this inequality as

$$\sum[(1 + k)x^4 - x^3y - 3x^2y^2 + 2xy^3 + (1 - k)y^4] \ge 0,$$

$$\sum(x - y)[x^3 - 3xy^2 - y^3 + k(x^3 + x^2y + xy^2 + y^3)] \ge 0.$$

Choosing $k = \frac{3}{4}$, we get the obvious inequality

$$\sum(x - y)^2(7x^2 + 10xy + y^2) \ge 0.$$

(c) Making the substitution

$$a = \frac{y}{x}, \quad b = \frac{z}{y}, \quad c = \frac{x}{z}, \quad x, y, z > 0,$$

we need to show that

$$3(x^3 + y^3 + z^3) + 6xyz \ge 5(x^2y + y^2z + z^2x).$$

Assuming that $x = \min\{x, y, z\}$ and substituting

$$y = x + p, \quad z = x + q, \quad p, q \geq 0,$$

the inequality turns into

$$(p^2 - pq + q^2)x + 3p^3 + 3q^3 - 5p^2q \geq 0.$$

This is true since, by the AM-GM inequality, we get

$$6p^3 + 6q^3 = 3p^3 + 3p^3 + 6q^3 \geq 3\sqrt[3]{3p^3 \cdot 3p^3 \cdot 6q^3} = 9\sqrt[3]{2}\, p^2q \geq 10p^2q.$$

The equality holds for $a = b = c = 1$.

□

P 1.52. *If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then*

(a)
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 2 + \frac{3}{ab + bc + ca};$$

(b)
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{9}{a + b + c}.$$

Solution. (a) By the Cauchy-Schwarz inequality, we have

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{(a + b + c)^2}{ab + bc + ca} = 2 + \frac{3}{ab + bc + ca}.$$

The equality holds for $a = b = c = 1$.

(b) Using the inequality in (a), it suffices to show that

$$2 + \frac{3}{ab + bc + ca} \geq \frac{9}{a + b + c}.$$

Let

$$t = \frac{a + b + c}{3}, \qquad t \leq 1.$$

Since

$$2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2) = 9t^2 - 3,$$

the inequality becomes

$$2 + \frac{2}{3t^2 - 1} \geq \frac{3}{t},$$

$$(t - 1)^2(2t + 1) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.53. *If a,b,c are positive real numbers such that $a^2+b^2+c^2=3$, then*

$$6\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+5(ab+bc+ca)\geq 33.$$

Solution. Write the inequality in the homogeneous form

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\geq\frac{5}{2}\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right).$$

We will prove the sharper inequality

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\geq m\left(1-\frac{ab+bc+ca}{a^2+b^2+c^2}\right),$$

where

$$m=4\sqrt{2}-3>\frac{5}{2}.$$

Write this inequality as follows:

$$\left(\sum a^2\right)\left(\sum ab^2\right)+mabc\sum ab-(m+3)abc\sum a^2\geq 0,$$

$$\sum ab^4+\sum a^3b^2+(m+1)abc\sum ab-(m+3)abc\sum a^2\geq 0,$$

$$\sum ab^4+\sum a^3b^2+2(2\sqrt{2}-1)abc\sum ab-4\sqrt{2}\,abc\sum a^2\geq 0,$$

On the other hand, from

$$\sum a(a-b)^2(b-kc)^2\geq 0,$$

we get

$$\sum ab^4+\sum a^3b^2+(k^2-2)\sum a^2b^3+k(4-k)abc\sum ab-4kabc\sum a^2\geq 0.$$

Choosing $k=\sqrt{2}$, we get the desired inequality. The equality holds for $a=b=c=1$.

□

P 1.54. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

(a) $$6\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3\geq 7(a^2+b^2+c^2);$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq a^2+b^2+c^2.$$

Solution. (a) Write the inequality in the homogeneous form

$$2\left(\sum a\right)^2\left(\sum ab^2\right)+abc\left(\sum a\right)^2 \ge 21abc\sum a^2,$$

which is equivalent to

$$\sum ab^4+\sum a^3b^2+2\sum a^2b^3+4abc\sum ab-8abc\sum a^2 \ge 0.$$

On the other hand, from

$$\sum a(a-b)^2(b-kc)^2 \ge 0,$$

we get

$$\sum ab^4+\sum a^3b^2+(k^2-2)\sum a^2b^3+k(4-k)abc\sum ab-4kabc\sum a^2 \ge 0.$$

Choosing $k=2$, we get the desired inequality. The equality holds for $a=b=c=1$.

(b) We get the desired inequality by adding the inequality in (a) and the obvious inequality

$$a^2+b^2+c^2 \ge 3.$$

The equality holds for $a=b=c=1$.

□

P 1.55. *If a,b,c are positive real numbers, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+2 \ge \frac{14(a^2+b^2+c^2)}{(a+b+c)^2}.$$

(Vo Quoc Ba Can, 2010)

Solution. By expanding, the inequality becomes as follows:

$$\left(\sum \frac{a}{b}\right)\left(\sum a^2+2\sum ab\right)+4\sum ab \ge 12\sum a^2,$$

$$\sum \frac{a^3}{b}+\sum \frac{a^2b}{c}+2\sum \frac{ab^2}{c}+7\sum ab \ge 10\sum a^2,$$

$$A+B \ge 10\sum a^2-10\sum ab,$$

where

$$A=\sum \frac{a^3}{b}+\sum \frac{a^2b}{c}-2\sum \frac{ab^2}{c}, \quad B=4\sum \frac{ab^2}{c}-3\sum ab.$$

Since

$$A=\sum\left(\frac{b^3}{c}+\frac{a^2b}{c}-\frac{2ab^2}{c}\right)=\sum \frac{b(a-b)^2}{c}$$

and

$$B=\sum\left(\frac{4ca^2}{b}-12ca+9bc\right)=\sum\frac{c(2a-3b)^2}{b},$$

we get

$$A+B=\sum\left[\frac{b(a-b)^2}{c}+\frac{c(2a-3b)^2}{b}\right]$$

$$\geq 2\sum(a-b)(2a-3b)=10\sum a^2-10\sum ab.$$

Thus, the proof is completed. For $a\geq b\geq c$, the equality holds for

$$b(a-b)=c(2a-3b),\quad c(b-c)=a(2b-3c),\quad a(c-a)=b(2c-3a),$$

which are equivalent to

$$\frac{a}{\sqrt{7}-\tan\frac{\pi}{7}}=\frac{b}{\sqrt{7}-\tan\frac{2\pi}{7}}=\frac{c}{\sqrt{7}-\tan\frac{4\pi}{7}}.$$

Notice that the equality conditions involve

$$a^2+b^2+c^2=2ab+2bc+2ca,$$

hence

$$\sqrt{a}=\sqrt{b}+\sqrt{c}.$$

Remark. Using the inequality in P 1.55, we can prove the weaker inequality

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{7(ab+bc+ca)}{a^2+b^2+c^2}\geq\frac{17}{2},$$

with equality for the same conditions. It suffices to show that

$$\frac{14(a^2+b^2+c^2)}{(a+b+c)^2}-2\geq\frac{17}{2}-\frac{7(ab+bc+ca)}{a^2+b^2+c^2}$$

which is equivalent to

$$(a^2+b^2+c^2-2ab-2bc-2ca)^2\geq 0.$$

Actually, the following statement is valid.

If a,b,c are positive real numbers, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{19(a^2+b^2+c^2)+2(ab+bc+ca)}{a^2+b^2+c^2+6(ab+bc+ca)},$$

with equality for $a=b=c$, and also for

$$\frac{a}{\sqrt{7}-\tan\frac{\pi}{7}}=\frac{b}{\sqrt{7}-\tan\frac{2\pi}{7}}=\frac{c}{\sqrt{7}-\tan\frac{4\pi}{7}}$$

(or any cyclic permutation).

This inequality is stronger than the inequality in P 1.55.

□

P 1.56. *Let a, b, c be positive real numbers such that $a+b+c=3$, and let*

$$x = 3a + \frac{1}{b}, \quad y = 3b + \frac{1}{c}, \quad z = 3c + \frac{1}{a}.$$

Prove that

$$xy + yz + zx \ge 48.$$

(Vasile C., 2007)

Solution. Write the inequality as follows:

$$3(ab+bc+ca) + \frac{1}{abc} + \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \ge 13.$$

We get this inequality by adding the inequality P 1.54-(a), namely

$$6\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) + 3 \ge 7(a^2+b^2+c^2),$$

and the inequality

$$18(ab+bc+ca) + \frac{6}{abc} + 7(a^2+b^2+c^2) \ge 81.$$

Since

$$a^2+b^2+c^2 = 9 - 2(ab+bc+ca),$$

the last inequality is equivalent to

$$2(ab+bc+ca) + \frac{3}{abc} \ge 9.$$

By the known inequality

$$(ab+bc+ca)^2 \ge 3abc(a+b+c),$$

we get

$$\frac{1}{abc} \ge \frac{9}{(ab+bc+ca)^2}.$$

Thus, it suffices to show that

$$2q + \frac{27}{q^2} \ge 9,$$

where $q = ab+bc+ca$. Indeed, by the AM-GM inequality, we have

$$2q + \frac{27}{q^2} = q + q + \frac{27}{q^2} \ge 3\sqrt[3]{q \cdot q \cdot \frac{27}{q^2}} = 9.$$

The equality holds for $a = b = c = 1$.

□

P 1.57. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a+1}{b}+\frac{b+1}{c}+\frac{c+1}{a} \geq 2(a^2+b^2+c^2).$$

Solution. We get the desired inequality by summing the inequality in P 1.54-(a), namely

$$6\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3 \geq 7(a^2+b^2+c^2),$$

and the inequality

$$6\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 5(a^2+b^2+c^2)+3.$$

Write the last inequality as $F(a,b,c) \geq 0$, where

$$F(a,b,c)=6\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-5(a^2+b^2+c^2)-3,$$

then assume that

$$a=\max\{a,b,c\}, \qquad b+c \leq 2.$$

and show that

$$F(a,b,c) \geq F\left(a, \frac{b+c}{2}, \frac{b+c}{2}\right) \geq 0.$$

Indeed, we have

$$F(a,b,c)-F\left(a,\frac{b+c}{2},\frac{b+c}{2}\right)=6\left(\frac{b+c}{bc}-\frac{4}{b+c}\right)-5\left[b^2+c^2-\frac{1}{2}(b+c)^2\right]$$

$$=(b-c)^2\left[\frac{6}{bc(b+c)}-\frac{5}{2}\right] \geq (b-c)^2\left[\frac{24}{(b+c)^3}-\frac{5}{2}\right] \geq 0.$$

Also,

$$F\left(a,\frac{b+c}{2},\frac{b+c}{2}\right)=F\left(a,\frac{3-a}{2},\frac{3-a}{2}\right)=\frac{3(a-1)^2(12-15a+5a^2)}{2a(3-a)} \geq 0.$$

The equality holds for $a=b=c=1$.

□

P 1.58. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+3 \geq 2(a^2+b^2+c^2).$$

(Pham Huu Duc, 2007)

First Solution. Assume that

$$a = \max\{a, b, c\},$$

then homogenize the inequality and write it as follows:

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + a + b + c \ge \frac{6(a^2 + b^2 + c^2)}{a + b + c},$$

$$\sum \left(\frac{b^2}{c} - 2b + c \right) \ge 6 \left(\frac{a^2 + b^2 + c^2}{a + b + c} - \frac{a + b + c}{3} \right),$$

$$\sum \frac{(b-c)^2}{c} \ge \frac{2}{a + b + c} \sum (b-c)^2,$$

$$(b-c)^2 A + (c-a)^2 B + (a-b)^2 C \ge 0,$$

where

$$A = \frac{a+b}{c} - 1 > 0, \quad B = \frac{b+c}{a} - 1, \quad C = \frac{c+a}{b} - 1 > 0.$$

By the Cauchy-Schwarz inequality, we have

$$(b-c)^2 A + (a-b)^2 C \ge \frac{[(b-c) + (a-b)]^2}{\frac{1}{A} + \frac{1}{C}} = \frac{AC}{A+C}(a-c)^2.$$

Therefore, it suffices to show that

$$\frac{AC}{A+C} + B \ge 0.$$

Indeed, by the third degree Schur's inequality, we get

$$AB + BC + CA = 3 + \frac{a^3 + b^3 + c^3 + 3abc - ab(a+b) - bc(b+c) - ca(c+a)}{abc} \ge 3.$$

The equality holds for $a = b = c = 1$.

Second Solution (by *Michael Rozenberg*). Write the inequality in the homogeneous form

$$\left(\sum a\right)\left(\sum ab^3\right) + abc\left(\sum a\right)^2 \ge 6abc \sum a^2.$$

By expanding, we get

$$\sum (ab^4 + a^2b^3 + 2ab^2c^2 - 4a^3bc) \ge 0,$$

which is equivalent to

$$\sum a(b^2 - 2bc + ac)^2 \ge 0.$$

□

P 1.59. *If a, b, c are positive real numbers, then*

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+2(ab+bc+ca)\geq 3(a^2+b^2+c^2).$$

(Michael Rozenberg, 2010)

Solution. Write the inequality as

$$\sum\left(\frac{a^3}{b}+ab-2a^2\right)\geq a^2+b^2+c^2-ab-bc-ca,$$

$$\frac{a(a-b)^2}{b}+\frac{b(b-c)^2}{c}+\frac{c(c-a)^2}{a}\geq a^2+b^2+c^2-ab-bc-ca.$$

Assume that $a=\max\{a,b,c\}$.

Case 1: $a\geq b\geq c$. By the Cauchy-Schwarz inequality, we have

$$\frac{a(a-b)^2}{b}+\frac{b(b-c)^2}{c}\geq\frac{[(a-b)+(b-c)]^2}{\frac{b}{a}+\frac{c}{b}}=\frac{ab(a-c)^2}{b^2+ac}.$$

On the other hand,

$$a^2+b^2+c^2-ab-bc-ca=(a-c)^2+(b-a)(b-c)\leq(a-c)^2.$$

Therefore, it suffice to show that

$$\frac{ab(a-c)^2}{b^2+ac}+\frac{c(a-c)^2}{a}\geq(a-c)^2,$$

which is true if

$$\frac{ab}{b^2+ac}+\frac{c}{a}\geq 1.$$

This inequality is equivalent to

$$a^2b+b^2c+c^2a-ab^2-ca^2\geq 0,$$

$$bc^2-(a-b)(b-c)(c-a)\geq 0.$$

Case 2: $a\geq c\geq b$. By the Cauchy-Schwarz inequality, we have

$$\frac{b(b-c)^2}{c}+\frac{c(c-a)^2}{a}\geq\frac{[(b-c)+(c-a)]^2}{\frac{c}{b}+\frac{a}{c}}=\frac{bc(a-b)^2}{c^2+ab}.$$

On the other hand,

$$a^2+b^2+c^2-ab-bc-ca=(a-b)^2+(c-a)(c-b)\leq(a-b)^2.$$

Therefore, it suffice to show that

$$\frac{a(a-b)^2}{b}+\frac{bc(a-b)^2}{c^2+ab}\ge (a-b)^2,$$

which is equivalent to

$$(a-b)^2(a^2b+b^2c+c^2a-ab^2-bc^2)\ge 0,$$

$$(a-b)^2[ab(a-b)+b^2c+c^2(a-b)]\ge 0.$$

The equality holds for $a=b=c$.

□

P 1.60. *If a,b,c are positive real numbers such that $a^4+b^4+c^4=3$, then*

(a) $$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge 3;$$

(b) $$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\ge \frac{3}{2}.$$

(Alexey Gladkich, 2005)

Solution. (a) By Hölder's inequality, we have

$$\left(\sum\frac{a^2}{b}\right)\left(\sum\frac{a^2}{b}\right)\left(\sum a^2b^2\right)\ge\left(\sum a^2\right)^3.$$

Therefore, it suffices to show that

$$\left(\sum a^2\right)^3\ge 9\sum a^2b^2,$$

which has the homogeneous form

$$\left(\sum a^2\right)^3\ge 3\left(\sum a^2b^2\right)\sqrt{3\sum a^4}.$$

Using the notation

$$x=\sum a^2,\quad y=\sum a^2b^2,$$

the inequality can be restated as

$$x^3\ge 3y\sqrt{3(x^2-2y)}.$$

By squaring, the inequality becomes

$$x^6-27x^2y^2+54y^3\ge 0,$$

which is true because

$$x^6-27x^2y^2+54y^3=(x^2-3y)^2(x^2+6y)\ge 0.$$

The equality holds for $a=b=c=1$.

(b) By Hölder's inequality, we have

$$\left(\sum\frac{a^2}{b+c}\right)\left(\sum\frac{a^2}{b+c}\right)\left[\sum a^2(b+c)^2\right]\ge\left(\sum a^2\right)^3.$$

Thus, it suffices to prove that

$$\left(\sum a^2\right)^3\ge\frac{9}{4}\sum a^2(b+c)^2.$$

Using the inequality from the proof of (a), namely

$$\left(\sum a^2\right)^3\ge 9\sum a^2b^2,$$

we still have to show that

$$\sum a^2b^2\ge\frac{1}{4}\sum a^2(b+c)^2.$$

This inequality is equivalent to

$$\sum a^2(b-c)^2\ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.61. *If a,b,c are positive real numbers, then*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge\frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}.$$

(Vo Quoc Ba Can, 2010)

Solution *(by Ta Minh Hoang)*. Assume that

$$a=\max\{a,b,c\},$$

and write the inequality as follows:

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-a-b-c\ge\frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}-a-b-c,$$

$$\sum \frac{(a-b)^2}{b} \geq \frac{1}{a^2+b^2+c^2}\sum(a+b)(a-b)^2,$$

$$(b-c)^2A+(c-a)^2B+(a-b)^2C \geq 0,$$

where

$$A=\frac{a^2+b^2-bc}{c}>0, \quad B=\frac{b^2+c^2-ca}{a}, \quad C=\frac{c^2+a^2-ab}{b}>0.$$

Consider the nontrivial case $B<0$; that is,

$$ac-b^2-c^2>0.$$

From

$$ac-b^2-c^2=c(a-2b)-(b-c)^2,$$

it follows that

$$c(a-2b)>(b-c)^2 \geq 0,$$

hence

$$a>2b.$$

By the Cauchy-Schwarz inequality, we have

$$(b-c)^2A+(a-b)^2C \geq \frac{[(b-c)+(a-b)]^2}{\dfrac{1}{A}+\dfrac{1}{C}}=\frac{AC}{A+C}\,(a-c)^2.$$

Therefore, it suffices to show that $\dfrac{AC}{A+C}+B \geq 0$; that is, $\dfrac{1}{A}+\dfrac{1}{B}+\dfrac{1}{C} \leq 0$, or

$$\frac{c}{a^2+b^2-bc}+\frac{b}{c^2+a^2-ab} \leq \frac{a}{ca-b^2-c^2}.$$

Case 1: $a \geq b \geq c$. Since

$$\begin{aligned} a^2+b^2-bc-(ca-b^2-c^2) &> a^2+b^2-bc-ca \\ &= a(a-c)+b(b-c) \geq 0, \end{aligned}$$

and

$$\begin{aligned} c^2+a^2-ab-(ca-b^2-c^2) &> a^2+b^2-a(b+c) \\ &\geq a^2+bc-a(b+c) \\ &= (a-b)(a-c) \geq 0, \end{aligned}$$

it suffices to show that $c+b \leq a$. Indeed, we have $a>2b \geq b+c$.

Case 2: $a \ge c \ge b$. Replacing b and c by c and b, respectively, we need to show that $a \ge b \ge c$ involves

$$\frac{a^2}{c} + \frac{c^2}{b} + \frac{b^2}{a} \ge \frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}.$$

According to the preceding case, we have

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge \frac{3(a^3+b^3+c^3)}{a^2+b^2+c^2}.$$

Therefore, it suffices to show that

$$\frac{a^2}{c} + \frac{c^2}{b} + \frac{b^2}{a} \ge \frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}.$$

This inequality is equivalent to

$$(a+b+c)(a-b)(b-c)(a-c) \ge 0,$$

which is clearly true for $a \ge b \ge c$.
The proof is completed. The equality holds for $a = b = c = 1$.

□

P 1.62. *If a, b, c are positive real numbers, then*

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} + a + b + c \ge 2\sqrt{(a^2+b^2+c^2)\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)}.$$

(Pham Huu Duc, 2006)

Solution. Without loss of generality, we may assume that b is between a and c; that is,

$$(b-a)(b-c) \le 0.$$

Since

$$\begin{aligned} 2\sqrt{(a^2+b^2+c^2)\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)} &= 2\sqrt{\frac{a^2+b^2+c^2}{b}\left(a + \frac{b^2}{c} + \frac{bc}{a}\right)} \\ &\le \frac{a^2+b^2+c^2}{b} + a + \frac{b^2}{c} + \frac{bc}{a} \\ &= \frac{a^2}{b} + \frac{b^2}{c} + a + b + \frac{bc}{a} + \frac{c^2}{b}, \end{aligned}$$

it suffices to prove that

$$\frac{c^2}{a} + c \ge \frac{bc}{a} + \frac{c^2}{b}.$$

This is true because

$$\frac{c^2}{a}+c-\frac{bc}{a}-\frac{c^2}{b}=\frac{c(a-b)(b-c)}{ab}\geq 0.$$

The proof is completed. The equality holds for $a=b=c$.

□

P 1.63. *If a,b,c are positive real numbers, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+32\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right)\geq 51.$$

(Vasile C., 2009)

Solution. Write the inequality as

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+45\geq 32\left(\frac{b}{a+b}+\frac{c}{b+c}+\frac{a}{c+a}\right).$$

Using the substitution

$$x=\frac{a}{b},\quad y=\frac{b}{c},\quad z=\frac{c}{a},$$

which involves $xyz=1$, the inequality becomes

$$x+y+z+45-32\left(\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\right)\geq 0.$$

We get this inequality by summing the inequalities

$$x-\frac{32}{x+1}+15\geq 9\ln x,$$

$$y-\frac{32}{y+1}+15\geq 9\ln y,$$

$$z-\frac{32}{z+1}+15\geq 9\ln z.$$

Let

$$f(x)=x-\frac{32}{x+1}+15-9\ln x,\quad x>0.$$

From the derivative

$$f'(x)=1+\frac{32}{(x+1)^2}-\frac{9}{x}=\frac{(x-1)(x-3)^2}{x(x+1)^2},$$

it follows that $f(x)$ is decreasing for $0<x\leq 1$ and increasing for $x\geq 1$. Therefore, we have $f(x)\geq f(1)=0$. The equality holds for $a=b=c$.

□

P 1.64. *Find the greatest positive real number K such that the inequalities below hold for any positive real numbers a,b,c:*

(a)
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\ge K\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}-\frac{3}{2}\right);$$

(b)
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3+K\left(\frac{a}{2a+b}+\frac{b}{2b+c}+\frac{c}{2c+a}-1\right)\ge 0.$$

(Vasile C., 2008)

Solution. (a) For
$$a=x^3,\quad b=x,\quad c=1,$$
the inequality becomes
$$x^2+x+\frac{1}{x^3}-3\ge K\left(\frac{x^3}{x+1}+\frac{x}{1+x^3}+\frac{1}{x^3+x}-\frac{3}{2}\right),$$
$$\frac{(1-K)x^3}{x+1}+\frac{x^2}{x+1}+x+\frac{1}{x^3}-3-K\left(\frac{x}{1+x^3}+\frac{1}{x^3+x}-\frac{3}{2}\right)\ge 0.$$
For $x\to\infty$, we get the necessary condition $1-K\ge 0$. We will show that the original inequality is true for $K=1$; that is,
$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge\frac{3}{2}+\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$
Write the inequality as
$$\left(\frac{c}{a}-\frac{c}{a+b}\right)+\left(\frac{a}{b}-\frac{a}{b+c}\right)+\left(\frac{b}{c}-\frac{b}{c+a}\right)\ge\frac{3}{2},$$
$$\frac{bc}{a(a+b)}+\frac{ca}{b(b+c)}+\frac{ab}{c(c+a)}\ge\frac{3}{2}.$$
By the Cauchy-Schwarz inequality, we have
$$\frac{bc}{a(a+b)}+\frac{ca}{b(b+c)}+\frac{ab}{c(c+a)}\ge\frac{(bc+ca+ab)^2}{abc(a+b)+abc(b+c)+abc(c+a)}$$
$$=\frac{(bc+ca+ab)^2}{2abc(a+b+c)}\ge\frac{3}{2}.$$

The equality holds for $a=b=c$.

(b) For $b=1$ and $c=a^2$, the inequality becomes
$$2a+\frac{1}{a^2}-3+K\left(\frac{2a}{2a+1}+\frac{1}{a^2+2}-1\right)\ge 0,$$
$$\frac{(a-1)^2(2a+1)}{a^2}-\frac{K(a-1)^2}{(2a+1)(a^2+2)}\ge 0.$$

This inequality holds for any positive a if and only if

$$\frac{2a+1}{a^2}-\frac{K}{(2a+1)(a^2+2)}\geq 0.$$

For $a=1$, this inequality involves $K\leq 27$. We will show that the original inequality is true for $K=27$. Using the substitution

$$x=\frac{a}{b},\quad y=\frac{b}{c},\quad z=\frac{c}{a},$$

which involves $xyz=1$, the inequality can be restated as

$$x+y+z-3-\frac{27}{2}\left(\frac{1}{2x+1}+\frac{1}{2y+1}+\frac{1}{2z+1}-1\right)\geq 0.$$

First Solution. We get the desired inequality by summing the inequalities

$$x-\frac{27}{2(2x+1)}+\frac{7}{2}\geq 4\ln x,$$

$$y-\frac{27}{2(2y+1)}+\frac{7}{2}\geq 4\ln y,$$

$$z-\frac{27}{2(2z+1)}+\frac{7}{2}\geq 4\ln z.$$

Let

$$f(x)=x-\frac{27}{2(2x+1)}+\frac{7}{2}-4\ln x,\quad x>0.$$

From the derivative

$$f'(x)=1+\frac{27}{(2x+1)^2}-\frac{4}{x}=\frac{4(x-1)^3}{x(2x+1)^2},$$

it follows that $f(x)$ is decreasing for $0<x\leq 1$ and increasing for $x\geq 1$. Therefore, we have $f(x)\geq f(1)=0$. The equality holds for $a=b=c$.

Second Solution. Replacing x,y,z by e^x,e^y,e^z, respectively, we need to show that

$$x+y+z=0$$

involves

$$f(x)+f(y)+f(z)\geq 3f\left(\frac{x+y+z}{3}\right),$$

where

$$f(u)=e^u-\frac{27}{2(2e^u+1)}.$$

If f is convex on $\mathbb{R}$, then this inequality is just Jensen's inequality. Indeed, f is convex because

$$e^{-u}f''(u)=1+\frac{27(1-2e^u)}{(2e^u+1)^3}=\frac{4(e^u-1)^2(2e^u+7)}{(2e^u+1)^3}\geq 0.$$

□

P 1.65. *If* $a,b,c \in \left[\frac{1}{2},2\right]$, *then*

(a) $$8\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge 5\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+9;$$

(b) $$20\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \ge 17\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

(Vasile C., 2008)

Solution. Without loss of generality, assume that

$$a = \max\{a,b,c\}.$$

Let

$$t = \sqrt{\frac{a}{c}}, \quad 1 \le t \le 2.$$

(a) Let

$$E(a,b,c) = 8\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) - 5\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) - 9.$$

We will show that

$$E(a,b,c) \ge E(a,\sqrt{ac},c) \ge 0.$$

We have

$$E(a,b,c) - E(a,\sqrt{ac},c) = 8\left(\frac{a}{b}+\frac{b}{c}-2\sqrt{\frac{a}{c}}\right) - 5\left(\frac{b}{a}+\frac{c}{b}-2\sqrt{\frac{c}{a}}\right)$$

$$= \frac{(b-\sqrt{ac})^2(8a-5c)}{abc} \ge 0.$$

Also,

$$\begin{aligned}
E(a,\sqrt{ac},c) &= 8\left(2\sqrt{\frac{a}{c}}+\frac{c}{a}-3\right) - 5\left(2\sqrt{\frac{c}{a}}+\frac{a}{c}-3\right)\\
&= 8\left(2t+\frac{1}{t^2}-3\right) - 5\left(\frac{2}{t}+t^2-3\right)\\
&= \frac{8}{t^2}(t-1^2(2t+1) - \frac{5}{t}(t-1)^2(t+2)\\
&= \frac{(t-1)^2(4+5t)(2-t)}{t^2} \ge 0.
\end{aligned}$$

The equality holds for $a=b=c$, and also for $a=2$, $b=1$ and $c=\frac{1}{2}$ (or any cyclic permutation).

(b) Let

$$E(a,b,c) = 20\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) - 17\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

We will show that

$$E(a,b,c) \ge E(a,\sqrt{ac},c) \ge 0.$$

We have

$$E(a,b,c) - E(a,\sqrt{ac},c) = 20\left(\frac{a}{b}+\frac{b}{c}-2\sqrt{\frac{a}{c}}\right) - 17\left(\frac{b}{a}+\frac{c}{b}-2\sqrt{\frac{c}{a}}\right)$$

$$= \frac{(b-\sqrt{ac})^2(20a-17c)}{abc} \ge 0.$$

Also, we have

$$\begin{aligned} E(a,\sqrt{ac},c) &= 20\left(2\sqrt{\frac{a}{c}}+\frac{c}{a}\right) - 17\left(2\sqrt{\frac{c}{a}}+\frac{a}{c}\right) \\ &= 20\left(2t+\frac{1}{t^2}\right) - 17\left(\frac{2}{t}+t^2\right) \\ &= \frac{20-34t+40t^3-17t^4}{t^2} \\ &= \frac{(2-t)(17t^3-6t^2-12t+10)}{t^2}. \end{aligned}$$

We need to show that $17t^3-6t^2-12t+10 \ge 0$ for $1 \le t \le 2$. Indeed, we have

$$17t^3-6t^2-12t+10 \ge 11t^2-12t+10 > 4t^2-12t+9 = (2t-3)^2 \ge 0.$$

The equality holds for $a = 2$, $b = 1$ and $c = \frac{1}{2}$ (or any cyclic permutation).

□

P 1.66. *If a,b,c are positive real numbers such that $a \le b \le c$, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge \frac{2a}{b+c}+\frac{2b}{c+a}+\frac{2c}{a+b}.$$

First Solution. Since

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} - \left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) = \left(\frac{a}{b}-1\right)\left(\frac{b}{c}-1\right)\left(\frac{c}{a}-1\right) \ge 0,$$

it suffices to show that

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) + \left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right) \ge \frac{4a}{b+c}+\frac{4b}{c+a}+\frac{4c}{a+b}.$$

This inequality is equivalent to

$$a\left(\frac{1}{b}+\frac{1}{c}-\frac{4}{b+c}\right) + b\left(\frac{1}{c}+\frac{1}{a}-\frac{4}{c+a}\right) + c\left(\frac{1}{a}+\frac{1}{b}-\frac{4}{a+b}\right) \ge 0,$$

$$\frac{a^2(b-c)^2}{b+c}+\frac{b^2(c-a)^2}{c+a}+\frac{c^2(a-b)^2}{a+b}\ge 0.$$

The equality holds for $a=b=c$.

Second Solution. The inequality is equivalent to

$$\frac{a(c-b)}{b(b+c)}-\frac{b(c-a)}{c(c+a)}+\frac{c(b-a)}{a(a+b)}\ge 0.$$

Taking account of

$$b(c-a)=c(b-a)+a(c-b),$$

we may rewrite the inequality as

$$c(b-a)\left[\frac{1}{a(a+b)}-\frac{1}{c(c+a)}\right]+a(c-b)\left[\frac{1}{b(b+c)}-\frac{1}{c(c+a)}\right]\ge 0.$$

Since

$$\frac{1}{a(a+b)}-\frac{1}{c(c+a)}=\frac{c^2-a^2+a(c-b)}{ac(a+b)(c+a)}\ge\frac{c-b}{c(a+b)(c+a)}$$

and

$$\frac{1}{b(b+c)}-\frac{1}{c(c+a)}=\frac{c^2-b^2+c(a-b)}{bc(b+c)(c+a)}\ge\frac{a-b}{b(b+c)(c+a)},$$

it suffices to show that

$$\frac{c(b-a)(c-b)}{c(a+b)(c+a)}+\frac{a(c-b)(a-b)}{b(b+c)(c+a)}\ge 0.$$

This inequality is true if

$$\frac{1}{a+b}-\frac{a}{b(b+c)}\ge 0.$$

Indeed,

$$\frac{1}{a+b}-\frac{a}{b(b+c)}\ge\frac{1}{a+b}-\frac{1}{b+c}=\frac{c-a}{(a+b)(b+c)}\ge 0.$$

□

P 1.67. *Let a,b,c be positive real numbers such that $abc=1$.*

(a) If $a\le b\le c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge a^{3/2}+b^{3/2}+c^{3/2};$$

(b) If $a\le 1\le b\le c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge a^{\sqrt{3}}+b^{\sqrt{3}}+c^{\sqrt{3}}.$$

(Vasile C., 2008)

Solution. (a) Since

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)=\left(\frac{a}{b}-1\right)\left(\frac{b}{c}-1\right)\left(\frac{c}{a}-1\right)\geq 0,$$

it suffices to show that

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)\geq 2(a^{3/2}+b^{3/2}+c^{3/2}).$$

Indeed, by the AM-GM inequality, we have

$$\sum\frac{a}{b}+\sum\frac{b}{a}=\sum a\left(\frac{1}{b}+\frac{1}{c}\right)\geq\sum\frac{2a}{\sqrt{bc}}=2\sum a^{3/2}.$$

The equality holds for $a=b=c=1$.

(b) Let $k=\sqrt{3}$ and

$$E(a,b,c)=\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-a^k-b^k-c^k.$$

We will show that

$$E(a,b,c)\geq E(a,\sqrt{bc},\sqrt{bc})\geq 0;$$

that is,

$$E(\frac{1}{bc},b,c)\geq E(\frac{1}{bc},\sqrt{bc},\sqrt{bc})\geq 0.$$

Substituting

$$t=\sqrt{bc},\quad t\geq 1,$$

we rewrite the right inequality as $f(t)\geq 0$, where

$$f(t)=\frac{1}{t^3}+1+t^3-\frac{1}{t^{2k}}-2t^k.$$

We have the derivative

$$\frac{f'(t)}{t^2}=g(t),\quad g(t)=\frac{-3}{t^6}+3+\frac{2k}{t^{2k+3}}-\frac{2k}{t^{3-k}}.$$

Since

$$\begin{aligned}\frac{1}{2}t^{2k+4}g'(t)&=9t^{2k-3}-k(2k+3)+k(3-k)t^{3k}\\&\geq 9-k(2k+3)+k(3-k)=9-3k^2=0,\end{aligned}$$

$g(t)$ is increasing for $t\geq 1$. Therefore, $g(t)\geq g(1)=0$, $f'(t)\geq 0$, $f(t)$ is increasing for $t\geq 1$, hence $f(t)\geq f(1)=0$.

Substituting $b=x^2$ and $c=y^2$, where $1\leq x\leq y$, the left inequality becomes

$$E\left(\frac{1}{x^2y^2},x^2,y^2\right)\geq E\left(\frac{1}{x^2y^2},xy,xy\right),$$

or, equivalently,

$$\frac{1}{x^4y^2}+\frac{x^2}{y^2}+x^2y^4-\frac{1}{x^3y^3}-1-x^3y^3\geq(y^k-x^k)^2.$$

We write this inequality as

$$(y-x)\left(x^2y^3+\frac{1}{x^4y^3}-\frac{x+y}{y^2}\right)\geq(y^k-x^k)^2,$$

and then show that

$$(y-x)\left(x^2y^3+\frac{1}{x^4y^3}-\frac{x+y}{y^2}\right)\geq(y-x)(y^3-x^3)\geq(y^k-x^k)^2. \qquad (*)$$

The left inequality (*) is true if $f(x,y)\geq 0$, where

$$f(x,y)=x^2y^3+\frac{1}{x^4y^3}-\frac{x+y}{y^2}-y^3+x^3.$$

We will show that

$$f(x,y)\geq f(1,y)\geq 0.$$

Since $1\leq x\leq y$, we have

$$\begin{aligned}f(x,y)-f(1,y)&=x^3-1+y^3(x^2-1)-\frac{1}{y^2}(x-1)-\frac{1}{y^3}\left(1-\frac{1}{x^4}\right)\\&\geq x^3-1+(x^2-1)-(x-1)-\left(1-\frac{1}{x^4}\right)\\&=(x^2-1)\left[\left(x-\frac{1}{x^2}\right)+\left(1-\frac{1}{x^4}\right)\right]\geq 0\end{aligned}$$

and

$$f(1,y)=\frac{1}{y^3}-\frac{1+y}{y^2}+1=\frac{(1+y)(1-y)^2}{y^3}\geq 0.$$

In order to prove the right inequality (*), we will prove that

$$(y-x)(y^3-x^3)\geq\frac{3}{4}(y^2-x^2)^2\geq(y^k-x^k)^2.$$

We have

$$4(y-x)(y^3-x^3)-3(y^2-x^2)^2=(y-x)^4\geq 0.$$

To complete the proof, we only need to show that

$$\frac{k}{2}(y^2-x^2)\geq y^k-x^k,\quad k=\sqrt{3}.$$

For fixed y, let

$$g(x)=x^k-y^k+\frac{k}{2}(y^2-x^2),\quad 1\leq x\leq y.$$

Since
$$g'(x) = kx(x^{k-2} - 1) \le 0,$$
$g(x)$ is decreasing, hence $g(x) \ge g(y) = 0$. The equality in (b) is an equality if and only if $a = b = c = 1$.

□

P 1.68. *If k and a, b, c are positive real numbers, then*
$$\frac{1}{(k+1)a+b} + \frac{1}{(k+1)b+c} + \frac{1}{(k+1)c+a} \ge \frac{1}{ka+b+c} + \frac{1}{kb+c+a} + \frac{1}{kc+a+b}.$$

(Vasile C., 2011)

First Solution. For $k = 1$, we need to show that
$$\frac{1}{2a+b} + \frac{1}{2b+c} + \frac{1}{2c+a} \ge \frac{3}{a+b+c}.$$
This follows immediately from the Cauchy-Schwarz inequality, as follows:
$$\frac{1}{2a+b} + \frac{1}{2b+c} + \frac{1}{2c+a} \ge \frac{9}{(2a+b)+(2b+c)+(2c+a)}$$
$$= \frac{3}{a+b+c}.$$
Further, consider two cases: $k > 1$ and $0 < k < 1$.

Case 1: $k > 1$. By the Cauchy-Schwarz inequality, we have
$$\frac{k-1}{(k+1)a+b} + \frac{1}{kc+a+b} \ge \frac{[(k-1)+1]^2}{(k-1)[(k+1)a+b]+(kc+a+b)}$$
$$= \frac{k}{ka+b+c}.$$
Adding this inequality and the similar ones yields the desired inequality.

Case 2: $0 < k < 1$. By the Cauchy-Schwarz inequality, we have
$$\frac{1-k}{(k+1)a+b} + \frac{k}{ka+b+c} \ge \frac{[(1-k)+k]^2}{(1-k)[(k+1)a+b]+k(ka+b+c)}$$
$$= \frac{1}{kc+a+b}.$$
Adding this inequality and the similar ones yields the desired inequality.
The equality holds for $a = b = c$.

Second Solution (by *Vo Quoc Ba Can*). By the Cauchy-Schwarz inequality, we have
$$\frac{1}{(k+1)a+b} + \frac{k}{(k+1)b+c} + \frac{k^2}{(k+1)c+a} \ge$$

$$\geq \frac{(1+k+k^2)^2}{[(k+1)a+b]+k[(k+1)b+c]+k^2[(k+1)c+a]}$$
$$= \frac{1+k+k^2}{kc+a+b}.$$

Therefore, we get in succession

$$\sum \frac{1}{(k+1)a+b} + \sum \frac{k}{(k+1)b+c} + \sum \frac{k^2}{(k+1)c+a} \geq \sum \frac{1+k+k^2}{kc+a+b},$$
$$(1+k+k^2)\sum \frac{1}{(k+1)a+b} \geq (1+k+k^2)\sum \frac{1}{ka+b+c},$$
$$\sum \frac{1}{(k+1)a+b} \geq \sum \frac{1}{ka+b+c}.$$

Third Solution. We have

$$\frac{1}{(k+1)a+b} - \frac{1}{ka+b+c} = \frac{c-a}{(ka+a+b)(ka+b+c)}$$
$$\geq \frac{c-a}{(kc+a+b)(ka+b+c)} = \frac{1}{k-1}\left(\frac{1}{ka+b+c} - \frac{1}{kc+a+b}\right),$$

hence

$$\sum \frac{1}{(k+1)a+b} - \sum \frac{1}{ka+b+c} \geq \frac{1}{k-1}\left(\sum \frac{1}{ka+b+c} - \sum \frac{1}{kc+a+b}\right) = 0.$$

□

P 1.69. *If a, b, c are positive real numbers, then*

(a) $$\frac{a}{\sqrt{2a+b}} + \frac{b}{\sqrt{2b+c}} + \frac{c}{\sqrt{2c+a}} \leq \sqrt{a+b+c};$$

(b) $$\frac{a}{\sqrt{a+2b}} + \frac{b}{\sqrt{b+2c}} + \frac{c}{\sqrt{c+2a}} \geq \sqrt{a+b+c}.$$

Solution. (a) By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{\sqrt{2a+b}} = \sum \left(\sqrt{a} \cdot \sqrt{\frac{a}{2a+b}}\right) \leq \sqrt{\left(\sum a\right)\left(\sum \frac{a}{2a+b}\right)}.$$

Therefore, it suffices to show that

$$\sum \frac{a}{2a+b} \leq 1.$$

This inequality is equivalent to

$$\sum \frac{b}{2a+b} \ge 1.$$

Applying the Cauchy-Schwarz inequality, we get

$$\sum \frac{b}{2a+b} \ge \frac{\left(\sum b\right)^2}{\sum b(2a+b)} = 1.$$

The equality holds for $a = b = c$.

(b) By Hölder's inequality, we have

$$\left(\sum \frac{a}{\sqrt{a+2b}}\right)^2 \ge \frac{\left(\sum a\right)^3}{\sum a(a+2b)} = \sum a.$$

From this, the desired inequality follows. The equality holds for $a = b = c$.

□

P 1.70. *Let a, b, c be nonnegative real numbers such that $a + b + c = 3$. Prove that*

$$a\sqrt{\frac{a+2b}{3}} + b\sqrt{\frac{b+2c}{3}} + c\sqrt{\frac{c+2a}{3}} \le 3.$$

First Solution. By the Cauchy-Schwarz inequality, we have

$$\sum a\sqrt{\frac{a+2b}{3}} \le \sqrt{\left(\sum a\right)\left[\sum \frac{a(a+2b)}{3}\right]} = \sqrt{\frac{\left(\sum a\right)^3}{3}} = 3.$$

The equality holds for $a = b = c = 1$, and also for $a = 3$ and $b = c = 0$ (or any cyclic permutation).

Second Solution. Applying Jensen's inequality to the concave function $f(x) = \sqrt{x}$, $x \ge 0$, we have

$$a\sqrt{a+2b} + b\sqrt{b+2c} + c\sqrt{c+2a} \le$$

$$\le (a+b+c)\sqrt{\frac{a(a+2b)+b(b+2c)+c(c+2a)}{a+b+c}}$$

$$= (a+b+c)\sqrt{a+b+c} = 3\sqrt{3}.$$

□

P 1.71. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$a\sqrt{1+b^3}+b\sqrt{1+c^3}+c\sqrt{1+a^3}\le 5.$$

(Pham Kim Hung, 2007)

Solution. Using the AM-GM inequality yields

$$\sqrt{1+b^3}=\sqrt{(1+b)(1-b+b^2)}\le\frac{(1+b)+(1-b+b^2)}{2}=1+\frac{b^2}{2}.$$

Therefore,

$$\sum a\sqrt{1+b^3}\le\sum a\left(1+\frac{b^2}{2}\right)=3+\frac{ab^2+bc^2+ca^2}{2}.$$

To complete the proof, it remains to show that

$$ab^2+bc^2+ca^2\le 4.$$

But this is just the inequality in P 1.1. The equality occurs for $a=0$, $b=1$ and $c=2$ (or any cyclic permutation).

□

P 1.72. *If a,b,c are positive real numbers such that $abc=1$, then*

(a)
$$\sqrt{\frac{a}{b+3}}+\sqrt{\frac{b}{c+3}}+\sqrt{\frac{c}{a+3}}\ge\frac{3}{2};$$

(b)
$$\sqrt[3]{\frac{a}{b+7}}+\sqrt[3]{\frac{b}{c+7}}+\sqrt[3]{\frac{c}{a+7}}\ge\frac{3}{2}.$$

Solution. (a) Putting

$$a=\frac{x}{y},\quad b=\frac{z}{x},\quad c=\frac{y}{z},$$

the inequality can be restated as

$$\frac{x}{\sqrt{y(3x+z)}}+\frac{y}{\sqrt{z(3y+x)}}+\frac{z}{\sqrt{x(3z+y)}}\ge\frac{3}{2}.$$

By Hölder's inequality, we have

$$\left[\sum\frac{x}{\sqrt{y(3x+z)}}\right]^2\left[\sum xy(3x+z)\right]\ge\left(\sum x\right)^3.$$

Therefore, it suffices to show that

$$4(x+y+z)^3 \geq 27(x^2y+y^2z+z^2x+xyz).$$

This is just the inequality (a) in P 1.9. The equality holds for $a=b=c=1$.

(b) Putting

$$a=\frac{x^4}{y^4}, \quad b=\frac{z^4}{x^4}, \quad c=\frac{y^4}{z^4},$$

the inequality becomes

$$\sum \sqrt[3]{\frac{x^8}{y^4(7x^4+z^4)}} \geq \frac{3}{2}.$$

By Hölder's inequality, we have

$$\left[\sum \sqrt[3]{\frac{x^8}{y^4(7x^4+z^4)}}\right]^3 \left[\sum (7x^4+z^4)\right] \geq \left(\sum \frac{x^2}{y}\right)^4.$$

Since $\sum(7x^4+z^4)=8\sum x^4$, it is enough to show that

$$\left(\frac{x^2}{y}+\frac{y^2}{z}+\frac{z^2}{x}\right)^4 \geq 27(x^4+y^4+z^4),$$

which is just the inequality in P 1.60-(a). The equality holds for $a=b=c=1$.

□

P 1.73. *If a,b,c are positive real numbers, then*

$$\left(1+\frac{4a}{a+b}\right)^2+\left(1+\frac{4b}{b+c}\right)^2+\left(1+\frac{4c}{c+a}\right)^2 \geq 27.$$

(Vasile C., 2012)

Solution. Let

$$x=\frac{a-b}{a+b}, \quad y=\frac{b-c}{b+c}, \quad z=\frac{c-a}{c+a}.$$

We have

$$-1<x,y,z<1$$

and

$$x+y+z+xyz=0.$$

Since

$$\frac{2a}{a+b}=x+1, \quad \frac{2b}{b+c}=y+1, \quad \frac{2c}{c+a}=z+1,$$

we can write the inequality as follows:

$$(2x+3)^2+(2y+3)^2+(2z+3)^2 \geq 27,$$

$$x^2 + y^2 + z^2 + 3(x + y + z) \geq 0,$$
$$x^2 + y^2 + z^2 \geq 3xyz.$$

By the AM-GM inequality, we have

$$x^2 + y^2 + z^2 \geq 3\sqrt[3]{x^2y^2z^2}.$$

Thus, it suffices to show that $|xyz| \leq 1$, which is clearly true. The equality holds for $a = b = c$.

□

P 1.74. *If a, b, c are positive real numbers, then*

$$\sqrt{\frac{2a}{a+b}} + \sqrt{\frac{2b}{b+c}} + \sqrt{\frac{2c}{c+a}} \leq 3.$$

(Vasile C., 1992)

First Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \sqrt{\frac{2a}{a+b}} \leq \sqrt{\left[\sum \frac{2a}{(a+b)(a+c)}\right]\left[\sum (a+c)\right]}.$$

Thus, it suffices to show that

$$\sum \frac{a}{(a+b)(a+c)} \leq \frac{9}{4(a+b+c)},$$

which is equivalent to

$$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \geq 0.$$

The equality occurs for $a = b = c$.

Second Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \sqrt{\frac{2a}{a+b}} \leq \sqrt{\left[\sum \frac{1}{(a+b)(b+c)}\right]\left[\sum 2a(b+c)\right]}.$$

Thus, it suffices to show that

$$\sum \frac{1}{(a+b)(b+c)} \leq \frac{9}{4(ab+bc+ca)},$$

which is equivalent to

$$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \geq 0.$$

□

P 1.75. *If a, b, c are nonnegative real numbers, then*

$$\sqrt{\frac{a}{4a+5b}}+\sqrt{\frac{b}{4b+5c}}+\sqrt{\frac{c}{4c+5a}} \le 1.$$

(Vasile C., 2004)

Solution. If one of a, b, c is zero, then the inequality is clearly true. Otherwise, using the substitution

$$u=\frac{b}{a}, \quad v=\frac{c}{b}, \quad w=\frac{a}{c},$$

we need to show that $uvw = 1$ involves

$$\frac{1}{\sqrt{4+5u}}+\frac{1}{\sqrt{4+5v}}+\frac{1}{\sqrt{4+5w}} \le 1.$$

Using the contradiction method, it suffices to show that

$$\frac{1}{\sqrt{4+5u}}+\frac{1}{\sqrt{4+5v}}+\frac{1}{\sqrt{4+5w}} > 1$$

involves $uvw < 1$. Let

$$x=\frac{1}{\sqrt{4+5u}}, \quad y=\frac{1}{\sqrt{4+5v}}, \quad z=\frac{1}{\sqrt{4+5w}},$$

where $x, y, z \in \left(0, \frac{1}{2}\right)$. Since

$$u=\frac{1-4x^2}{5x^2}, \quad v=\frac{1-4y^2}{5y^2}, \quad w=\frac{1-4z^2}{5z^2},$$

we have to prove that $x+y+z > 1$ involves

$$(1-4x^2)(1-4y^2)(1-4z^2) < 125x^2y^2z^2.$$

Since

$$1-4x^2 < (x+y+z)^2-4x^2 = (-x+y+z)(3x+y+z),$$

it suffices to prove the homogeneous inequality

$$(3x+y+z)(3y+z+x)(3z+x+y)(-x+y+z)(-y+z+x)(-z+x+y) \le 125x^2y^2z^2.$$

By the AM-GM inequality, we have

$$(3x+y+z)(3y+z+x)(3z+x+y) \le 125\left(\frac{x+y+z}{3}\right)^3.$$

Therefore, it is enough to show that

$$\left(\frac{x+y+z}{3}\right)^3(-x+y+z)(-y+z+x)(-z+x+y) \le x^2y^2z^2.$$

Using the substitution

$$a = -x + y + z, \quad b = -y + z + x, \quad c = -z + x + y,$$

where $a, b, c > 0$, the inequality can be restated as

$$64abc(a+b+c)^3 \le 27(b+c)^2(c+a)^2(a+b)^2.$$

The known inequality

$$9(b+c)(c+a)(a+b) \ge 8(a+b+c)(ab+bc+ca),$$

equivalent to

$$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \ge 0,$$

involves

$$81(b+c)^2(c+a)^2(a+b)^2 \ge 64(a+b+c)^2(ab+bc+ca)^2.$$

Thus, it suffices to show that

$$3abc(a+b+c) \le (ab+bc+ca)^2.$$

which is also a known inequality, equivalent to

$$a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2 \ge 0.$$

Thus, the proof is completed. The equality occurs for $a = b = c$.

□

P 1.76. *If a, b, c are positive real numbers, then*

$$\frac{a}{\sqrt{4a^2+ab+4b^2}} + \frac{b}{\sqrt{4b^2+bc+4c^2}} + \frac{c}{\sqrt{4c^2+ca+4a^2}} \le 1.$$

(Bin Zhao, 2006)

Solution. By the AM-GM inequality, we have

$$ab + 4b^2 \ge 5\sqrt[5]{ab \cdot b^8} = 5\sqrt[5]{ab^9},$$

$$\frac{a}{\sqrt{4a^2+ab+4b^2}} \le \frac{a}{\sqrt{4a^2+5\sqrt[5]{ab^9}}} = \sqrt{\frac{a^{9/5}}{4a^{9/5}+5b^{9/5}}}.$$

Therefore, it suffices to show that

$$\sqrt{\frac{a^{9/5}}{4a^{9/5}+5b^{9/5}}} + \sqrt{\frac{b^{9/5}}{4b^{9/5}+5c^{9/5}}} + \sqrt{\frac{c^{9/5}}{4c^{9/5}+5a^{9/5}}} \le 1.$$

Replacing $a^{9/5}, b^{9/5}, c^{9/5}$ by a, b, c, respectively, we get the inequality in P 1.75. The equality holds for $a = b = c$.

□

P 1.77. *If a,b,c are positive real numbers, then*

$$\sqrt{\frac{a}{a+b+7c}}+\sqrt{\frac{b}{b+c+7a}}+\sqrt{\frac{c}{c+a+7b}}\ge 1.$$

(Vasile C., 2006)

Solution. Substituting

$$x=\sqrt{\frac{a}{a+b+7c}},\quad y=\sqrt{\frac{b}{b+c+7a}},\quad z=\sqrt{\frac{c}{c+a+7b}},$$

we have

$$\begin{cases} (x^2-1)a+x^2b+7x^2c=0 \\ (y^2-1)b+y^2c+7y^2a=0 \\ (z^2-1)c+z^2a+7z^2b=0 \end{cases},$$

which involves

$$\begin{vmatrix} x^2-1 & x^2 & 7x^2 \\ 7y^2 & y^2-1 & y^2 \\ z^2 & 7z^2 & z^2-1 \end{vmatrix}=0\ ;$$

that is,

$$F(x,y,z)=0,$$

where

$$F(x,y,z)=324x^2y^2z^2+6\sum x^2y^2+\sum x^2-1.$$

We need to show that $F(x,y,z)=0$ involves $x+y+z\ge 1$, where $x,y,z>0$. To do this, we use the contradiction method. Assume that $x+y+z<1$ and show that $F(x,y,z)<0$. Since $F(x,y,z)$ is strictly increasing in each of its arguments, it is enough to prove that $x+y+z=1$ involves $F(x,y,z)\le 0$. We have

$$\begin{aligned} F(x,y,z) &= 324x^2y^2z^2+6\left(\sum xy\right)^2-12xyz\sum x+\left(\sum x\right)^2-2\sum xy-1 \\ &= 324x^2y^2z^2+6\left(\sum xy\right)^2-12xyz-2\sum xy \\ &= 12xyz(27xyz-1)+2\left(\sum xy\right)\left(3\sum xy-1\right). \end{aligned}$$

Because

$$27xyz\le\left(\sum x\right)^3=1$$

and

$$3\sum xy\le\left(\sum x\right)^2=1,$$

the conclusion follows. The equality occurs for $a=b=c$.

□

P 1.78. *If a,b,c are nonnegative real numbers, no two of which are zero, then*

(a)
$$\sqrt{\frac{a}{3b+c}}+\sqrt{\frac{b}{3c+a}}+\sqrt{\frac{c}{3a+b}}\ge\frac{3}{2};$$

(b)
$$\sqrt{\frac{a}{2b+c}}+\sqrt{\frac{b}{2c+a}}+\sqrt{\frac{c}{2a+b}}\ge\sqrt[4]{8}.$$

(Vasile Cîrtoaje and Pham Kim Hung, 2006)

Solution. Consider the inequality

$$\sqrt{\frac{(k+1)a}{kb+c}}+\sqrt{\frac{(k+1)b}{kc+a}}+\sqrt{\frac{(k+1)c}{ka+b}}\ge A_k,\quad k>0,$$

and use the substitution

$$x=\sqrt{\frac{(k+1)a}{kb+c}},\quad y=\sqrt{\frac{(k+1)b}{kc+a}},\quad z=\sqrt{\frac{(k+1)c}{ka+b}}.$$

From the identity

$$(kb+c)(kc+a)(ka+b)=(k^3+1)abc+kbc(kb+c)+kca(kc+a)+kab(ka+b),$$

written as

$$\frac{kb+c}{(k+1)a}\cdot\frac{kc+a}{(k+1)b}\cdot\frac{ka+b}{(k+1)c}=\frac{k^2-k+1}{(k+1)^2}+\frac{k}{(k+1)^2}\left[\frac{kb+c}{(k+1)a}+\frac{kc+a}{(k+1)b}+\frac{ka+b}{(k+1)c}\right],$$

we get

$$\frac{1}{x^2y^2z^2}=\frac{k^2-k+1}{(k+1)^2}+\frac{k}{(k+1)^2}\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right),$$

which is equivalent to $F(x,y,z)=0$, where

$$F(x,y,z)=k(x^2y^2+y^2z^2+z^2x^2)+(k^2-k+1)x^2y^2z^2-(k+1)^2.$$

So, we need to show that $F(x,y,z)=0$ yields $x+y+z\ge A_k$. To do this, we use the contradiction method. Assume that $x+y+z<A_k$ and show that $F(x,y,z)<0$. Since $F(x,y,z)$ is strictly increasing in each of its variables, it suffices to prove that $x+y+z=A_k$ involves $F(x,y,z)\le 0$. Let

$$k_1=\frac{49+9\sqrt{17}}{32}\approx 2.691.$$

(a) We need to show that $F(x,y,z)\le 0$ for $x+y+z=A_k=3$ and $k=3$. We will show a more general inequality, namely $F(x,y,z)\le 0$ for $k\ge k_1$ and all nonnegative numbers x,y,z satisfying $x+y+z=3$. The AM-GM inequality $x+y+z\ge 3\sqrt[3]{xyz}$ involves $xyz\le 1$. On the other hand, by Schur's inequality

$$(x+y+z)^3+9xyz\ge 4(x+y+z)(xy+yz+zx)$$

we get

$$4(xy+yz+zx)\le 9+3xyz,$$

hence

$$(xy+yz+zx)^2-9\le\frac{(9+3xyz)^2}{16}-9=\frac{9}{16}(xyz-1)(xyz+7).$$

Therefore,

$$\begin{aligned}F(x,y,z)&=k[(xy+yz+zx)^2-6xyz]+(k^2-k+1)x^2y^2z^2-(k+1)^2\\&=k[(xy+yz+zx)^2-9]+(k^2-k+1)(x^2y^2z^2-1)-6k(xyz-1)\\&\le\frac{9k}{16}(xyz-1)(xyz+7)+(k^2-k+1)(x^2y^2z^2-1)-6k(xyz-1)\\&=\frac{1}{16}(xyz-1)\left[(16k^2-7k+16)xyz+16k^2-49k+16\right]\le 0.\end{aligned}$$

Since $xyz-1\le 0$ and $16k^2-7k+16>0$, it suffices to show that $16k^2-49k+16\ge 0$; indeed, this inequality is true for $k\ge k_1$.
The equality occurs for $a=b=c$. In addition, when $k=k_1$, the equality occurs also for $a=0$ and $b/c=\sqrt{k}$ (or any cyclic permutation).

(b) We need to show that $F(x,y,z)\le 0$ for $A_k=\sqrt[4]{72}$ and $k=2$. We will show a more general inequality, that $F(x,y,z)\le 0$ for $1\le k\le k_1$ and all nonnegative numbers x,y,z satisfying

$$x+y+z=A_k=2\sqrt[4]{\frac{(k+1)^2}{k}}.$$

From

$$\begin{aligned}F(x,y,z)&=k(x^2y^2+y^2z^2+z^2x^2)+(k^2-k+1)x^2y^2z^2-(k+1)^2\\&=k(xy+yz+zx)^2-2kA_kxyz+(k^2-k+1)x^2y^2z^2-(k+1)^2,\end{aligned}$$

it follows that for fixed xyz, $F(x,y,z)$ is maximal when $xy+yz+zx$ is maximal; that is, according to P 3.58 in Volume 1, when two of x,y,z are equal. Due to symmetry, we only need to show that $F(x,y,z)\le 0$ for $y=z$. Write the inequality $F(x,y,z)\le 0$ as follows:

$$k(x^2y^2+y^2z^2+z^2x^2)+(k^2-k+1)x^2y^2z^2-k\left(\frac{x+y+z}{2}\right)^4\le 0,$$

$$k\left[\left(\frac{x+y+z}{2}\right)^4-x^2y^2-y^2z^2-z^2x^2\right]\ge(k^2-k+1)x^2y^2z^2,$$

$$k\sqrt{k}\ (x+y+z)^2\left[(x+y+z)^4-16(x^2y^2+y^2z^2+z^2x^2\right]\ge 64(k^3+1)x^2y^2z^2.$$

Due to homogeneity, we may only consider the cases $y=z=0$ and $y=z=1$. In the non-trivial case $y=z=1$, the inequality becomes

$$k\sqrt{k}\ x(x+2)^2(x^3+8x^2-8x+32)\ge 64(k^3+1)x^2.$$

This is true because

$$297k\sqrt{k}\ge 64(k^3+1)$$

for $1\le k\le k_1$, and

$$x(x+2)^2(x^3+8x^2-8x+32)\ge 297x^2.$$

Notice that

$$x(x+2)^2(x^3+8x^2-8x+32)-297x^2=x(x-1)^2(x^3+14x^2+55x+128)\ge 0.$$

If $1\le k<k_1$, then the equality occurs only for $a=0$ and $b/c=\sqrt{k}$ (or any cyclic permutation). Therefore, if $k=2$, then the equality holds for $a=0$ and $b/c=\sqrt{2}$ (or any cyclic permutation).

Remark. From the proof above, it follows that the following more general statement holds:

- *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k>0$, then*

$$\sqrt{\frac{a}{kb+c}}+\sqrt{\frac{b}{kc+a}}+\sqrt{\frac{c}{ka+b}}\ge \min\left\{\frac{3}{\sqrt{k+1}},\frac{2}{\sqrt[4]{k}}\right\}.$$

For $k=1$, we get the known inequality

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\ge 2,$$

with equality for $a=0$ and $b=c$ (or any cyclic permutation). We can get this inequality by summing the inequalities

$$\sqrt{\frac{a}{b+c}}\ge\frac{2a}{a+b+c},\quad \sqrt{\frac{b}{c+a}}\ge\frac{2b}{a+b+c},\quad \sqrt{\frac{c}{a+b}}\ge\frac{2c}{a+b+c}.$$

□

P 1.79. *If a,b,c are positive real numbers such that $ab+bc+ca=3$, then*

(a) $$\frac{1}{(a+b)(3a+b)}+\frac{1}{(b+c)(3b+c)}+\frac{1}{(c+a)(3c+a)}\ge\frac{3}{8};$$

(b) $$\frac{1}{(2a+b)^2}+\frac{1}{(2b+c)^2}+\frac{1}{(2c+a)^2}\ge\frac{1}{3}.$$

(Vasile Cîrtoaje and Pham Kim Hung, 2007)

Solution. (a) Using the Cauchy-Schwarz inequality and the inequality in P 1.78-(a) gives

$$\sum \frac{1}{(a+b)(3a+b)} = \sum \frac{1}{(b+c)(3b+c)}$$
$$\geq \frac{\left(\sum \sqrt{\frac{a}{3b+c}}\right)^2}{\sum a(b+c)}$$
$$\geq \frac{9}{8(ab+bc+ca)} = \frac{3}{8}.$$

The equality holds for $a = b = c$.

(b) We consider two cases (*Vo Quoc Ba Can*).

Case 1: $4(ab + bc + ca \geq a^2 + b^2 + c^2$. By the Cauchy-Schwarz inequality, we get

$$\sum \frac{1}{(2a+b)^2} \geq \frac{9\left(\sum a\right)^2}{\sum (2a+b)^2(b+2c)^2}.$$

Thus, it suffices to show that

$$9p^2q \geq \sum (2a+b)^2(b+2c)^2,$$

where $p = a + b + c$, $q = ab + bc + ca$. Since

$$(2a+b)(b+2c) = pb + q + 3ac,$$

we have

$$\sum (2a+b)^2(b+2c)^2 = p^2 \sum a^2 + 3q^2 + 9\sum a^2b^2 + 2p^2q + 18abcp + 6q^2$$
$$= p^2(p^2-2q) + 9q^2 + 9(q^2-2abcp) + 2p^2q + 18abcp = p^4 + 18q^2,$$

and the inequality becomes

$$9p^2q \geq p^4 + 18q^2,$$
$$(p^2-3q)(6q-p^2) \geq 0.$$

The last inequality is true since $p^2 - 3q \geq 0$ and

$$6q - p^2 = 4(ab+bc+ca) - a^2 - b^2 - c^2 \geq 0.$$

Case 2: $4(ab + bc + ca < a^2 + b^2 + c^2$. Assume that $a = \max\{a, b, c\}$. From

$$a^2 - 4(b+c)a + (b+c)^2 > 6bc > 0,$$

we get

$$a > (2+\sqrt{3})(b+c) > 2(b+c).$$

Since

$$\frac{1}{(2a+b)^2}+\frac{1}{(2b+c)^2}+\frac{1}{(2c+a)^2}>\frac{1}{(2b+c)^2}+\frac{1}{(2c+a)^2}\ge\frac{2}{(2b+c)(2c+a)},$$

it suffices to show that

$$\frac{2}{(2b+c)(2c+a)}\ge\frac{1}{ab+bc+ca}.$$

This is equivalent to the obvious inequality

$$c(a-2b-2c)\ge 0.$$

The proof is completed. The equality holds for $a=b=c$.

Conjecture. *Let a,b,c be nonnegative real numbers, no two of which are zero. If $k>0$, then*

(a) $$\frac{1}{(a+b)(ka+b)}+\frac{1}{(b+c)(kb+c)}+\frac{1}{(c+a)(kc+a)}\ge\frac{9}{2(k+1)(ab+bc+ca)};$$

(b) $$\frac{1}{(ka+b)^2}+\frac{1}{(kb+c)^2}+\frac{1}{(kc+a)^2}\ge\frac{9}{(k+1)^2(ab+bc+ca)}.$$

For $k=1$, from (a) and (b), we get the well-known inequality (Iran 96):

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\ge\frac{9}{4(ab+bc+ca)}.$$

□

P 1.80. *If a,b,c are nonnegative real numbers, then*

$$a^4+b^4+c^4+15(a^3b+b^3c+c^3a)\ge\frac{47}{4}(a^2b^2+b^2c^2+c^2a^2).$$

(Vasile C., 2011)

Solution. Without loss of generality, assume that $a=\min\{a,b,c\}$. There are two cases to consider: $a\le b\le c$ and $a\le c\le b$.

Case 1: $a\le b\le c$. For $a=0$, the inequality is true because is equivalent to

$$b^4+c^4+15b^3c-\frac{47}{4}b^2c^2\ge 0,$$

$$\left(b-\frac{c}{2}\right)^2(b^2+16bc+4c^2)\ge 0.$$

Based on this result, it suffices to prove that

$$a^4+15(a^3b+c^3a)\geq \frac{47}{4}a^2(b^2+c^2).$$

This inequality is true if

$$a^3b+c^3a\geq a^2(b^2+c^2).$$

Indeed,

$$\begin{aligned}a^2b+c^3-a(b^2+c^2)&=c^2(c-a)-ab(b-a)\geq c^2(b-a)-ab(b-a)\\&=(c^2-ab)(b-a)\geq 0.\end{aligned}$$

Case 2: $a\leq c\leq b$. It suffices to show that

$$a^3b+b^3c+c^3a\geq a^2b^2+b^2c^2+c^2a^2.$$

Since

$$ab^3+bc^3+ca^3-(a^3b+b^3c+c^3a)=(a+b+c)(a-b)(b-c)(c-a)\leq 0,$$

we have

$$\sum a^3b\geq \frac{1}{2}(\sum a^3b+\sum ab^3)=\frac{1}{2}\sum ab(a^2+b^2)\geq \sum a^2b^2.$$

The equality holds for $a=0$ and $2b=c$ (or any cyclic permutation).

□

P 1.81. *If a,b,c are nonnegative real numbers such that $a+b+c=4$, then*

$$a^3b+b^3c+c^3a\leq 27.$$

Solution. Assume that $a=\max\{a,b,c\}$. There are two possible cases: $a\geq b\geq c$ and $a\geq c\geq b$.

Case 1: $a\geq b\geq c$. Using the AM-GM inequality gives

$$\begin{aligned}3(a^3b+b^3c+c^3a)&\leq 3ab(a^2+ac+c^2)\leq 3ab(a+c)^2\\&=a\cdot 3b\cdot(a+c)\cdot(a+c)\leq\left[\frac{a+3b+(a+c)+(a+c)}{4}\right]^4\\&=\left(\frac{3a+3b+2c}{4}\right)^4\leq\left(\frac{3a+3b+3c}{4}\right)^4=81.\end{aligned}$$

Case 2: $a\geq c\geq b$. Since

$$ab^3+bc^3+ca^3-(a^3b+b^3c+c^3a)=(a+b+c)(a-b)(b-c)(c-a)\geq 0,$$

it suffices to prove that

$$a^3b + b^3c + c^3a + (ab^3 + bc^3 + ca^3) \le 54.$$

Indeed,

$$\begin{aligned}\sum a^3b + \sum ab^3 &\le (a^2 + b^2 + c^2)(ab + bc + ca)\\ &\le \frac{1}{8}[a^2 + b^2 + c^2 + 2(ab + bc + ca)]^2\\ &= \frac{1}{8}(a + b + c)^4 = 32 < 54.\end{aligned}$$

The equality holds for $a = 3,\ b = 1$ and $c = 0$ (or any cyclic permutation).

Remark. The following sharper inequality holds (*Michael Rozenberg*).

- *If a, b, c are nonnegative real numbers such that $a + b + c = 4$, then*

$$a^3b + b^3c + c^3a + \frac{473}{64}\,abc \le 27,$$

with equality for $a = b = c = 4/3$, and also for $a = 3,\ b = 1$ and $c = 0$ (or any cyclic permutation).

Write the inequality in the homogeneous form

$$27(a + b + c)^4 \ge 256(a^3b + b^3c + c^3a) + 473abc(a + b + c).$$

Assuming that $c = \min\{a, b, c\}$ and using the substitution

$$a = c + p, \quad b = c + q, \quad p, q \ge 0,$$

this inequality can be restated as

$$Ac^2 + Bc + C \ge 0,$$

where

$$A = 217(p^2 - pq + q^2) \ge 0,$$
$$B = 68p^3 - 269p^2q + 499pq^2 + 68q^3 \ge 60p(p^2 - 5pq + 8q^2) \ge 0,$$
$$C = (p - 3q)^2(27p^2 + 14pq + 3q^2) \ge 0.$$

□

P 1.82. *Let a, b, c be nonnegative real numbers such that*

$$a^2 + b^2 + c^2 = \frac{10}{3}(ab + bc + ca).$$

Prove that

$$a^4 + b^4 + c^4 \ge \frac{82}{27}(a^3b + b^3c + c^3a).$$

(Vasile C., 2011)

Solution *(by Vo Quoc Ba Can).* We see that the equality holds for $a = 3$, $b = 1$, $c = 0$. From

$$a^4 + b^4 + c^4 + 2(ab + bc + ca)^2 = (a^2 + b^2 + c^2)^2 + 4abc(a + b + c),$$

we get

$$\begin{aligned} a^4 + b^4 + c^4 &\geq (a^2 + b^2 + c^2)^2 - 2(ab + bc + ca)^2 \\ &= \frac{82}{9}(ab + bc + ca)^2. \end{aligned}$$

Therefore, it suffices to show that

$$3(ab + bc + ca)^2 \geq a^3b + b^3c + c^3a.$$

In addition, since

$$ab + bc + ca = \frac{3(a^2 + b^2 + c^2) + 6(ab + bc + ca)}{16} = 3\left(\frac{a + b + c}{4}\right)^2,$$

it suffices to show that

$$27\left(\frac{a + b + c}{4}\right)^4 \geq a^3b + b^3c + c^3a,$$

which is the inequality from the preceding P 1.81. The equality holds for $a = 3b$ and $c = 0$ (or any cyclic permutation).

□

P 1.83. *If a, b, c are positive real numbers, then*

$$\frac{a^3}{2a^2 + b^2} + \frac{b^3}{2b^2 + c^2} + \frac{c^3}{2c^2 + a^2} \geq \frac{a + b + c}{3}.$$

(Vasile C., 2005)

Solution. We write the inequality as

$$\left(\frac{a^3}{2a^2 + b^2} - \frac{a}{3}\right) + \left(\frac{b^3}{2b^2 + c^2} - \frac{b}{3}\right) + \left(\frac{c^3}{2c^2 + a^2} - \frac{c}{3}\right) \geq 0,$$

$$\frac{a(a^2 - b^2)}{2a^2 + b^2} + \frac{b(b^2 - c^2)}{2b^2 + c^2} + \frac{c(c^2 - a^2)}{2c^2 + a^2} \geq 0.$$

Taking into account that

$$\frac{a(a^2 - b^2)}{2a^2 + b^2} - \frac{b(a^2 - b^2)}{2b^2 + a^2} = \frac{(a + b)(a - b)^2(a^2 - ab + b^2)}{(2a^2 + b^2)(2b^2 + a^2)} \geq 0,$$

it suffices to show that

$$\frac{b(a^2-b^2)}{2b^2+a^2}+\frac{b(b^2-c^2)}{2b^2+c^2}+\frac{c(c^2-a^2)}{2c^2+a^2}\geq 0.$$

Since

$$\frac{b(a^2-b^2)}{2b^2+a^2}+\frac{b(b^2-c^2)}{2b^2+c^2}=\frac{3b^2(a^2-c^2)}{(2b^2+a^2)(2b^2+c^2)},$$

the last inequality is equivalent to

$$(c^2-a^2)(c-b)[a^2(3b^2+bc+c^2)+2b^2c(c-2b)]\geq 0. \tag{*}$$

Similarly, the desired inequality is true if

$$(a^2-b^2)(a-c)[b^2(3c^2+ca+a^2)+2c^2a(a-2c)]\geq 0. \tag{**}$$

Without loss of generality, assume that

$$c=\max\{a,b,c\}.$$

According to (*), the desired inequality is true if

$$a^2(3b^2+bc+c^2)+2b^2c(c-2b)\geq 0.$$

We claim that this inequality holds for for $a\geq b$, and also for $2ac\geq\sqrt{3}\,b^2$. If $a\geq b$, then

$$\begin{aligned}a^2(3b^2+bc+c^2)+2b^2c(c-2b)&\geq b^2(3b^2+bc+c^2)+2b^2c(c-2b)\\&=3b^2[b^2+c(c-b)]>0;\end{aligned}$$

also, if $2ac\geq\sqrt{3}\,b^2$, then

$$\begin{aligned}a^2(3b^2+bc+c^2)+2b^2c(c-2b)&\geq\frac{3b^4}{4c^2}(3b^2+bc+c^2)+2b^2c(c-2b)\\&=\frac{b^2}{4c^2}(8c^4-16bc^3+3b^2c^2+3b^3c+9b^4)\\&=\frac{b^2}{4c^2}[2c(c+b)(2c-3b)^2+9b^2(c-b)^2+3b^3c]>0.\end{aligned}$$

Consequently, we only need to consider that $a<b\leq c$ and $\sqrt{3}\,b^2>2ac$. According to (**), the desired inequality is true if

$$b^2(3c^2+ca+a^2)+2c^2a(a-2c)\geq 0.$$

We have

$$\begin{aligned}b^2(3c^2+ca+a^2)+2c^2a(a-2c)&>\frac{4ac}{3}(3c^2+ca+a^2)+2c^2a(a-2c)\\&=\frac{2a^2c(2a+5c)}{3}>0.\end{aligned}$$

This completes the proof. The equality occurs for $a=b=c$.

□

P 1.84. *If a,b,c are positive real numbers, then*

$$\frac{a^4}{a^3+b^3}+\frac{b^4}{b^3+c^3}+\frac{c^4}{c^3+a^3}\ge\frac{a+b+c}{2}.$$

(Vasile C., 2005)

Solution *(by Vo Quoc Ba Can).* Multiplying by $a^3+b^3+c^3$, the inequality becomes

$$\sum a^4+\sum\frac{a^4c^3}{a^3+b^3}\ge\frac{1}{2}\left(\sum a\right)\left(\sum a^3\right).$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^4c^3}{a^3+b^3}\ge\frac{\left(\sum a^2c^2\right)^2}{\sum c(a^3+b^3)}=\frac{\left(\sum a^2b^2\right)^2}{\sum a(b^3+c^3)}.$$

According to the inequality

$$\frac{x^2}{y}\ge x-\frac{y}{4},\quad x,y>0,$$

we have

$$\frac{(\sum a^2b^2)^2}{\sum a(b^3+c^3)}\ge\sum a^2b^2-\frac{1}{4}\sum a(b^3+c^3).$$

Therefore, it suffices to show that

$$\sum a^4+\sum a^2b^2-\frac{1}{4}\sum a(b^3+c^3)\ge\frac{1}{2}\left(\sum a\right)\left(\sum a^3\right),$$

which is equivalent to

$$2\sum a^4+4\sum a^2b^2\ge 3\sum ab(a^2+b^2),$$

$$\sum[a^4+b^4+4a^2b^2-3ab(a^2+b^2)]\ge 0,$$

$$\sum(a-b)^2(a^2-ab+b^2)\ge 0.$$

This completes the proof. The equality occurs for $a=b=c$.

□

P 1.85. *If a,b,c are positive real numbers such that $abc=1$, then*

(a) $$3\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)+4\left(\frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\right)\ge 7(a^2+b^2+c^2);$$

(b) $$8\left(\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}\right)+5\left(\frac{b}{a^3}+\frac{c}{b^3}+\frac{a}{c^3}\right)\ge 13(a^3+b^3+c^3).$$

(Vasile C., 1992)

Solution. (a) We use the AM-GM inequality, as follows:

$$3\sum\frac{a^2}{b}+4\sum\frac{b}{a^2}=\sum\left(3\frac{a^2}{b}+\frac{c}{b^2}+3\frac{a}{c^2}\right)\ge 7\sum\sqrt[7]{\left(\frac{a^2}{b}\right)^3\cdot\frac{c}{b^2}\cdot\left(\frac{a}{c^2}\right)^3}$$

$$=7\sum\sqrt[7]{\frac{a^9}{b^5c^5}}=7\sum a^2.$$

The equality holds for $a=b=c=1$.

(b) By the AM-GM inequality, we have

$$8\sum\frac{a^3}{b}+5\sum\frac{b}{a^3}=\sum\left(8\frac{a^3}{b}+\frac{c}{b^3}+4\frac{a}{c^3}\right)\ge 13\sum\sqrt[13]{\left(\frac{a^3}{b}\right)^8\cdot\frac{c}{b^3}\cdot\left(\frac{a}{c^3}\right)^4}$$

$$=13\sum\sqrt[13]{\frac{a^{28}}{b^{11}c^{11}}}=13\sum a^3.$$

The equality holds for $a=b=c=1$.

□

P 1.86. *If a,b,c are positive real numbers, then*

$$\frac{ab}{b^2+bc+c^2}+\frac{bc}{c^2+ca+a^2}+\frac{ca}{a^2+ab+b^2}\le\frac{a^2+b^2+c^2}{ab+bc+ca}.$$

(Tran Quoc Anh, 2007)

Solution. Write the inequality as follows:

$$\sum\left(\frac{a^2}{ab+bc+ca}-\frac{ab}{b^2+bc+c^2}\right)\ge 0,$$

$$\sum\frac{ac(ac-b^2)}{b^2+bc+c^2}\ge 0,$$

$$\sum\left[\frac{ac(ac-b^2)}{b^2+bc+c^2}+ac\right]\ge\sum ac,$$

$$\sum\frac{ac^2(a+b+c)}{b^2+bc+c^2}\ge\sum ac,$$

$$\sum\frac{ac^2}{b^2+bc+c^2}\ge\frac{ab+bc+ca}{a+b+c}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{ac^2}{b^2+bc+c^2}\ge\frac{\left(\sum ac\right)^2}{\sum a(b^2+bc+c^2)}=\frac{ab+bc+ca}{a+b+ca}.$$

The equality holds for $a=b=c$.

□

P 1.87. *If a, b, c are positive real numbers, then*

$$\frac{a-b}{b(2b+c)}+\frac{b-c}{c(2c+a)}+\frac{c-a}{a(2a+b)}\ge 0.$$

Solution. Write the inequality as follows:

$$\sum \frac{ac(a-b)}{2b+c}\ge 0,$$

$$\sum \left[\frac{ac(a-b)}{2b+c}+ac\right]\ge ab+bc+ca,$$

$$\sum \frac{ac}{2b+c}\ge \frac{ab+bc+ca}{a+b+c}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{ac}{2b+c}\ge \frac{\left(\sum ac\right)^2}{\sum ac(2b+c)}=\frac{\left(\sum ab\right)^2}{6abc+\sum a^2b}.$$

Thus, it suffices to prove that

$$\frac{\sum ab}{6abc+\sum a^2b}\ge \frac{1}{\sum a},$$

which is equivalent to

$$\sum ab^2\ge 3abc.$$

Clearly, the last inequality follows immediately from the AM-GM inequality. The equality holds for $a=b=c$.

□

P 1.88. *If a, b, c are positive real numbers, then*

(a) $$\frac{a^2+6bc}{ab+2bc}+\frac{b^2+6ca}{bc+2ca}+\frac{c^2+6ab}{ca+2ab}\ge 7;$$

(b) $$\frac{a^2+7bc}{ab+bc}+\frac{b^2+7ca}{bc+ca}+\frac{c^2+7ab}{ca+ab}\ge 12.$$

(Vasile C., 2012)

Solution. (a) Write the inequality as follows:

$$\sum ac(a^2+6bc)(b+2a)(c+2b) \ge 7abc(a+2c)(b+2a)(c+2b),$$

$$2\sum a^2b^4 + abc\left(72abc + 4\sum a^3 + 26\sum a^2b + 7\sum ab^2\right) \ge$$
$$\ge 7abc\left(9abc + 4\sum a^2b + 2\sum ab^2\right),$$
$$2\left(\sum a^2b^4 - abc\sum a^2b\right) + abc\left(4\sum a^3 + 9abc - 7\sum ab^2\right) \ge 0.$$

Since
$$2\left(\sum a^2b^4 - abc\sum a^2b\right) = \sum (ab^2 - bc^2)^2 \ge 0,$$
it suffices to show that

$$4\sum a^3 + 9abc - 7\sum ab^2 \ge 0.$$

Assume that $a = \min\{a, b, c\}$. Using the substitution

$$b = a + x, \quad c = a + y, \quad x, y \ge 0,$$

we have

$$4\sum a^3 + 9abc - 7\sum ab^2 = 5(x^2 - xy + y^2)a + 4x^3 + 4y^3 - 7xy^2 \ge 0,$$

since

$$4x^3 + 4y^3 = 4x^3 + 2y^3 + 2y^3 \ge 3\sqrt[3]{4x^3 \cdot 2y^3 \cdot 2y^3} = 6\sqrt[3]{2}\, xy^2 \ge 7xy^2.$$

The equality holds for $a = b = c$.

(b) Write the inequality as follows:

$$\sum ac(a^2+7bc)(b+a)(c+b) \ge 12abc(a+c)(b+a)(c+b),$$

$$\sum a^2b^4 + abc\left(21abc + \sum a^3 + 15\sum a^2b + 8\sum ab^2\right) \ge$$
$$\ge 12abc\left(2abc + \sum a^2b + \sum ab^2\right),$$
$$\left(\sum a^2b^4 - abc\sum a^2b\right) + abc\left(\sum a^3 - 3abc + 4\sum a^2b - 4\sum ab^2\right) \ge 0.$$

Since
$$\sum a^2b^4 - abc\sum a^2b = \frac{1}{2}\sum (ab^2 - bc^2)^2 \ge 0,$$
it suffices to show that

$$\sum a^3 - 3abc + 4\sum a^2b - 4\sum ab^2 \ge 0,$$

which is equivalent to

$$\frac{1}{2}(a+b+c)\sum(a-b)^2-4(a-b)(b-c)(c-a)\ge 0.$$

Assume that $a=\min\{a,b,c\}$. Making the substitution

$$b=a+x,\quad c=a+y,\quad x,y\ge 0,$$

we have

$$\frac{1}{2}(a+b+c)\sum(a-b)^2-4(a-b)(b-c)(c-a)=$$

$$=(x^2-xy+y^2)(3a+x+y)+4xy(x-y)$$

$$=3(x^2-xy+y^2)a+x^3+y^3+4xy(x-y)$$

$$=3(x^2-xy+y^2)a+x^3+y(2x-y)^2\ge 0.$$

The equality holds for $a=b=c$.

□

P 1.89. *If a,b,c are positive real numbers, then*

(a)
$$\frac{ab}{2b+c}+\frac{bc}{2c+a}+\frac{ca}{2a+b}\le\frac{a^2+b^2+c^2}{a+b+c};$$

(b)
$$\frac{ab}{b+c}+\frac{bc}{c+a}+\frac{ca}{a+b}\le\frac{3(a^2+b^2+c^2)}{2(a+b+c)};$$

(c)
$$\frac{ab}{4b+5c}+\frac{bc}{4c+5a}+\frac{ca}{4a+5b}\le\frac{a^2+b^2+c^2}{3(a+b+c)}.$$

(Vasile C., 2012)

Solution. (a) ***First Solution.*** Since

$$\frac{2ab}{2b+c}=a-\frac{ac}{2b+c},$$

we can write the inequality as

$$\sum\frac{ac}{2b+c}+\frac{2(a^2+b^2+c^2)}{a+b+c}\ge a+b+c.$$

By the Cauchy-Schwarz inequality,

$$\sum\frac{ac}{2b+c}\ge\frac{\left(\sum\sqrt{ac}\right)^2}{\sum(2b+c)}=\frac{(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2}{3(a+b+c)}.$$

Then, it suffices to show that

$$\frac{(\sqrt{ab}+\sqrt{bc}+\sqrt{ca})^2+6(a^2+b^2+c^2)}{3(a+b+c)} \ge a+b+c,$$

which is equivalent to

$$3(a^2+b^2+c^2)+2\sqrt{abc}\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right) \ge 5(ab+bc+ca).$$

Using the substitution

$$x=\sqrt{a}, \quad y=\sqrt{b}, \quad z=\sqrt{c},$$

the inequality can be restated as

$$3(x^4+y^4+z^4)+2xyz(x+y+z) \ge 5(x^2y^2+y^2z^2+z^2x^2).$$

We can get it by summing Schur's inequality of degree four

$$2(x^4+y^4+z^4)+2xyz(x+y+z) \ge 2\sum xy(x^2+y^2)$$

and

$$x^4+y^4+z^4+2\sum xy(x^2+y^2) \ge 5(x^2y^2+y^2z^2+z^2x^2),$$

the last being equivalent to the obvious inequality

$$(x^4+y^4+z^4-x^2y^2-y^2z^2-z^2x^2)+2\sum xy(x-y)^2 \ge 0.$$

The equality holds for $a=b=c$.

Second Solution. By the Cauchy-Schwarz inequality, we have

$$\frac{1}{2b+c}=\frac{1}{b+b+c} \le \frac{a^2/b+b+c}{(a+b+c)^2}=\frac{a^2+b^2+bc}{b(a+b+c)^2},$$

$$\frac{ab}{2b+c} \le \frac{a(a^2+b^2+bc)}{(a+b+c)^2},$$

$$\sum \frac{ab}{2b+c} \le \frac{\sum a^3+\sum ab^2+3abc}{(a+b+c)^2}.$$

Since $3abc \le \sum a^2b$ (by the AM-GM inequality), we get

$$\sum \frac{ab}{2b+c} \le \frac{\sum a^3+\sum ab^2+\sum a^2b}{(a+b+c)^2}=\frac{a^2+b^2+c^2}{a+b+c}.$$

Third Solution. Write the inequality as

$$\sum \frac{ab(a+b+c)}{2b+c} \le a^2+b^2+c^2.$$

Since

$$2ab(a+b+c) = (a^2+2ab)(2b+c) - 2ab^2 - a^2c,$$

we can write the inequality as

$$\sum \frac{2ab^2}{2b+c} + \sum \frac{a^2c}{2b+c} + p \ge 2q,$$

where

$$p = a^2+b^2+c^2, \ \ q = ab+bc+ca, \quad p \ge q.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{ab^2}{2b+c} \ge \frac{\left(\sum ab\right)^2}{\sum a(2b+c)} = \frac{q}{3}$$

and

$$\sum \frac{a^2c}{2b+c} \ge \frac{\left(\sum ac\right)^2}{\sum c(2b+c)} = \frac{q^2}{p+2q}.$$

Thus, it suffices to show that

$$\frac{2q}{3} + \frac{q^2}{p+2q} + p \ge 2q,$$

which is equivalent to the obvious inequality

$$(p-q)(3p+5q) \ge 0.$$

(b) Write the inequality as

$$\frac{3}{2}(a^2+b^2+c^2) \ge \sum \frac{ab(a+b+c)}{b+c}.$$

Since

$$\frac{ab(a+b+c)}{b+c} = \frac{a^2b}{b+c} + ab = a^2 + ab - \frac{a^2c}{b+c},$$

the inequality can be written as

$$\sum \frac{a^2c}{b+c} + \frac{1}{2}(a^2+b^2+c^2) \ge ab+bc+ca.$$

By the Cauchy-Schwarz inequality,

$$\sum \frac{a^2c}{b+c} \ge \frac{\left(\sum ac\right)^2}{\sum c(b+c)} = \frac{q^2}{p+q},$$

where

$$p = a^2+b^2+c^2, \ \ q = ab+bc+ca, \quad p \ge q.$$

Therefore, we have

$$\sum \frac{a^2c}{b+c} + \frac{1}{2}(a^2+b^2+c^2) - (ab+bc+ca) \geq \frac{q^2}{p+q} + \frac{p}{2} - q = \frac{p(p-q)}{2(p+q)} \geq 0.$$

The equality holds for $a = b = c$.

(c) Since

$$\frac{4ab}{4b+5c} = a - \frac{5ac}{4b+5c},$$

we can write the inequality as

$$5\sum \frac{ac}{4b+5c} + \frac{4(a^2+b^2+c^2)}{3(a+b+c)} \geq a+b+c.$$

By the Cauchy-Schwarz inequality,

$$\sum \frac{ac}{4b+5c} \geq \frac{\left(\sum ac\right)^2}{\sum ac(4b+5c)} = \frac{(ab+bc+ca)^2}{12abc+5(a^2b+b^2c+c^2a)}.$$

Therefore, it suffices to show that

$$\frac{5(ab+bc+ca)^2}{12abc+5(a^2b+b^2c+c^2a)} + \frac{4(a^2+b^2+c^2)}{3(a+b+c)} \geq a+b+c.$$

Due to homogeneity, we may assume that $a+b+c=3$. Using the notation

$$q = ab+bc+ca, \quad q \leq 3,$$

this inequality becomes

$$\frac{5q^2}{5(a^2b+b^2c+c^2a+abc)+7abc} + \frac{4(9-2q)}{9} \geq 3.$$

According to the inequality (a) in P 1.9, we have

$$a^2b+b^2c+c^2a+abc \leq 4.$$

On the other hand, from

$$(ab+bc+ca)^2 \geq 3abc(a+b+c),$$

we get

$$abc \leq \frac{q^2}{9}.$$

Thus, it suffices to prove that

$$\frac{5q^2}{20+7q^2/9} + \frac{4(9-2q)}{9} \geq 3,$$

which is equivalent to

$$(q-3)(14q^2-75q+135)\le 0.$$

This is true since $q-3\le 0$ and

$$14q^2-75q+135>3(4q^2-25q+39)=3(3-q)(13-4q)\ge 0.$$

The equality holds for $a=b=c$.

□

P 1.90. *If a,b,c are positive real numbers, then*

(a) $$a\sqrt{b^2+8c^2}+b\sqrt{c^2+8a^2}+c\sqrt{a^2+8b^2}\le (a+b+c)^2;$$

(b) $$a\sqrt{b^2+3c^2}+b\sqrt{c^2+3a^2}+c\sqrt{a^2+3b^2}\le a^2+b^2+c^2+ab+bc+ca.$$

(Vo Quoc Ba Can, 2007)

Solution. (a) By the AM-GM inequality, we have

$$\sqrt{b^2+8c^2}=\frac{\sqrt{(b^2+8c^2)(b+2c)^2}}{b+2c}\le\frac{(b^2+8c^2)+(b+2c)^2}{2(b+2c)}$$

$$=\frac{b^2+2bc+6c^2}{b+2c}=b+3c-\frac{3bc}{b+2c},$$

hence

$$a\sqrt{b^2+8c^2}\le ab+3ac-\frac{3abc}{b+2c},$$

$$\sum a\sqrt{b^2+8c^2}\le 4\sum ab-3abc\sum\frac{1}{b+2c}.$$

Therefore, it suffices to show that

$$\left(\sum a\right)^2+3abc\sum\frac{1}{b+2c}\ge 4\sum ab.$$

Since

$$\sum\frac{1}{b+2c}\ge\frac{9}{\sum(b+2c)}=\frac{3}{\sum a},$$

it is enough to prove that

$$\left(\sum a\right)^3+9abc\ge 4\left(\sum a\right)\left(\sum ab\right).$$

This is Shur's inequality of degree three. The equality holds for $a=b=c$.

(b) Similarly, we have

$$\sqrt{b^2+3c^2} = \frac{\sqrt{(b^2+3c^2)(b+c)^2}}{b+c} \le \frac{(b^2+3c^2)+(b+c)^2}{2(b+c)}$$

$$= \frac{b^2+bc+2c^2}{b+c} = b+2c-\frac{2bc}{b+c},$$

hence

$$a\sqrt{b^2+3c^2} \le ab+2ac-\frac{2abc}{b+c},$$

$$\sum a\sqrt{b^2+3c^2} \le 3\sum ab - 2abc\sum\frac{1}{b+c}.$$

Thus, it suffices to show that

$$\left(\sum a\right)^2 + 2abc\sum\frac{1}{b+c} \ge 4\sum ab.$$

Since

$$\sum\frac{1}{b+c} \ge \frac{9}{\sum(b+c)} = \frac{9}{2\sum a},$$

it is enough to prove that

$$\left(\sum a\right)^3 + 9abc \ge 4\left(\sum a\right)\left(\sum ab\right),$$

which is just Shur's inequality of degree three. The equality holds for $a=b=c$.

□

P 1.91. *If a,b,c are positive real numbers, then*

(a)
$$\frac{1}{a\sqrt{a+2b}}+\frac{1}{b\sqrt{b+2c}}+\frac{1}{c\sqrt{c+2a}} \ge \sqrt{\frac{3}{abc}};$$

(b)
$$\frac{1}{a\sqrt{a+8b}}+\frac{1}{b\sqrt{b+8c}}+\frac{1}{c\sqrt{c+8a}} \ge \sqrt{\frac{1}{abc}}.$$

(Vasile C., 2007)

Solution. (a) Write the inequality as

$$\sum\sqrt{\frac{bc}{3a(a+2b)}} \ge 1.$$

Replacing a,b,c by $\frac{1}{x},\frac{1}{y},\frac{1}{z}$, respectively, the inequality can be restated as

$$\sum\frac{x}{\sqrt{3z(2x+y)}} \ge 1.$$

Since
$$\sqrt{3z(2x+y)} \le \frac{3z+(2x+y)}{2},$$
it suffices to show that
$$\sum \frac{x}{2x+y+3z} \ge \frac{1}{2}.$$
Indeed, using the Cauchy-Schwarz inequality gives
$$\sum \frac{x}{2x+y+3z} \ge \sum \frac{(\sum x)^2}{\sum x(2x+y+3z)} = \frac{1}{2}.$$
The equality holds for $a=b=c$.

(b) Write the inequality as
$$\sum \sqrt{\frac{bc}{a(a+8b)}} \ge 1.$$
Replacing a,b,c by $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$, respectively, the inequality becomes
$$\sum \frac{x^2}{z\sqrt{8x^2+y^2}} \ge 1.$$
Applying the Cauchy-Schwarz inequality yields
$$\sum \frac{x^2}{z\sqrt{8x^2+y^2}} \ge \frac{\left(\sum x\right)^2}{\sum z\sqrt{8x^2+y^2}}.$$
Therefore, it suffices to show that
$$\sum z\sqrt{8x^2+y^2} \le (x+y+z)^2,$$
which is just the inequality in P 1.90-(a). The equality holds for $a=b=c$.

□

P 1.92. *If a,b,c are positive real numbers, then*
$$\frac{a}{\sqrt{5a+4b}} + \frac{b}{\sqrt{5b+4c}} + \frac{c}{\sqrt{5c+4a}} \le \sqrt{\frac{a+b+c}{3}}.$$
(Vasile C., 2012)

Solution. By the Cauchy-Schwarz inequality, we have

$$\left(\sum \frac{a}{\sqrt{5a+4b}}\right)^2 \le \left(\sum \frac{a}{4a+4b+c}\right)\left(\sum \frac{a(4a+4b+c)}{5a+4b}\right).$$

It suffices to show that

$$\sum \frac{a}{4a+4b+c} \le \frac{1}{3}$$

and

$$\sum \frac{a(4a+4b+c)}{5a+4b} \le a+b+c.$$

The first is just the inequality in P 1.18, while the second is equivalent to

$$\sum a\left(1-\frac{4a+4b+c}{5a+4b}\right) \ge 0,$$

$$\sum \frac{a(a-c)}{5a+4b} \ge 0,$$

$$\sum a(a-c)(5b+4c)(5c+4a) \ge 0,$$

$$\sum a^2b^2 + 4\sum ab^3 \ge 5abc\sum a.$$

The last inequality follows from the well-known inequality

$$\sum a^2b^2 \ge abc\sum a$$

and the known inequality

$$\sum ab^3 \ge abc\sum a,$$

which follows from the Cauchy-Schwarz inequality, as follows:

$$\left(\sum c\right)\left(\sum ab^3\right) \ge \left(\sum \sqrt{ab^3c}\right)^2 = abc\left(\sum b\right)^2.$$

The equality holds for $a=b=c$.

□

P 1.93. *If a,b,c are positive real numbers, then*

(a)
$$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}} \ge \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{2}};$$

(b)
$$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}} \ge \sqrt[4]{\frac{27(ab+bc+ca)}{4}}.$$

(Lev Buchovsky - 1995, Pham Huu Duc - 2007)

Solution. (a) By squaring, the inequality becomes

$$\sum \frac{a^2}{a+b} + 2\sum \frac{ab}{\sqrt{(a+b)(b+c)}} \ge \frac{1}{2}\sum a + \sum \sqrt{ab}.$$

The sequences

$$\left\{\frac{1}{\sqrt{a+b}}, \quad \frac{1}{\sqrt{b+c}}, \quad \frac{1}{\sqrt{c+a}}\right\}$$

and

$$\left\{\frac{ab}{\sqrt{a+b}}, \quad \frac{bc}{\sqrt{b+c}}, \quad \frac{ca}{\sqrt{c+a}}\right\}$$

are always reversely ordered; therefore, according to the rearrangement inequality, we have

$$\frac{1}{\sqrt{a+b}} \cdot \frac{ab}{\sqrt{a+b}} + \frac{1}{\sqrt{b+c}} \cdot \frac{bc}{\sqrt{b+c}} + \frac{1}{\sqrt{c+a}} \cdot \frac{ca}{\sqrt{c+a}} \le$$

$$\le \frac{1}{\sqrt{a+b}} \cdot \frac{ca}{\sqrt{c+a}} + \frac{1}{\sqrt{b+c}} \cdot \frac{ab}{\sqrt{a+b}} + \frac{1}{\sqrt{c+a}} \cdot \frac{bc}{\sqrt{b+c}},$$

$$\sum \frac{ab}{a+b} \le \sum \frac{ab}{\sqrt{(a+b)(b+c)}}.$$

Thus, it suffices to show that

$$\sum \frac{a^2}{a+b} + 2\sum \frac{ab}{a+b} \ge \frac{1}{2}\sum a + \sum \sqrt{ab}.$$

Since

$$\sum \frac{a^2}{a+b} + \sum \frac{ab}{a+b} = \sum a,$$

the inequality becomes as follows:

$$\sum a + \sum \frac{ab}{a+b} \ge \frac{1}{2}\sum a + \sum \sqrt{ab},$$

$$\sum \frac{a+b}{2} + \sum \frac{2ab}{a+b} \ge 2\sum \sqrt{ab},$$

$$\sum \left(\sqrt{\frac{a+b}{2}} - \sqrt{\frac{2ab}{a+b}}\right)^2 \ge 0.$$

The equality holds for $a = b = c$.

(b) By Hölder's inequality, we have

$$\left(\sum \frac{a}{\sqrt{a+b}}\right)^2 \sum a(a+b) \ge \left(\sum a\right)^3.$$

Thus, it suffices to show that

$$\left(\sum a\right)^3 \geq \frac{3}{2}\left(\sum a^2 + \sum ab\right)\sqrt{3(ab+bc+ca)},$$

which is equivalent to

$$2p^3 + q^3 \geq 3p^2q,$$

where $p = a+b+c$ and $q = \sqrt{3(ab+bc+ca)}$. By the AM-GM inequality, we have

$$2p^3 + q^3 \geq 3\sqrt[3]{p^6q^3} = 3p^2q.$$

The equality holds for $a = b = c$.

□

P 1.94. *If a, b, c are nonnegative real numbers such that $a+b+c = 3$, then*

$$\sqrt{3a+b^2} + \sqrt{3b+c^2} + \sqrt{3c+a^2} \geq 6.$$

First Solution. Assume that $a = \max\{a, b, c\}$. We can get the desired inequality by summing the inequalities

$$\sqrt{3b+c^2} + \sqrt{3c+a^2} \geq \sqrt{3a+c^2} + b + c$$

and

$$\sqrt{3a+b^2} + \sqrt{3a+c^2} \geq 2a + b + c.$$

By squaring two times, the first inequality becomes in succession

$$\sqrt{(3b+c^2)(3c+a^2)} \geq (b+c)\sqrt{3a+c^2},$$

$$[b(a+b+c)+c^2][c(a+b+c)+a^2] \geq (b+c)^2[a(a+b+c)+c^2],$$

$$b(a-b)(a-c)(a+b+c) \geq 0.$$

Similarly, the second inequality becomes

$$\sqrt{(3a+b^2)(3a+c^2)} \geq (a+b)(a+c),$$

$$[a(a+b+c)+b^2][a(a+b+c)+c^2] \geq (a+b)^2(a+c)^2,$$

$$a(a+b+c)(b-c)^2 \geq 0.$$

The original inequality becomes an equality when $a = b = c$, and also when two of a, b, c are zero.

Second Solution. Write the inequality as

$$\sqrt{X} + \sqrt{Y} + \sqrt{Z} \leq \sqrt{A} + \sqrt{B} + \sqrt{C},$$

where

$$X=(b+c)^2,\quad Y=(c+a)^2,\quad Z=(a+b)^2,$$

$$A=3a+b^2,\quad B=3b+c^2,\quad C=3c+a^2.$$

According to Lemma from the proof of P 2.11 in Volume 2, since

$$X+Y+Z=A+B+C,$$

it suffices to show that

$$\max\{X,Y,Z\}\ge\max\{A,B,C\},\qquad \min\{X,Y,Z\}\le\min\{A,B,C\}.$$

To show that $\max\{X,Y,Z\}\ge\max\{A,B,C\}$, we assume that

$$a=\min\{a,b,c\},\qquad \max\{X,Y,Z\}=X.$$

From

$$X-A=(c^2-a^2)+b(c-a)+c(b-a)\ge 0,$$

$$X-B=b(c-a)\ge 0,$$

$$X-C=(b^2-a^2)+c(b-a)\ge 0,$$

the conclusion follows. Similarly, to show that $\min\{X,Y,Z\}\le\min\{A,B,C\}$, we assume that

$$a=\max\{a,b,c\},\qquad \min\{X,Y,Z\}=X,$$

when

$$A-X=(a^2-c^2)+b(a-c)+c(a-b)\ge 0,$$

$$B-X=b(a-c)\ge 0,$$

$$C-X=(a^2-b^2)+c(a-b)\ge 0.$$

□

P 1.95. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{a^2+b^2+2bc}+\sqrt{b^2+c^2+2ca}+\sqrt{c^2+a^2+2ab}\ge 2(a+b+c).$$

(Vasile C., 2012)

First Solution *(by Nguyen Van Quy).* Assume that $a=\max\{a,b,c\}$. We can get the desired inequality by summing the inequalities

$$\sqrt{a^2+b^2+2bc}+\sqrt{b^2+c^2+2ca}\ge\sqrt{a^2+b^2+2ca}+b+c$$

and

$$\sqrt{c^2+a^2+2ab}+\sqrt{a^2+b^2+2ca}\ge 2a+b+c.$$

By squaring two times, the first inequality becomes

$$\sqrt{(a^2+b^2+2bc)(b^2+c^2+2ca)} \geq (b+c)\sqrt{a^2+b^2+2ca},$$

$$c(a-b)(a^2-c^2) \geq 0.$$

Similarly, the second inequality becomes

$$\sqrt{(c^2+a^2+2ab)(a^2+b^2+2ca)} \geq (a+b)(a+c),$$

$$a(b+c)(b-c)^2 \geq 0.$$

The original inequality becomes an equality when $a=b=c$, and also when two of a,b,c are zero.

Second Solution. Let $\{x,y,z\}$ be a permutation of $\{ab,bc,ca\}$. We will prove that

$$2(a+b+c) \leq \sqrt{b^2+c^2+2x} + \sqrt{c^2+a^2+2y} + \sqrt{a^2+b^2+2z}.$$

Due to symmetry, assume that $a \geq b \geq c$. Using the substitution

$$X = a^2+b^2+2ab, \quad Y = c^2+a^2+2ca, \quad Z = b^2+c^2+2bc,$$

$$A = b^2+c^2+2x, \quad B = c^2+a^2+2y, \quad C = a^2+b^2+2z,$$

we can write the inequality as

$$\sqrt{X}+\sqrt{Y}+\sqrt{Z} \leq \sqrt{A}+\sqrt{B}+\sqrt{C}.$$

Since $X+Y+Z = A+B+C$, $X \geq Y \geq Z$ and

$$X \geq \max\{A,B,C\}, \quad Z \leq \min\{A,B,C\},$$

the conclusion follow by Lemma from the proof of P 2.11 in Volume 2.

□

P 1.96. *If a,b,c are nonnegative real numbers, then*

$$\sqrt{a^2+b^2+7bc} + \sqrt{b^2+c^2+7ca} + \sqrt{c^2+a^2+7ab} \geq 3\sqrt{3(ab+bc+ca)}.$$

(Vasile C., 2012)

Solution. Assume that $a = \max\{a,b,c\}$. We can get the desired inequality by summing the inequalities

$$\sqrt{a^2+b^2+7bc} + \sqrt{b^2+c^2+7ca} \geq \sqrt{a^2+b^2+7ca} + \sqrt{b^2+c^2+7bc}$$

and

$$\sqrt{a^2+c^2+7ab}+\sqrt{a^2+b^2+7ac}\ge 3\sqrt{3(ab+bc+ca)}-\sqrt{b^2+c^2+7bc}.$$

By squaring, the first inequality becomes

$$(a^2+b^2+7b)(b^2+c^2+7ca)\ge(a^2+b^2+7ca)(b^2+c^2+7bc),$$

$$c(a-b)(a^2-c^2)\ge 0.$$

Similarly, the second inequality becomes

$$a^2+\sqrt{E}+3\sqrt{3F}\ge 10a(b+c)+17bc,$$

where

$$\begin{aligned}E&=(a^2+c^2+7ab)(a^2+b^2+7ac)\\&=a^4+7(b+c)a^3+(b^2+c^2+49bc)a^2+7(b^3+c^3)a+b^2c^2\end{aligned}$$

and

$$F=(ab+bc+ca)(b^2+c^2+7bc).$$

Due to homogeneity, we may assume that $b+c=1$. Let us denote $x=bc$. We need to show that $f(x)\ge 0$ for $0\le x\le\frac{1}{4}$ and $a\ge\frac{1}{2}$, where

$$f(x)=a^2-10a-17x+\sqrt{g(x)}+3\sqrt{3h(x)},$$

with

$$\begin{aligned}g(x)&=a^4+7a^3+(1+47x)a^2+7(1-3x)a+x^2\\&=x^2+a(47a-21)x+a^4+7a^3+a^2+7a,\end{aligned}$$

$$h(x)=(a+x)(1+5x)=5x^2+(5a+1)x+a.$$

We have the derivatives

$$\begin{aligned}f'(x)&=-17+\frac{g'}{2\sqrt{g}}+\frac{3\sqrt{3}h'}{2\sqrt{h}}\\&=-17+\frac{2x+a(47a-21)}{2\sqrt{g}}+\frac{3\sqrt{3}(10x+5a+1)}{2\sqrt{h}},\end{aligned}$$

$$\begin{aligned}f''(x)&=\frac{2g''g-(g')^2}{4g\sqrt{g}}+\frac{3\sqrt{3}[2h''h-(h')^2]}{4h\sqrt{h}}\\&=\frac{a(28-45a)(7a-1)^2}{4g\sqrt{g}}-\frac{3\sqrt{3}(5a-1)^2}{4h\sqrt{h}}.\end{aligned}$$

We will show that $g \geq 3h$. Since $0 \leq x \leq \dfrac{1}{4}$ and $a \geq \dfrac{1}{2}$, we have

$$\begin{aligned} g - 3h &= -14x^2 + (47a^2 - 36a - 3)x + a^4 + 7a^3 + a^2 + 4a \\ &\geq -\frac{7}{8} + (47a^2 - 36a - 3)x + a^4 + 7a^3 + a^2 + 4a. \end{aligned}$$

For the non-trivial case $47a^2 - 36a - 3 < 0$, we get

$$\begin{aligned} g - 3h &\geq -\frac{7}{8} + \frac{47a^2 - 36a - 3}{4} + a^4 + 7a^3 + a^2 + 4a \\ &= \frac{(2a-1)(4a^3 + 30a^2 + 66a + 13)}{8} \geq 0. \end{aligned}$$

We will prove now that $f''(x) < 0$. This is clearly true for $a \geq \dfrac{28}{45}$. Otherwise, for $\dfrac{1}{2} \leq a \leq \dfrac{28}{45}$, we have

$$f''(x) \leq \frac{a(28 - 45a)(7a-1)^2 - 27(5a-1)^2}{4g\sqrt{g}} < 0,$$

since

$$a(28 - 45a)(7a-1)^2 - 27(5a-1)^2 < \left(28 - \frac{45}{2}\right)(7a-1)^2 - 27(5a-1)^2$$

$$< \frac{27}{4}(7a-1)^2 - 27(5a-1)^2 = \frac{27(1-3a)(17a-3)}{4} < 0.$$

Since f is concave, it suffices to show that $f(0) \geq 0$ and $f\left(\dfrac{1}{4}\right) \geq 0$.

From

$$f(0) = \sqrt{a}\left(a\sqrt{a} - 10\sqrt{a} + 3\sqrt{3} + \sqrt{a^3 + 7a^2 + a + 7}\right),$$

it follows that $f(0) \geq 0$ for all $a \geq \dfrac{1}{2}$ if and only if

$$\sqrt{a^3 + 7a^2 + a + 7} \geq -a\sqrt{a} + 10\sqrt{a} - 3\sqrt{3}.$$

This is true if

$$a^3 + 7a^2 + a + 7 \geq (-a\sqrt{a} + 10\sqrt{a} - 3\sqrt{3})^2,$$

which is equivalent to

$$(\sqrt{3a} - 2)^2(9a + 10\sqrt{a} - 5) \geq 0.$$

Clearly, this inequality holds for $a \geq \dfrac{1}{2}$.

Since
$$g\left(\frac{1}{4}\right)=\left(\frac{4a^2+14a+1}{4}\right)^2$$
and
$$h\left(\frac{1}{4}\right)=\frac{9(4a+1)}{16},$$
we get
$$f\left(\frac{1}{4}\right)=\frac{8a^2-26a-16+9\sqrt{3(4a+1)}}{4}.$$
Using the substitution
$$x=\sqrt{\frac{4a+1}{3}}, \quad x\ge 1,$$
we find
$$f\left(\frac{1}{4}\right)=\frac{9x^4-45x^2+54x-18}{8}=\frac{(x-1)^2(9x^2+18x-18)}{8}\ge 0.$$
Thus, the proof is completed. The equality holds for $a=b=c$, and also for $3a=4b$ and $c=0$ (or any cyclic permutation).

□

P 1.97. *If a,b,c are positive real numbers, then*
$$\frac{a^2+3ab}{(b+c)^2}+\frac{b^2+3bc}{(c+a)^2}+\frac{c^2+3ca}{(a+b)^2}\ge 3.$$

Solution. Write the inequality as
$$\sum\frac{a(a+b)}{(b+c)^2}+2\sum\frac{ab}{(b+c)^2}\ge 3.$$
The sequences
$$\{bc,\ \ ca,\ \ ab\}$$
and
$$\left\{\frac{1}{(b+c)^2},\ \ \frac{1}{(c+a)^2},\ \ \frac{1}{(a+b)^2}\right\}$$
are reversely ordered. Thus, by the rearrangement inequality, we have
$$\sum\frac{bc}{(b+c)^2}\le\sum\frac{ab}{(b+c)^2}.$$
Therefore, it suffices to show that
$$\sum\frac{a(a+b)}{(b+c)^2}+\sum\frac{b(c+a)}{(b+c)^2}\ge 3,$$

which is equivalent to

$$\sum a\left[\frac{a+b}{(b+c)^2}+\sum\frac{b+c}{(a+b)^2}\right]\geq 3.$$

By the AM-GM inequality, we have

$$\frac{a+b}{(b+c)^2}+\frac{b+c}{(a+b)^2}\geq\frac{2}{\sqrt{(a+b)(b+c)}}\geq\frac{4}{(a+b)+(b+c)}.$$

Thus, it is enough to prove that

$$\sum\frac{a}{a+2b+c}\geq\frac{3}{4}.$$

Indeed, by the Cauchy-Schwarz inequality, we get

$$\sum\frac{a}{a+2b+c}\geq\frac{\left(\sum a\right)^2}{\sum a(a+2b+c)}=\frac{\sum a^2+2\sum ab}{\sum a^2+3\sum ab}\geq\frac{3}{4}.$$

The equality holds for $a=b=c$.

□

P 1.98. *If a,b,c are positive real numbers, then*

$$\frac{a^2b+1}{a(b+1)}+\frac{b^2c+1}{b(c+1)}+\frac{c^2a+1}{c(a+1)}\geq 3.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$(a^2b+1)\left(\frac{1}{b}+1\right)\geq(a+1)^2,$$

hence

$$\frac{a^2b+1}{a(b+1)}\geq\frac{b(a+1)^2}{a(b+1)^2}.$$

Therefore, it suffices to prove that

$$\sum\frac{b(a+1)^2}{a(b+1)^2}\geq 3.$$

This inequality follows immediately from the AM-GM inequality:

$$\sum\frac{b(a+1)^2}{a(b+1)^2}\geq 3\sqrt[3]{\prod\frac{b(a+1)^2}{a(b+1)^2}}=3.$$

The equality holds for $a=b=c=1$.

□

P 1.99. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$\sqrt{a^3+3b}+\sqrt{b^3+3c}+\sqrt{c^3+3a}\ge 6.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$(a^3+3b)(a+3b)\ge(a^2+3b)^2.$$

Thus, it suffices to show that

$$\sum\frac{a^2+3b}{\sqrt{a+3b}}\ge 6.$$

By Hölder's inequality, we have

$$\left(\sum\frac{a^2+3b}{\sqrt{a+3b}}\right)^2\left[\sum(a^2+3b)(a+3b)\right]\ge\left[\sum(a^2+3b)\right]^3=\left(\sum a^2+9\right)^3.$$

Therefore, it is enough to show that

$$\left(\sum a^2+9\right)^3\ge 36\sum(a^2+3b)(a+3b).$$

Let

$$p=a+b+c=3,\qquad q=ab+bc+ca,\qquad q\le 3.$$

We have

$$\sum a^2+9=p^2-2q+9=2(9-q),$$

$$\begin{aligned}\sum(a^2+3b)(a+3b)&=\sum a^3+3\sum a^2b+9\sum a^2+3\sum ab\\&=(p^3-3pq+3abc)+3\sum a^2b+9(p^2-2q)+3q\\&=108-24q+3\left(abc+\sum a^2b\right).\end{aligned}$$

Since $abc+\sum a^2b\le 4$ (see the inequality (a) in P 1.9), we get

$$\sum(a^2+3b)(a+3b)\le 24(5-q).$$

Thus, it suffices to show that

$$(9-q)^3\ge 108(5-q),$$

which is equivalent to

$$(3-q)^2(21-q)\ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.100. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$\sqrt{\frac{a}{a+6b+2bc}}+\sqrt{\frac{b}{b+6c+2ca}}+\sqrt{\frac{c}{c+6a+2ab}}\geq 1.$$

(Nguyen Van Quy and Vasile Cîrtoaje, 2013)

Solution. By Hölder's inequality, we have

$$\left(\sum\sqrt{\frac{a}{a+6b+2bc}}\right)^2\left[\sum a(a+6b+2bc)\right]\geq\left(\sum a^{2/3}\right)^3.$$

Therefore, it suffices to show that

$$\left(\sum a^{2/3}\right)^3\geq\sum a^2+6\sum ab+6,$$

which is equivalent to

$$3\sum(ab)^{2/3}(a^{2/3}+b^{2/3})\geq 6\sum ab.$$

Since

$$a^{2/3}+b^{2/3}\geq 2(ab)^{1/3},$$

the desired conclusion follows. The equality holds for $a = b = c = 1$.

□

P 1.101. *If a, b, c are positive real numbers such that $abc = 1$, then*

$$\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2\geq 6(a+b+c-1).$$

(Marius Stanean, 2014)

Solution *(by Michael Rozenberg)*. By the AM-GM inequality, we have

$$\begin{aligned}\sum\left(a+\frac{1}{b}\right)^2+6&=\sum(a+ac)^2+6\\&=\sum(a^2+a^2c^2+2a^2c)+6\\&=\sum(a^2+a^2b^2+2a^2c+2)\\&\geq 6\sum\sqrt[6]{a^2\cdot a^2b^2\cdot a^2c\cdot a^2c\cdot 1\cdot 1}=6\sum a.\end{aligned}$$

The equality holds for $a = b = c = 1$.

□

P 1.102. *If a,b,c are positive real numbers, then*

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\ge\frac{a+b+c}{a+b+c-\sqrt[3]{abc}}.$$

(Michael Rozenberg, 2014)

Solution. There are two cases to consider.

Case 1: $ab+bc+ca\ge\sqrt[3]{abc}\,(a+b+c)$. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a}{a+b}\ge\frac{\left(\sum a\right)^2}{\sum a(a+b)}=\frac{(a+b+c)^2}{(a+b+c)^2-(ab+bc+ca)}.$$

Therefore, it suffices to show that

$$\frac{(a+b+c)^2}{(a+b+c)^2-(ab+bc+ca)}\ge\frac{a+b+c}{a+b+c-\sqrt[3]{abc}},$$

which is equivalent to

$$ab+bc+ca-\sqrt[3]{abc}\,(a+b+c)\ge 0.$$

Case 2: $\sqrt[3]{abc}\,(a+b+c)\ge ab+bc+ca$. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a}{a+b}\ge\frac{\left(\sum ac\right)^2}{\sum ac^2(a+b)}=\frac{(ab+bc+ca)^2}{(ab+bc+ca)^2-abc(a+b+c)}.$$

Thus, it suffices to show that

$$\frac{(ab+bc+ca)^2}{(ab+bc+ca)^2-abc(a+b+c)}\ge\frac{a+b+c}{a+b+c-\sqrt[3]{abc}},$$

which is equivalent to

$$\left[\sqrt[3]{abc}\,(a+b+c)\right]^2\ge(ab+bc+ca)^2,$$

$$\sqrt[3]{abc}\,(a+b+c)\ge ab+bc+ca.$$

The proof is completed. The equality does not hold.

□

P 1.103. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$a\sqrt{b^2+b+1}+b\sqrt{c^2+c+1}+c\sqrt{a^2+a+1}\le 3\sqrt{3}.$$

(Nguyen Van Quy, 2014)

Solution. From

$$4(b^2+b+1)=2(b+1)^2+2(b^2+1)\ge 3(b+1)^2,$$

we get

$$\sqrt{b^2+b+1}\ge \frac{\sqrt{3}}{2}(b+1),$$

hence

$$\sum a\sqrt{b^2+b+1}=\sum \frac{a(b^2+b+1)}{\sqrt{b^2+b+1}}\le \sum \frac{2a(b^2+b+1)}{\sqrt{3}(b+1)}.$$

Therefore, it suffices to prove that

$$\sum \frac{a(b^2+b+1)}{b+1}\le \frac{9}{2},$$

which is equivalent to

$$\sum \frac{ab^2}{b+1}\le \frac{3}{2}.$$

In addition, since $b+1\ge 2\sqrt{b}$, it is enough to show that

$$\sum ab^{3/2}\le 3.$$

Replacing a,b,c by a^2,b^2,c^2, respectively, we need to show that $a^2+b^2+c^2=3$ involves $a^2b^3+b^2c^3+c^2a^3\le 3$, which is just the inequality in P 1.7. The equality holds for $a=b=c$.

□

P 1.104. *If a,b,c are positive real numbers, then*

$$\frac{1}{b(a+2b+3c)^2}+\frac{1}{c(b+2c+3a)^2}+\frac{1}{a(c+2a+3b)^2}\le \frac{1}{12abc}.$$

(Vo Quoc Ba Can, 2012)

Solution. Assume that $a=\max\{a,b,c\}$, and write the inequality as

$$\frac{ca}{(a+2b+3c)^2}+\frac{ab}{(b+2c+3a)^2}+\frac{bc}{(c+2a+3b)^2}\le \frac{1}{12}.$$

Case 1: $a\ge b\ge c$. By the AM-GM inequality, we have

$$(a+2b+3c)^2\ge 4(2b+c)(2c+a);$$

thus, it suffices to show that

$$\sum \frac{ca}{(2b+c)(2c+a)}\le \frac{1}{3},$$

which is equivalent to

$$3\sum ca(2a+b) \le (2a+b)(2b+c)(2c+a),$$

$$ab^2+bc^2+ca^2 \le a^2b+b^2c+c^2a,$$

$$(a-b)(b-c)(c-a) \le 0.$$

Clearly, the last inequality is true.

Case 2: $a \ge c \ge b$. Since, by the AM-GM inequality,

$$(a+2b+3c)^2 \ge 12c(a+2b),$$

$$(b+2c+3a)^2 \ge 4(2a+b)(2c+a),$$

$$(c+2a+3b)^2 \ge 4(a+2b)(a+b+c),$$

it suffices to show that

$$\frac{a}{3(a+2b)}+\frac{ab}{(2a+b)(2c+a)}+\frac{bc}{(a+2b)(a+b+c)} \le \frac{1}{3},$$

which is equivalent to

$$\frac{ab}{(2a+b)(2c+a)}+\frac{bc}{(a+2b)(a+b+c)} \le \frac{2b}{3(a+2b)},$$

$$\frac{a}{(2a+b)(2c+a)}+\frac{c}{(a+2b)(a+b+c)} \le \frac{2}{3(a+2b)},$$

$$\frac{a(a+2b)}{(2a+b)(2c+a)}+\frac{c}{a+b+c} \le \frac{2}{3},$$

$$\frac{a(a+2b)}{2a+b}+\frac{c(2c+a)}{a+b+c} \le \frac{2(2c+a)}{3},$$

$$\frac{c(2c+a)}{a+b+c}-\frac{2(2c+a)}{3} \le \frac{3a^2}{2a+b}-2a,$$

$$f(c) \le f(a),$$

where

$$f(x)=\frac{x(2x+a)}{a+b+x}-\frac{2(2x+a)}{3}.$$

We have

$$\begin{aligned} f(a)-f(c) &= (a-c)\left[\frac{3a^2+4ac+b(3a+2c)}{(a+b+c)(2a+b)}-\frac{4}{3}\right] \\ &= \frac{(a-c)[a^2-3ab-4b^2+2c(2a+b)]}{3(a+b+c)(2a+b)} \ge 0, \end{aligned}$$

because

$$a^2-3ab-4b^2+2c(2a+b) \ge a^2-3ab-4b^2+2b(2a+b)=(a-b)(a+2b) \ge 0.$$

The equality holds for $a=b=c$.

□

P 1.105. *Let a, b, c be positive real numbers such that $a+b+c=3$. Prove that*

(a)
$$\frac{a^2+9b}{b+c}+\frac{b^2+9c}{c+a}+\frac{c^2+9a}{a+b}\ge 15;$$

(b)
$$\frac{a^2+3b}{a+b}+\frac{b^2+3c}{b+c}+\frac{c^2+3a}{c+a}\ge 6.$$

Solution. (a) Write the inequality as follows:

$$\sum\frac{a^2+3b(a+b+c)}{b+c}\ge 5(a+b+c),$$

$$\sum\left[\frac{a^2+3b(a+b+c)}{b+c}-3b\right]\ge 2(a+b+c),$$

$$\sum\frac{a^2+3ab}{b+c}\ge 2(a+b+c),$$

$$\sum\left(\frac{a^2+3ab}{b+c}-2a\right)\ge 0,$$

$$\sum\frac{a(a+b-2c)}{b+c}\ge 0,$$

$$\sum\frac{a(a-c)}{b+c}+\sum\frac{a(b-c)}{b+c}\ge 0,$$

$$\sum\frac{a(a-c)}{b+c}+\sum\frac{b(c-a)}{c+a}\ge 0,$$

$$\sum(a-c)\left(\frac{a}{b+c}-\frac{b}{c+a}\right)\ge 0,$$

$$(a+b+c)\sum\frac{(a-b)(a-c)}{(b+c)(c+a)}\ge 0.$$

Therefore, we need to show that

$$\sum(a^2-b^2)(a-c)\ge 0,$$

which is equivalent to the obvious inequality

$$\sum a(a-c)^2\ge 0.$$

The equality holds for $a=b=c$.

(b) Write the inequality as follows:

$$\sum\frac{a^2+b(a+b+c)}{a+b}\ge 2(a+b+c),$$

$$\sum \frac{a^2+bc}{a+b} \ge a+b+c,$$

$$\sum \left(\frac{a^2+bc}{a+b} - a \right) \ge 0,$$

$$\sum \frac{b(c-a)}{a+b} \ge 0,$$

$$\sum \frac{bc}{a+b} \ge \sum \frac{ab}{a+b}.$$

Since the sequences

$$\{ab, \quad bc, \quad ca\}$$

and

$$\left\{ \frac{1}{a+b}, \quad \frac{1}{b+c}, \quad \frac{1}{c+a} \right\}$$

are reversely ordered, the inequality follows from the rearrangement inequality. The equality holds for $a=b=c$.

□

P 1.106. *If $a,b,c \in [0,1]$, then*

(a)
$$\frac{bc}{2ab+1} + \frac{ca}{2bc+1} + \frac{ab}{2ca+1} \le 1.$$

(b)
$$\frac{a}{ab+1} + \frac{b}{bc+1} + \frac{c}{ca+1} \le \frac{3}{2}.$$

(Vasile C., 2010)

Solution. (a) ***First Solution.*** It suffices to prove that

$$\frac{bc}{2abc+1} + \frac{ca}{2abc+1} + \frac{ab}{2abc+1} \le 1;$$

that is,

$$2abc+1 \ge ab+bc+ca,$$

$$1-bc \ge a(b+c-2bc).$$

Since $a \le 1$ and

$$b+c-2bc = b(1-c)+c(1-b) \ge 0,$$

it suffices to show that

$$1-bc \ge b+c-2bc,$$

which is equivalent to

$$(1-b)(1-c) \ge 0.$$

The equality holds for $a = b = c = 1$, and for $a = 0$ and $b = c = 1$ (or any cyclic permutation).

Second Solution. Assume that $a = \max\{a, b, c\}$. It suffices to show that

$$\frac{bc}{2bc+1} + \frac{ca}{2bc+1} + \frac{ab}{2bc+1} \le 1;$$

that is,

$$a(b+c) \le 1 + bc.$$

We have

$$1 + bc - a(b+c) \ge 1 + bc - (b+c) = (1-b)(1-c) \ge 0.$$

(b) We will show that

$$E(a,b,c) \le E(1,b,c) \le E(1,1,c) = \frac{3}{2},$$

where

$$E(a,b,c) = \frac{a}{ab+1} + \frac{b}{bc+1} + \frac{c}{ca+1}.$$

Write the inequality $E(a,b,c) \le E(1,b,c)$ as follows:

$$\frac{a}{ab+1} + \frac{c}{ca+1} \le \frac{1}{b+1} + \frac{c}{c+1},$$

$$(1-a)\left[\frac{1}{(b+1)(ab+1)} - \frac{c^2}{(c+1)(ca+1)}\right] \ge 0,$$

$$(1-a)[(c+1)(ca+1) - (b+1)(ab+1)c^2] \ge 0.$$

Since $1 - a \ge 0$ and $c \le 1$, it suffices to show that

$$(c+1)(ca+1) - (b+1)(ab+1)c \ge 0,$$

which is true because

$$\begin{aligned}(c+1)(ca+1) - (b+1)(ab+1)c &\ge (c+1)(ca+1) - 2(a+1)c \\ &= (1-c)(1-ac) \ge 0.\end{aligned}$$

Setting $a = 1$ in the similar inequality

$$E(a,b,c) \le E(a,1,c),$$

it follows that

$$E(1,b,c) \le E(1,1,c).$$

Finally,

$$E(1,1,c) = \frac{1}{2} + \frac{1}{c+1} + \frac{c}{c+1} = \frac{3}{2}.$$

The equality holds for $a = b = 1$ (or any cyclic permutation).

□

P 1.107. *If a,b,c are nonnegative real numbers, then*

$$a^4+b^4+c^4+5(a^3b+b^3c+c^3a)\geq 6(a^2b^2+b^2c^2+c^2a^2).$$

Solution. Assume that $a=\min\{a,b,c\}$ and use the substitution

$$b=a+p,\quad c=a+q,\qquad p,q\geq 0.$$

The inequality becomes

$$9Aa^2+3Ba+C\geq 0,$$

where

$$A=p^2-pq+q^2,\quad B=3p^3+p^2q-4pq^2+3q^3,$$
$$C=p^4+5p^3q-6p^2q^2+q^4.$$

Since

$$A\geq 0,$$
$$B=3p(p-q)^2+q(7p^2-7pq+3q^2)\geq 0,$$
$$C=(p-q)^4+pq(3p-2q)^2\geq 0,$$

the inequality is obviously true. The equality occurs for $a=b=c$.

□

P 1.108. *If a,b,c are positive real numbers, then*

$$a^5+b^5+c^5-a^4b-b^4c-c^4a\geq 2abc(a^2+b^2+c^2-ab-bc-ca).$$

(Vasile C., 2006)

Solution. Since

$$5\left(\sum a^5-\sum a^4b\right)=\sum(4a^5+b^5-5a^4b)=\sum(a-b)^2(4a^3+3a^2b+2ab^2+b^3)$$

and

$$2\left(\sum a^2-\sum ab\right)=\sum(a-b)^2,$$

we can write the inequality in the form

$$A(a-b)^2+B(b-c)^2+C(c-a)^2\geq 0,$$

where

$$A=4a^3+3a^2b+2ab^2+b^3-5abc,$$
$$B=4b^3+3b^2c+2bc^2+c^3-5abc,$$
$$C=4c^3+3c^2a+2ca^2+a^3-5abc.$$

Without loss of generality, assume that $a = \max\{a, b, c\}$. We have

$$A > a(4a^2 + 3ab - 5bc) > a(4c^2 + 3b^2 - 5bc) > 0,$$

$$C > a(3c^2 + 2ca + a^2 - 5bc) > a(3c^2 - 3ca + a^2) > 0,$$

$$\begin{aligned} A + B &> 4a^3 + 5b^3 + c^3 + 3a^2b + 2bc^2 - 10abc \\ &\geq 3\sqrt[3]{4a^3 \cdot 5b^3 \cdot c^3} + 2\sqrt{3a^2b \cdot 2bc^2} - 10abc \\ &= (3\sqrt[3]{20} + 2\sqrt{6} - 10)abc > 0, \end{aligned}$$

$$\begin{aligned} B + C &> a^3 + 4b^3 + 5c^3 + 3b^2c + 2ca^2 - 10abc \\ &\geq 3\sqrt[3]{a^3 \cdot 4b^3 \cdot 5c^3} + 2\sqrt{3b^2c \cdot 2ca^2} - 10abc \\ &= (3\sqrt[3]{20} + 2\sqrt{6} - 10)abc > 0. \end{aligned}$$

If $a \geq b \geq c$, then

$$\sum A(a-b)^2 \geq B(b-c)^2 + C(a-c)^2 \geq (B+C)(b-c)^2 \geq 0.$$

If $a \geq c \geq b$, then

$$\sum A(a-b)^2 \geq A(a-b)^2 + B(c-b)^2 \geq (A+B)(c-b)^2 \geq 0.$$

The equality holds for $a = b = c$.

□

P 1.109. *If a, b, c are positive real numbers such that $a^2 + b^2 + c^2 = 3$, then*

$$\frac{a}{1+b} + \frac{b}{1+c} + \frac{c}{1+a} \geq \frac{3}{2}.$$

(Vasile C., 2005)

Solution. Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad p^2 = 3 + 2q.$$

First Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{1+b} \geq \frac{\left(\sum a\right)^2}{\sum a(1+b)} = \frac{3+2q}{p+q}.$$

Thus, it suffices to prove that

$$6 + q \geq 3p.$$

Indeed,

$$2(6+q-3p)=12+(p^2-3)-6p=(p-3)^2\geq 0.$$

The equality holds for $a=b=c=1$.

Second Solution. By the AM-GM inequality, we have

$$\sum\frac{a}{1+b}=\sum\frac{a(a+c)}{(1+b)(a+c)}\geq\sum\frac{4a(a+c)}{[(1+b)+(a+c)]^2}$$
$$=\frac{4\left(\sum a^2+\sum ac\right)}{(1+p)^2}=\frac{4(3+q)}{(1+p)^2}=\frac{6+2p^2}{(1+p)^2}.$$

Therefore, it suffices to show that

$$\frac{6+2p^2}{(1+p)^2}\geq\frac{3}{2},$$

which is equivalent to $(p-3)^2\geq 0$.

Conjecture. *If a,b,c are positive real numbers such that $a^2+b^2+c^2=3$, then*

$$\frac{a}{5+4b}+\frac{b}{5+4c}+\frac{c}{5+4a}\geq\frac{1}{3}.$$

□

P 1.110. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$a\sqrt{a+b}+b\sqrt{b+c}+c\sqrt{c+a}\geq 3\sqrt{2}.$$

(Hong Ge Chen, 2011)

First Solution. Denote

$$q=\sqrt{\frac{ab+bc+ca}{3}},\qquad q\leq 1.$$

By squaring, the inequality turns into

$$\sum a^3+\sum a^2b+2\sum ac\sqrt{a^2+3q^2}\geq 18.$$

Since

$$2\sqrt{a^2+3q^2}\geq a+3q,$$

we have

$$2\sum ac\sqrt{a^2+3q^2}\geq\sum ac(a+3q)=\sum ab^2+9q^3.$$

Thus, it suffices to show that

$$\sum a^3 + \sum ab(a+b) + 9q^3 \geq 18,$$

which is equivalent to

$$(a+b+c)(a^2+b^2+c^2) + 9q^3 \geq 18,$$

$$3(9-6q^2) + 9q^3 \geq 0,$$
$$1 - 2q^2 + q^3 \geq 0,$$
$$(1-q^2)^2 + q^3(1-q) \geq 0.$$

Clearly, the last inequality is true. The equality holds for $a = b = c = 1$.

Second Solution. Using the substitution

$$\sqrt{\frac{a+b}{2}} = \frac{x+y}{2}, \quad \sqrt{\frac{b+c}{2}} = \frac{y+z}{2}, \quad \sqrt{\frac{c+a}{2}} = \frac{z+x}{2}$$

gives

$$x = \sqrt{\frac{a+b}{2}} + \sqrt{\frac{a+c}{2}} - \sqrt{\frac{b+c}{2}} \geq 0,$$
$$a = \left(\frac{x+y}{2}\right)^2 + \left(\frac{x+z}{2}\right)^2 - \left(\frac{y+z}{2}\right)^2 = \frac{x(x+y+z) - yz}{2}.$$

In addition, $a + b + c = 3$ involves

$$x^2 + y^2 + z^2 + xy + yz + zx = 6,$$

which is equivalent to

$$p^2 - q = 6,$$

where

$$p = x + y + z, \qquad q = xy + yz + zx.$$

From

$$18 - 2p^2 = 3(x^2 + y^2 + z^2 + xy + yz + zx) - 2(x+y+z)^2$$
$$= x^2 + y^2 + z^2 - xy - yz - zx \geq 0,$$

it follows that

$$p \leq 3.$$

The desired inequality is equivalent to

$$\sum (xp - yz)(x+y) \geq 12,$$

$$p \sum (x^2 + xy) \geq 3xyz + \sum y^2 z + 12,$$

$$6p \ge 3xyz + \sum y^2z + 12,$$

$$6p + \sum yz^2 \ge pq + 12.$$

Since

$$\left(\sum yz^2\right)\left(\sum y\right) \ge \left(\sum yz\right)^2$$

(by the Cauchy-Schwarz inequality), it suffices to show that

$$6p + \frac{q^2}{p} \ge pq + 12.$$

Indeed,

$$6p + \frac{q^2}{p} - pq = \frac{p^2(6-q)+q^2}{p} = \frac{(6+q)(6-q)+q^2}{p} = \frac{36}{p} \ge 12.$$

Conjecture. *If a, b, c are nonnegative real numbers such that $a+b+c=3$, then*

$$a\sqrt{4a+5b} + b\sqrt{4b+5c} + c\sqrt{4c+5a} \ge 9.$$

□

P 1.111. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a}{2b^2+c} + \frac{b}{2c^2+a} + \frac{c}{2a^2+b} \ge 1.$$

(Vasile Cîrtoaje and Nguyen Van Quy, 2007)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{2b^2+c} \ge \frac{\left(\sum a\sqrt{a+c}\right)^2}{\sum a(a+c)(2b^2+c)}.$$

Since $\sum a\sqrt{a+c} \ge 3\sqrt{2}$ (see the preceding P 1.110), it suffices to prove that

$$\sum a(a+c)(2b^2+c) \le 18,$$

which is equivalent to

$$2\sum a^2b^2 + 6abc + \sum ac(a+c) \le 18,$$

$$2\sum a^2b^2 + 3abc + \left(\sum a\right)\left(\sum ab\right) \le 18.$$

Denoting

$$q = ab + bc + ca,$$

the inequality becomes

$$9abc + 18 \geq 2q^2 + 3q.$$

This inequality is true for $q < 2$, because $18 > 2q^2 + 3q$. Since $q \leq p^2/3 = 3$, consider further the case $2 \leq q \leq 3$. By Schur's inequality of degree three, we have

$$9abc \geq 4pq - p^3 = 12q - 27.$$

Therefore,

$$\begin{aligned} 9abc + 18 - (2q^2 + 3q) &\geq (12q - 27) + 18 - (2q^2 + 3q) \\ &= -2q^2 + 9q - 9 = (3 - q)(2q - 3) \geq 0. \end{aligned}$$

This completes the proof. The equality holds for $a = b = c = 1$.

□

P 1.112. *If a, b, c are positive real numbers such that $a + b + c = ab + bc + ca$, then*

$$\frac{1}{a^2 + b + 1} + \frac{1}{b^2 + c + 1} + \frac{1}{c^2 + a + 1} \leq 1.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$\frac{1}{a^2 + b + 1} \leq \frac{1 + b + c^2}{(a + b + c)^2},$$

hence

$$\sum \frac{1}{a^2 + b + 1} \leq \sum \frac{1 + b + c^2}{(a + b + c)^2} = \frac{3 + a + b + c + a^2 + b^2 + c^2}{(a + b + c)^2}.$$

It suffices to show that

$$3 + a + b + c \leq 2(ab + bc + ca),$$

which is equivalent to

$$a + b + c \geq 3.$$

We can get this inequality from the known inequality

$$(a + b + c)^2 \geq 3(ab + bc + ca).$$

The equality holds for $a = b = c = 1$.

□

P 1.113. *If a,b,c are positive real numbers, then*

$$\frac{1}{(a+2b+3c)^2}+\frac{1}{(b+2c+3a)^2}+\frac{1}{(c+2a+3b)^2}\le\frac{1}{4(ab+bc+ca)}.$$

Solution. By the AM-GM inequality, we have

$$(a+2b+3c)^2=[(a+c)+2(b+c)]^2=(a+c)^2+4(b+c)^2+4(a+c)(b+c)$$
$$\ge 3(b+c)^2+6(a+c)(b+c)=3(b+c)(2a+b+3c).$$

Thus, it suffices to show that

$$\sum\frac{1}{(b+c)(2a+b+3c)}\le\frac{3}{4(ab+bc+ca)}.$$

Write this inequality as follows:

$$\frac{3}{4}-\sum\frac{ab+bc+ca}{(b+c)(2a+b+3c)}\ge 0,$$

$$\sum\left[1-\frac{2(ab+bc+ca)}{(b+c)(2a+b+3c)}\right]\ge\frac{3}{2},$$

$$\sum\frac{(b+c)^2+2c^2}{(b+c)(2a+b+3c)}\ge\frac{3}{2},$$

$$\sum\frac{b+c}{2a+b+3c}+\sum\frac{2c^2}{(b+c)(2a+b+3c)}\ge\frac{3}{2}.$$

Applying the Cauchy-Schwarz inequality, we get

$$\sum\frac{b+c}{2a+b+3c}\ge\frac{[\sum(b+c)]^2}{\sum(b+c)(2a+b+3c)}=\frac{4\left(\sum a\right)^2}{4\left(\sum a\right)^2}=1$$

and

$$\sum\frac{c^2}{(b+c)(2a+b+3c)}\ge\frac{\left(\sum c\right)^2}{\sum(b+c)(2a+b+3c)}=\frac{1}{4},$$

from where the conclusion follows. The equality holds for $a=b=c$.

□

P 1.114. *If a,b,c are positive real numbers, then*

$$\sqrt{\frac{a}{a+b+2c}}+\sqrt{\frac{b}{b+c+2a}}+\sqrt{\frac{c}{c+a+2b}}\le\frac{3}{2}.$$

Solution. Apply the Cauchy-Schwarz inequality as follows:

$$\left(\sum\sqrt{\frac{a}{a+b+2c}}\right)^2 \le \left[\sum(b+c+2a)\right]\left[\sum\frac{a}{(b+c+2a)(a+b+2c)}\right]$$
$$=\frac{4\left(\sum a\right)\left[\sum a(c+a+2b)\right]}{(b+c+2a)(c+a+2b)(a+b+2c)}.$$

Thus, it suffices to show that

$$16\left(\sum a\right)\left[\sum a(c+a+2b)\right] \le 9(b+c+2a)(c+a+2b)(a+b+2c).$$

Denoting

$$p=a+b+c, \qquad q=ab+bc+ca,$$

the inequality becomes

$$16p(p^2+q) \le 9(p+a)(p+b)(p+c),$$

$$16p(p^2+q) \le 9(2p^3+pq+abc),$$
$$2p^3-7pq+9abc \ge 0.$$

Using Schur's inequality of degree three

$$p^3+9abc \ge 4pq,$$

we have

$$2p^3-7pq+9abc=(p^3+9abc-4pq)+p(p^2-3q) \ge 0.$$

The equality holds for $a=b=c$.

□

P 1.115. *If a,b,c are positive real numbers, then*

$$\sqrt{\frac{5a}{a+b+3c}}+\sqrt{\frac{5b}{b+c+3a}}+\sqrt{\frac{5c}{c+a+3b}} \le 3.$$

Solution. Substituting

$$x=\sqrt{\frac{5a}{a+b+3c}}, \quad y=\sqrt{\frac{5b}{b+c+3a}}, \quad z=\sqrt{\frac{5c}{c+a+3b}},$$

we have

$$\begin{cases} (x^2-5)a+x^2b+3x^2c=0 \\ 3y^2a+(y^2-5)b+y^2c=0 \\ z^2a+3z^2b+(z^2-5)c=0 \end{cases},$$

which involves

$$\begin{vmatrix} x^2-5 & x^2 & 3x^2 \\ 3y^2 & y^2-5 & y^2 \\ z^2 & 3z^2 & z^2-5 \end{vmatrix} = 0\ ;$$

that is,

$$F(x,y,z)=0,$$

where

$$F(x,y,z)=4x^2y^2z^2+2\sum x^2y^2+5\sum x^2-25.$$

We need to show that $F(x,y,z)=0$ involves $x+y+z\le 3$, where $x,y,z>0$. According to the contradiction method, assume that $x+y+z>3$ and show that $F(x,y,z)>0$. Since $F(x,y,z)$ is strictly increasing in each of its arguments, it is enough to prove that

$$x+y+z=3$$

involves

$$F(x,y,z)\ge 0.$$

Denote

$$q=xy+yz+zx, \qquad r=xyz.$$

Since

$$\sum x^2y^2=q^2-6r, \qquad \sum x^2=9-2q,$$

we have

$$F(x,y,z)=4r^2+2(q^2-6r)+5(9-2q)-25=2(2r^2-6r+q^2-5q+10),$$

$$\frac{1}{2}F(x,y,z)=2(r-1)^2+q^2-5q+8-2r.$$

It suffices to show that

$$q^2-5q+8\ge 2r.$$

From the known inequality

$$(xy+yz+zx)^2\ge 3xyz(x+y+z),$$

it follows that $q^2\ge 9r$. Therefore, it suffices to prove that

$$q^2-5q+8\ge \frac{2q^2}{9},$$

which is equivalent to

$$(3-q)(24-7q)\ge 0.$$

Since

$$q\le \frac{1}{3}(x+y+z)^2=3,$$

the conclusion follows. The original inequality is an equality for $a=b=c$.

□

P 1.116. *If* $a,\ b,\ c \in [0,\ 1]$, *then*

$$ab^2+bc^2+ca^2+\frac{5}{4} \geq a+b+c.$$

(Ji Chen, 2007)

Solution. We use the substitution

$$a=1-x, \quad b=1-y, \quad c=1-z,$$

where $x,y,z \in [0,\ 1]$. Since

$$\begin{aligned}\sum a(1-b^2) &= \sum y(1-x)(2-y) = \sum y(2-2x-y+xy) \\ &= 2\sum x - \left(\sum x\right)^2 + \sum xy^2,\end{aligned}$$

the inequality can be written as

$$\frac{5}{4} \geq 2\sum x - \left(\sum x\right)^2 + \sum xy^2.$$

According to the known inequality in P 1.1, we have

$$\sum xy^2 \leq \frac{4}{27}\left(\sum x\right)^3.$$

Thus, it suffices to prove the following inequality

$$\frac{5}{4} \geq 2t - t^2 + \frac{4}{27}t^3,$$

where

$$t = x+y+z \leq 3.$$

This inequality is equivalent to

$$(15-4t)(3-2t)^2 \geq 0,$$

which is obviously true for $t \leq 3$. The proof is completed. The equality occurs for $a=0$, $b=1$ and $c=\frac{1}{2}$ (or any cyclic permutation thereof).

□

P 1.117. *If* $a,\ b,\ c$ *are nonnegative real numbers such that*

$$a+b+c=3, \qquad a \leq b \leq 1 \leq c,$$

then

$$a^2b+b^2c+c^2a \leq 3.$$

Solution. Since

$$ab^2+bc^2+ca^2-(a^2b+b^2c+c^2)=(a-b)(b-c)(c-a)\ge 0,$$

it suffices to prove that

$$a^2b+b^2c+c^2+(ab^2+bc^2+ca^2)\le 6;$$

that is,

$$(a+b+c)(ab+bc+ca)-3abc\le 6,$$

$$ab+bc+ca-abc\le 2,$$

$$1-(a+b+c)+ab+bc+ca-abc\le 0,$$

$$(1-a)(1-b)(1-c)\le 0.$$

The equality occurs for $a=b=c=1$.

□

P 1.118. *Let a, b, c be nonnegative real numbers such that*

$$a+b+c=3,\qquad a\le 1\le b\le c.$$

Prove that

(a) $$a^2b+b^2c+c^2a\ge ab+bc+ca;$$

(b) $$a^2b+b^2c+c^2a\ge abc+2;$$

(c) $$\frac{1}{abc}+2\ge\frac{9}{a^2b+b^2c+c^2a};$$

(d) $$ab^2+bc^2+ca^2\ge 3.$$

(Vasile C., 2008)

Solution. (a) We have

$$\begin{aligned}a^2b+b^2c+c^2a-ab-bc-ca&=ab(a-1)+bc(b-1)+ca(c-1)\\&=-ab[(b-1)+(c-1)]+bc(b-1)+ca(c-1)\\&=b(b-1)(c-a)+a(c-1)(c-b)\ge 0.\end{aligned}$$

The equality holds for $a=b=c=1$, and also for $a=0$, $b=1$ and $c=2$.

(b) Since

$$a(b-a)(b-c)\le 0,$$

we have

$$a^2b+b^2c+c^2a \geq a^2b+b^2c+c^2a+a(b-a)(b-c)$$
$$= b^2(a+c)+ac(a+c-b).$$

Thus, it suffices to prove that

$$b^2(a+c)+ac(a+c-b) \geq abc+2.$$

This inequality is equivalent to

$$b^2(a+c)-2 \geq ac(2b-a-c),$$

$$b^2(3-b)-2 \geq ac(3b-3).$$

From $(b-a)(b-c) \leq 0$, it follows that

$$ac \leq b(a+c-b) = b(3-2b).$$

Thus, it suffices to show that

$$b^2(3-b)-2 \geq b(3-2b)(3b-3),$$

which is equivalent to the obvious inequality

$$(5b-2)(b-1)^2 \geq 0.$$

The equality holds for $a=b=c=1$, and also for $a=0$, $b=1$ and $c=2$.

(c) According to the inequality in (a), it suffices to show that

$$\frac{1}{abc}+2 \geq \frac{9}{abc+2},$$

which is equivalent to

$$(abc-1)^2 \geq 0.$$

The equality holds for $a=b=c=1$.

(d) Since

$$ab^2+bc^2+ca^2-(a^2b+b^2c+c^2) = (a-b)(b-c)(c-a) \geq 0,$$

it suffices to prove that

$$ab^2+bc^2+ca^2+(a^2b+b^2c+c^2) \geq 6;$$

that is,

$$(a+b+c)(ab+bc+ca)-3abc \geq 6,$$
$$ab+bc+ca-abc \geq 2,$$

$$1-(a+b+c)+ab+bc+ca-abc \ge 0,$$

$$(1-a)(1-b)(1-c) \ge 0.$$

The equality holds for $a=b=c=1$.

Remark 1. For

$$a+b+c=3, \qquad 0<a\le 1\le b\le c,$$

the following *open inequality* holds

$$\frac{1}{abc}+6 \ge \frac{21}{a^2b+b^2c+c^2a},$$

which is sharper than the inequality in (c).

Remark 2. From the proof of the inequality in (d), the following identity follows for $a+b+c=3$:

$$2(ab^2+bc^2+ca^2-3)=3(1-a)(1-b)(1-c)+(a-b)(b-c)(c-a).$$

□

P 1.119. *If a, b, c are nonnegative real numbers such that*

$$a+b+c=3, \qquad a\le 1\le b\le c,$$

then

(a)
$$\frac{5-2a}{1+b}+\frac{5-2b}{1+c}+\frac{5-2c}{1+a} \ge \frac{9}{2};$$

(b)
$$\frac{3-2b}{1+a}+\frac{3-2c}{1+b}+\frac{3-2a}{1+c} \le \frac{3}{2}.$$

(Vasile C., 2008)

Solution. (a) Write the inequality as follows:

$$2\sum(5-2a)(1+c)(1+a) \ge 9(1+a)(1+b)(1+c),$$

$$2\left(21+7\sum ab-2\sum ab^2\right) \ge 9\left(4+\sum ab+abc\right),$$

$$6+5\sum ab \ge 9abc+4\sum ab^2.$$

By P 1.9-(a), we have

$$\sum ab^2 \le 4-abc.$$

Therefore, it suffices to prove that

$$6+5\sum ab \ge 9abc+4(4-abc),$$

which is equivalent to

$$\sum ab \geq 2 + abc,$$

$$(1-a)(1-b)(1-c) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0,\ b = 1,\ c = 2$.

(b) Write the inequality as follows:

$$2\sum(3-2b)(1+b)(1+c) \leq 3(1+a)(1+b)(1+c),$$

$$2\left(3+5\sum ab - 2\sum a^2b\right) \leq 3\left(4+\sum ab + abc\right),$$

$$6+3abc+4\sum a^2b \geq 7\sum ab,$$

$$6+3abc+4\sum ab(a+b) \geq 7\sum ab + 4\sum ab^2,$$

$$6+3abc+4\left(\sum a\right)\left(\sum ab\right) - 12abc \geq 7\sum ab + 4\sum ab^2,$$

$$6+5\sum ab \geq 9abc + 4\sum ab^2.$$

By P 1.9-(a), we have

$$\sum ab^2 \leq 4 - abc.$$

Therefore, it suffices to prove that

$$6+5\sum ab \geq 9abc + 4(4-abc),$$

which is equivalent to

$$\sum ab \geq 2 + abc,$$

$$(1-a)(1-b)(1-c) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0,\ b = 1,\ c = 2$.

□

P 1.120. *If $a,\ b,\ c$ are nonnegative real numbers such that*

$$ab + bc + ca = 3, \qquad a \leq 1 \leq b \leq c,$$

then

(a)
$$a^2b + b^2c + c^2a \geq 3;$$

(b)
$$ab^2 + bc^2 + ca^2 + 3(\sqrt{3}-1)abc \geq 3\sqrt{3}.$$

(Vasile C., 2008)

Solution. (a) Since

$$a(b-a)(b-c) \le 0,$$

we have

$$\begin{aligned} a^2b + b^2c + c^2a &\ge a^2b + b^2c + c^2a + a(b-a)(b-c) \\ &= b^2(a+c) + ac(a+c-b). \end{aligned}$$

Thus, it suffices to prove that

$$b^2(a+c) + ac(a+c-b) \ge 3.$$

Denote

$$x = a + c.$$

From $ab + bc + ca = 3$, we get

$$ac = 3 - bx$$

and

$$x = \frac{3-ac}{b} \le \frac{3}{b} \le 3.$$

Thus, we need to show that

$$b^2x + (3-bx)(x-b) \ge 3,$$

$$2b^2x - (x^2+3)b + 3x - 3 \ge 0.$$

Since

$$\begin{aligned} 2b^2x - (x^2+3)b + 3x - 3 &= 2(b^2 - 2b + 1)x + 2(2b-1)x - (x^2+3)b + 3x - 3 \\ &= 2(b-1)^2x + (3-x)(bx-b-1) \\ &\ge (3-x)(bx-b-1), \end{aligned}$$

it is enough to prove that

$$bx - b - 1 \ge 0.$$

From the inequality $(b-a)(b-c) \le 0$, we get

$$bx \ge b^2 + ac = b^2 + 3 - bx, \quad bx \ge \frac{b^2+3}{2}.$$

Therefore,

$$bx - b - 1 \ge \frac{b^2+3}{2} - b - 1 = \frac{(b-1)^2}{2} \ge 0.$$

The proof is completed. The equality holds for $a = b = c = 1$, and for $a = 0$, $b = 1$ and $c = 3$.

(b) Since

$$ab^2 + bc^2 + ca^2 - (a^2b + b^2c + c^2) = (a-b)(b-c)(c-a) \geq 0,$$

it suffices to prove that

$$ab^2 + bc^2 + ca^2 + (a^2b + b^2c + c^2) + 6(\sqrt{3}-1)abc \geq 6\sqrt{3};$$

that is,

$$(a+b+c)(ab+bc+ca) + 3(2\sqrt{3}-3)abc \geq 6\sqrt{3},$$

$$a+b+c+(2\sqrt{3}-3)abc \geq 2\sqrt{3},$$

$$a[1+(2\sqrt{3}-3)bc] + b + c \geq 2\sqrt{3},$$

$$a[1+(2\sqrt{3}-3)p] + 2(s-\sqrt{3}) \geq 0,$$

where

$$s = \frac{b+c}{2}, \quad p = bc, \quad s^2 \geq p \geq 1.$$

From $ab + bc + ca = 3$, we get

$$a = \frac{3-p}{2s}, \quad p \leq 3.$$

Therefore, we need to show that $F(s,p) \geq 0$, where

$$F(s,p) = (3-p)[1+(2\sqrt{3}-3)p] + 4s(s-\sqrt{3}).$$

Since the inequality $F(s,p) \geq 0$ is true for $s - \sqrt{3} \geq 0$, consider further the case

$$s \leq \sqrt{3}.$$

We will show that

$$F(s,p) \geq F(s,s^2) \geq 0.$$

We have

$$F(s,p) - F(s,s^2) = (2\sqrt{3}-3)(s^4-p^2) - (6\sqrt{3}-10)(s^2-p)$$
$$= (s^2-p)[(2\sqrt{3}-3)(s^2+p) - 6\sqrt{3}+10].$$

Since $s^2 - p \geq 0$ and

$$(2\sqrt{3}-3)(s^2+p) - 6\sqrt{3}+10 \geq (2\sqrt{3}-3)(1+1) - 6\sqrt{3}+10 = 4-2\sqrt{3} > 0,$$

the left inequality is true. The right inequality is also true because

$$F(s,s^2) = (3-s^2)[1+(2\sqrt{3}-3)s^2] + 4s(s-\sqrt{3})$$
$$= (\sqrt{3}-s)[(\sqrt{3}+s)(1+(2\sqrt{3}-3)s^2) - 4s]$$
$$= (\sqrt{3}-s)[\sqrt{3}(1-s)^2(1+2s) - 3s(1-s)^2]$$
$$= (\sqrt{3}-s)(1-s)^2[\sqrt{3}+(2\sqrt{3}-3)s] \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = \sqrt{3}$.

□

P 1.121. *If a, b, c are nonnegative real numbers such that*

$$a^2+b^2+c^2=3, \quad a \le 1 \le b \le c,$$

then

(a) $$a^2b+b^2c+c^2a \ge 2abc+1;$$

(b) $$2(ab^2+bc^2+ca^2) \ge 3abc+3.$$

(Vasile C., 2008)

Solution. (a) Let

$$x=a+c, \quad x \ge b.$$

From $a^2+b^2+c^2=3$, we get

$$ac=\frac{b^2+x^2-3}{2},$$

and from $(b-a)(b-c) \le 0$, we get

$$bx \ge b^2+ac,$$

$$bx \ge b^2+\frac{x^2+b^2-3}{2},$$

$$(x-b)^2 \le 3-2b^2, \quad b \le \sqrt{\frac{3}{2}},$$

$$x \le b+d, \quad d=\sqrt{3-2b^2}.$$

Since

$$a(b-a)(b-c) \le 0,$$

we have

$$\begin{aligned} a^2b+b^2c+c^2a &\ge a^2b+b^2c+c^2a+a(b-a)(b-c) \\ &= b^2x-ac(b-x). \end{aligned}$$

Thus, it suffices to prove that

$$b^2x-ac(3b-x) \ge 1,$$

which is equivalent to $f(x,b) \ge 0$, where

$$\begin{aligned} f(x,b) &= 2b^2x-(x^2+b^2-3)(3b-x)-2 \\ &= x^3-3bx^2+3(b^2-1)x-3b^3+9b-2. \end{aligned}$$

We will show that

$$f(x,b) \ge f(b+d,b) \ge 0.$$

Since $x \le b+d$ and

$$f(x,b)-f(b+d,b) = (x-b-d)[x^2+x(b+d)+(b+d)^2-3b(x+b+d)+3b^2-3]$$
$$= (x-b-d)[x^2-(2b-d)x-b^2-bd],$$

we need to show that $g(x) \le 0$, where

$$g(x) = x^2-(2b-d)x-b^2-bd = (x-2b)(x+d)+b(d-b).$$

Since $d-b \le 0$, it suffices to show that $x-2b \le 0$. Indeed, we have

$$x^2 = (a+c)^2 \le 2(a^2+c^2) = 2(3-b^2) \le 4,$$

hence

$$x \le 2 \le 2b.$$

To prove the right inequality $f(b+d,b) \ge 0$, we have

$$f(b+d,b) = 2b^2(b+d)-2bd(2b-d)-2 = 2(3b-b^3-1-b^2d).$$

We need to show that

$$3b-b^3-1 \ge b^2\sqrt{3-2b^2}$$

for

$$1 \le b \le \sqrt{\frac{3}{2}}.$$

We have

$$3b-b^3-1 \ge 3b-\frac{3b}{2}-1 = \frac{3b-2}{2} \ge 0.$$

By squaring, the inequality becomes

$$(3b-b^3-1)^2 \ge b^4(3-2b^2),$$

$$3b^6-9b^4+2b^3+9b^2-6b+1 \ge 0,$$

$$(b-1)^2(3b^4+6b^3-4b+1) \ge 0.$$

The original inequality is an equality for $a=b=c=1$.

(b) Denote

$$p = a+b+c, \qquad q = ab+bc+ca.$$

Since

$$ab^2+bc^2+ca^2-(a^2b+b^2c+c^2) = (a-b)(b-c)(c-a) \ge 0,$$

it suffices to prove that

$$ab^2+bc^2+ca^2+(a^2b+b^2c+c^2) \ge 3abc+3;$$

that is,

$$pq \geq 6abc + 3.$$

From

$$(a-1)(b-1)(c-1) \geq 0,$$

we get

$$abc \geq 1 - p + q,$$

therefore

$$\begin{aligned} pq - 6abc - 3 &\geq pq - 6(1 - p + q) - 3 \\ &= (p-6)q + 6p - 9 \\ &= \frac{(p-6)(p^2-3)}{2} + 6p - 9 \\ &= \frac{p(p-3)^2}{2} \geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$.

□

P 1.122. *If a, b, c are nonnegative real numbers such that*

$$ab + bc + ca = 3, \qquad a \leq b \leq 1 \leq c,$$

then

$$ab^2 + bc^2 + ca^2 + 3abc \geq 6.$$

(Vasile C., 2008)

Solution. Denote

$$p = a + b + c.$$

Since

$$ab^2 + bc^2 + ca^2 - (a^2b + b^2c + c^2) = (a-b)(b-c)(c-a) \geq 0,$$

it suffices to prove that

$$ab^2 + bc^2 + ca^2 + (a^2b + b^2c + c^2) + 6abc \geq 12;$$

that is,

$$(a + b + c)(ab + bc + ca) + 3abc \geq 12,$$

$$a + b + c + abc \geq 4,$$

which is equivalent to

$$(a-1)(b-1)(c-1) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 1.123. *If a, b, c are nonnegative real numbers such that*

$$a^2+b^2+c^2=3, \qquad a\le b\le 1\le c,$$

then

$$2(a^2b+b^2c+c^2a)\le 3abc+3.$$

(Vasile C., 2008)

Solution. Consider two cases.

Case 1: $a+c\ge 2b$. Denote

$$x=a+c, \qquad x\ge 2b.$$

From $a^2+b^2+c^2=3$ and $(b-a)(b-c)\le 0$, we get in succession

$$ac=\frac{b^2+x^2-3}{2},$$

$$bx\ge b^2+ac,$$

$$bx\ge b^2+\frac{x^2+b^2-3}{2},$$

$$(x-b)^2\le 3-2b^2,$$

$$x\le b+d, \qquad d=\sqrt{3-2b^2}.$$

Since

$$ab^2+bc^2+ca^2-(a^2b+b^2c+c^2)=(a-b)(b-c)(c-a)\ge 0,$$

it suffices to prove that

$$a^2b+b^2c+c^2a+(ab^2+bc^2+ca^2)\le 3abc+3;$$

that is,

$$(a+b+c)(ab+bc+ca)\le 6abc+3,$$

$$(x+b)(bx+ac)\le 6abc+3,$$

$$ac(x-5b)+bx(x+b)-3\le 0.$$

Thus, we need to show that $f(x,b)\le 0$, where

$$\begin{aligned}f(x,b)&=(x^2+b^2-3)(x-5b)+2bx(x+b)-6\\&=x^3-3bx^2+3(b^2-1)x-5b^3+15b-6.\end{aligned}$$

We will show that

$$f(x,b)\le f(b+d,b)\le 0.$$

Since $x \le b+d$ and

$$f(x,b)-f(b+d,b)=(x-b-d)[x^2+x(b+d)+(b+d)^2-3b(x+b+d)+3b^2-3]$$
$$=(x-b-d)[x^2-(2b-d)x-b^2-bd],$$

we need to show that $g(x) \ge 0$, where

$$g(x)=x^2-(2b-d)x-b^2-bd.$$

Since $x-2b \ge 0$ and $d-b \ge 0$, we have

$$g(x)=(x-2b)(x+d)+b(d-b) \ge 0.$$

To prove the right inequality $f(b+d,b) \le 0$, from

$$f(b+d,b)=2bd(d-4b)+2b(b+d)(2b+d)-6=2(6b-2b^3-3-b^2d),$$

it follows that we need to show that

$$6b-2b^3-3 \le b^2\sqrt{3-2b^2}$$

for $0 \le b \le 1$. This inequality is true for $b \le \dfrac{1}{2}$ because

$$6b-2b^3-3 \le 3(2b-1) \le 0.$$

So, it suffices to prove the inequality for $1/2 < b \le 1$. By squaring, the inequality becomes

$$(6b-2b^3-3)^2 \le b^4(3-2b^2),$$
$$2b^6-9b^4+4b^3+12b^2-12b+3 \le 0,$$
$$(b-1)^3(2b^3+6b^2+3b-3) \le 0.$$

We only need to show that

$$2b^3+6b^2+3b-3 \ge 0.$$

Indeed,

$$2b^3+6b^2+3b-3 > 3(2b^2+b-1)=3(2b-1)(b+1) > 0.$$

Case 2: $a+c \le 2b$. Consider the nontrivial case $a<c$, denote

$$b_1=\frac{a+c}{2}, \quad b_2=\sqrt{\frac{a^2+c^2}{2}} \quad (b_1<b_2),$$

and write the inequality in the homogeneous form $E(a,b,c) \le 0$, where

$$E(a,b,c)=2(a^2b+b^2c+c^2a)-3abc-3\left(\frac{a^2+b^2+c^2}{3}\right)^{3/2}.$$

From $a^2+b^2+c^2=3$ and $b\le 1$, it follows that $b\le b_2$. For fixed a and c, consider the function

$$f(b)=E(a,b,c), \qquad b\in[b_1,b_2].$$

We will show that

$$f(b)\le f(b_2))\le 0.$$

The left inequality is true if $f'(b)\ge 0$ for $b\in[b_1,b_2]$. Since

$$\begin{aligned}
f'(b) &= 2a^2+4bc-3ac-3b\left(\frac{a^2+b^2+c^2}{3}\right)^{1/2}\\
&= 2a^2+4bc-3ac-3b = 2a^2-3ac+b(4c-3)\\
&\ge 2a^2-3ac+\frac{(a+c)(4c-3)}{2}\\
&= \frac{(a-c)^2+3(a^2+c^2-a-c)}{2}\\
&\ge \frac{3(a^2+c^2-a-c)}{2},
\end{aligned}$$

it suffices to show that

$$a^2+c^2\ge a+c.$$

From $a^2+b^2+c^2=3$ and $b\le 1$, it follows that $a^2+c^2\ge 2$. If $a+c\le 2$, then

$$a^2+b^2\ge 2\ge a+c.$$

Also, if $a+c\ge 2$, then

$$a^2+b^2\ge \frac{1}{2}(a+c)^2\ge a+c.$$

To prove the right inequality $f(b_2)\le 0$, we see that

$$\begin{aligned}
f(b_2) &= 2a^2b_2+(a^2+c^2)c+2c^2a-3ab_2c-3b_2\frac{a^2+c^2}{2}\\
&= c(a+c)^2-\frac{(3c^2+6ac-a^2)}{2}b_2\\
&= c(a+c)^2-\frac{(3c^2+6ac-a^2)}{2}\sqrt{\frac{a^2+c^2}{2}}.
\end{aligned}$$

Thus, we need to show that

$$c^2(c+a)^4\le \frac{(3c^2+6ac-a^2)^2(c^2+a^2)}{8},$$

which is equivalent to

$$c^6+4ac^5-9a^2c^4-8a^3c^3+23a^4c^2-12a^5c+a^6\ge 0,$$

$$(c-a)^3(c^3+7c^2a+9ca^2-a^3)\le 0.$$

The proof is completed. The equality holds for $a=b=c=1$.

□

P 1.124. *If a, b, c are nonnegative real numbers such that*

$$a^2+b^2+c^2=3, \qquad a \le b \le 1 \le c,$$

then

$$2(a^3b+b^3c+c^3a) \le abc+5.$$

(Vasile C., 2008)

Solution. Let

$$p=a+b+c, \qquad q=ab+bc+ca.$$

Since

$$ab^3+bc^3+ca^3-(a^3b+b^3c+c^3)=(a+b+c)(a-b)(b-c)(c-a) \ge 0,$$

it suffices to prove that

$$(a^3b+b^3c+c^3a)+(ab^3+bc^3+ca^3) \le abc+5,$$

which is equivalent to

$$(a^2+b^2+c^2)(ab+bc+ca) \le abc(a+b+c+1)+5,$$

$$3q \le abc(p+1)+5.$$

From

$$(a-1)(b-1)(c-1) \ge 0,$$

we get

$$abc \ge q-p+1.$$

Therefore, it suffices to show that

$$3q \le (q-p+1)(p+1)+5,$$

which is equivalent to

$$6-p^2 \ge q(2-p),$$

$$12-2p^2 \ge (p^2-3)(2-p),$$

$$p^3-4p^2-3p+18 \ge 0,$$

$$(p-3)^2(p+2) \ge 0.$$

The proof is completed. The equality holds for $a=b=c=1$.

□

P 1.125. *If a, b, c are real numbers, then*

$$(a^2+b^2+c^2)^2 \ge 3(a^3b+b^3c+c^3a).$$

(Vasile C., 1992)

First Solution. Write the inequality as

$$E_1 - 2E_2 \geq 0,$$

where

$$E_1 = a^3(a-b) + b^3(b-c) + c^3(c-a),$$
$$E_2 = a^2b(a-b) + b^2c(b-c) + c^2a(c-a).$$

Using the substitution

$$b = a + p, \quad c = a + q,$$

we have

$$\begin{aligned}
E_1 &= a^3(a-b) + b^3[(b-a) + (a-c)] + c^3(c-a) \\
&= (a-b)^2(a^2+ab+b^2) + (a-c)(b-c)(b^2+bc+c^2) \\
&= p^2(a^2+ab+b^2) - q(p-q)(b^2+bc+c^2) \\
&= 3(p^2-pq+q^2)a^2 + 3(p^3-p^2q+q^3)a + p^4 - p^3q + q^4
\end{aligned}$$

and

$$\begin{aligned}
E_2 &= a^2b(a-b) + b^2c[(b-a)+(a-c)] + c^2a(c-a) \\
&= (a-b)b(a^2-bc) + (a-c)c(b^2-ca) \\
&= pb(bc-a^2) + qc(ca-b^2) \\
&= (p^2-pq+q^2)a^2 + (p^3+p^2q-2pq^2+q^3)a + p^3q - p^2q^2.
\end{aligned}$$

Thus, the inequality can be rewritten as

$$Aa^2 + Ba + C \geq 0,$$

where

$$A = p^2 - pq + q^2,$$
$$B = p^3 - 5p^2q + 4pq^2 + q^3,$$
$$C = p^4 - 3p^3q + 2p^2q^2 + q^4.$$

For the non-trivial case $A > 0$, it is enough to show that $\delta \leq 0$, where $\delta = B^2 - 4AC$ is the discriminant of the quadratic function $Aa^2 + Ba + C$. Indeed, we have

$$\begin{aligned}
\delta &= -3(p^6 - 2p^5q - 3p^4q^2 + 6p^3q^3 + 2p^2q^4 - 4pq^5 + q^6) \\
&= -3(p^3 - p^2q - 2pq^2 + q^3)^2 \leq 0.
\end{aligned}$$

The equality holds for $a = b = c$, and also for

$$\frac{a}{\sin^2 \frac{4\pi}{7}} = \frac{b}{\sin^2 \frac{2\pi}{7}} = \frac{c}{\sin^2 \frac{\pi}{7}}$$

(or any cyclic permutation).

Second Solution. Let us denote

$$x = a^2 - ab + bc,$$

$$y = b^2 - bc + ca,$$

$$z = c^2 - ca + ab.$$

We have

$$x^2 + y^2 + z^2 = \sum a^4 + 2\sum a^2b^2 - 2\sum a^3b$$

and

$$xy + yz + zx = \sum a^3b.$$

From the known inequality

$$x^2 + y^2 + z^2 \ge xy + yz + zx,$$

the desired inequality follows.

Third Solution. Let us denote

$$x = a(a - 2b - c),$$

$$y = b(b - 2c - a),$$

$$z = c(c - 2a - b).$$

We have

$$x^2 + y^2 + z^2 = \sum a^4 + 5\sum a^2b^2 + 4abc\sum a - 4\sum a^3b - 2\sum ab^3$$

and

$$xy + yz + zx = 3\sum a^2b^2 + 4abc\sum a - \sum a^3b - 2\sum ab^3.$$

The known inequality

$$x^2 + y^2 + z^2 \ge xy + yz + zx$$

leads to the desired inequality.

Remark 1. Let

$$E = (a^2 + b^2 + c^2)^2 - 3(a^3b + b^3c + c^3a).$$

Using the notations from the first solution, the formula

$$4A(Aa^2 + Ba + C) = (2Aa + B)^2 - \delta,$$

leads to the following identity

$$4E_1E = (A_1 - 5B_1 + 4C_1)^2 + 3(A_1 - B_1 - 2C_1 + 2D_1)^2,$$

where

$$A_1 = a^3 + b^3 + c^3, \quad B_1 = a^2b + b^2c + c^2a, \quad C_1 = ab^2 + bc^2 + ca^2, \quad D_1 = 3abc,$$

$$E_1 = a^2 + b^2 + c^2 - ab - bc - ca.$$

Remark 2. Let

$$E = (a^2 + b^2 + c^2)^2 - 3(a^3b + b^3c + c^3a),$$

The identity

$$x^2 + y^2 + z^2 - xy - yz - zx = \frac{1}{2}\sum(x-y)^2,$$

where x, y, z are defined in the second or third solution, leads to the identity

$$2E = \sum(a^2 - b^2 - ab + 2bc - ca)^2.$$

In addition, the following similar identities hold:

$$6E = \sum(2a^2 - b^2 - c^2 - 3ab + 3bc)^2,$$

$$4E = (2a^2 - b^2 - c^2 - 3ab + 3bc)^2 + 3(b^2 - c^2 - ab - bc + 2ca)^2.$$

Remark 3. The inequality in P 1.125 is known as *Vasc's inequality*, after the author's username on the Art of Problem Solving website.

□

P 1.126. *If a, b, c are real numbers, then*

$$a^4 + b^4 + c^4 + ab^3 + bc^3 + ca^3 \ge 2(a^3b + b^3c + c^3a).$$

(Vasile C., 1992)

First Solution. Making the substitution

$$b = a + p, \quad c = a + q,$$

the inequality turns into

$$Aa^2 + Ba + C \ge 0,$$

where

$$A = 3(p^2 - pq + q^2), \quad B = 3(p^3 - 2p^2q + pq^2 + q^3), \quad C = p^4 - 2p^3q + pq^3 + q^4.$$

Since the discriminant of the quadratic trinomial $Aa^2 + Ba + C$ is nonpositive,

$$\begin{aligned}\delta = B^2 - 4AC &= -3(p^6 - 6p^4q + 2p^3q^3 + 9p^2q^4 - 6pq^5 + q^6)\\ &= -3(p^3 - 3pq^2 + q^3)^2 \le 0,\end{aligned}$$

the conclusion follows. The equality holds for $a = b = c$, and also for

$$\frac{a}{\sin\frac{\pi}{9}} = \frac{b}{\sin\frac{7\pi}{9}} = \frac{c}{\sin\frac{13\pi}{9}}$$

(or any cyclic permutation).

Second Solution. Let us denote

$$x = a(a-b),$$

$$y = b(b-c),$$

$$z = c(c-a).$$

We have

$$x^2 + y^2 + z^2 = \sum a^4 + \sum a^2b^2 - 2\sum a^3b$$

and

$$xy + yz + zx = \sum a^2b^2 - \sum ab^3.$$

Applying the known inequality

$$x^2 + y^2 + z^2 \ge xy + yz + zx,$$

the desired inequality follows.

Third Solution. Let

$$x = a^2 + bc + ca,$$

$$y = b^2 + ca + ab,$$

$$z = c^2 + ab + bc.$$

We have

$$x^2 + y^2 + z^2 = \sum a^4 + 2\sum a^2b^2 + 4abc\sum a + 2\sum ab^3$$

and

$$xy + yz + zx = 2\sum a^2b^2 + 4abc\sum a + 2\sum a^3b + \sum ab^3.$$

The known inequality

$$x^2 + y^2 + z^2 \ge xy + yz + zx$$

leads to the desired inequality.

Remark 1. The inequality is more interesting in the case $abc < 0$. If a, b, c are positive, then the inequality is less sharp than Vasc's inequality in P 1.125, because it can be obtained by adding Vasc's inequality and

$$ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2 \ge 0.$$

On the other hand, if a, b, c are positive, then the inequality

$$3(a^4 + b^4 + c^4) + 4(ab^3 + bc^3 + ca^3) \geq 7(a^3b + b^3c + c^3a)$$

is a refinement of the inequality in P 1.126. To prove this inequality, we write it as

$$3(a^4 + b^4 + c^4 - a^3b - b^3c - c^3a) + 4(ab^3 + bc^3 + ca^3 - a^3b - b^3c - c^3a) \geq 0,$$

consider $a = \min\{a, b, c\}$ and use the substitution

$$b = a + p, \quad c = a + q, \quad a > 0, \ p \geq 0, \ q \geq 0.$$

Since

$$\sum a^4 - \sum a^3b = \sum a^3(a - b)$$
$$= 3(p^2 - pq + q^2)a^2 + 3(p^3 - p^2q + q^3)a + p^4 - p^3q + q^4$$

and

$$\sum ab^3 - \sum a^3b = (a + b + c)(a - b)(b - c)(c - a)$$
$$= pq(q - p)(3a + p + q),$$

the inequality becomes

$$Aa^2 + Ba + C \geq 0,$$

where

$$A = 9(p^2 - pq + q^2), \quad B = 3(3p^3 - 7p^2q + 4pq^2 + 3q^3),$$
$$C = 3p^4 - 7p^3q + 4pq^3 + 3q^4.$$

The inequality $Aa^2 + Ba + C \geq 0$ is true for $a > 0$ and $p, q \geq 0$, because

$$A \geq 0,$$

$$B = p(3p - 4q)^2 + q(p - 3q)^2 + 2pq(p + q) \geq 0,$$

$$3C = p(p + q)(3p - 5q)^2 + 5q^2\left(p - \frac{13q}{10}\right)^2 + \frac{11}{20}q^4 \geq 0.$$

Remark 2. Let

$$E = a^4 + b^4 + c^4 + ab^3 + bc^3 + ca^3 - 2(a^3b + b^3c + c^3a).$$

Using the notations from the first solution, the formula

$$4A(Aa^2 + Ba + C) = (2Aa + B)^2 - \delta$$

leads to the following identity

$$4E_1E = (A_1 - 3C_1 + 2D_1)^2 + 3(A_1 - 2B_1 + C_1)^2,$$

where

$$A_1 = a^3 + b^3 + c^3, \quad B_1 = a^2b + b^2c + c^2a, \quad C_1 = ab^2 + bc^2 + ca^2, \quad D_1 = 3abc,$$

$$E_1 = a^2 + b^2 + c^2 - ab - bc - ca.$$

Remark 3. Let

$$E = a^4 + b^4 + c^4 + ab^3 + bc^3 + ca^3 - 2(a^3b + b^3c + c^3a).$$

The identity

$$x^2 + y^2 + z^2 - xy - yz - zx = \frac{1}{2}\sum(x-y)^2,$$

where x, y, z are defined in the second or third solution, leads to the identity

$$2E = \sum(a^2 - b^2 - ab + bc)^2.$$

In addition, the following similar identities hold:

$$6E = \sum(2a^2 - b^2 - c^2 - 2ab + bc + ca)^2,$$

$$4E = (2a^2 - b^2 - c^2 - 2ab + bc + ca)^2 + 3(b^2 - c^2 - bc + ca)^2.$$

Remark 4. The inequalities in P 1.125 and P 1.126 are particular cases of the following more general statement (*Vasile Cîrtoaje*, 2007).

- *Let*

$$f_4(a,b,c) = \sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3,$$

where A, B, C, D are real constants such that

$$1 + A + B + C + D = 0, \qquad 3(1+A) \ge C^2 + CD + D^2.$$

If a, b, c are real numbers, then

$$f_4(a,b,c) \ge 0.$$

Note that the following identity holds:

$$4Sf_4(a,b,c) = [U+V+(C+D)S]^2 + 3\left(U - V + \frac{C-D}{3}S\right)^2 + \frac{4}{3}(3+3A-C^2-CD-D^2)S^2,$$

where

$$S = \sum a^2b^2 - \sum a^2bc,$$

$$U = \sum a^3b - \sum a^2bc,$$

$$V = \sum ab^3 - \sum a^2bc.$$

For the main case

$$3(1+A)=C^2+CD+D^2,$$

the inequality $f_4(a,b,c)\geq 0$ is equivalent to each of the following two inequalities

$$\sum[2a^2-b^2-c^2+Cab-(C+D)bc+Dca]^2\geq 0,$$

$$\sum[3b^2-3c^2+(C+2D)ab+(C-D)bc-(2C+D)ca]^2\geq 0.$$

□

P 1.127. *If a,b,c are positive real numbers, then*

(a) $$\frac{a^2}{ab+2c^2}+\frac{b^2}{bc+2a^2}+\frac{c^2}{ca+2b^2}\geq 1;$$

(b) $$\frac{a^3}{a^2b+2c^3}+\frac{b^3}{b^2c+2a^3}+\frac{c^3}{c^2a+2b^3}\geq 1.$$

Solution. (a) By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^2}{ab+2c^2}\geq\frac{\left(\sum a^2\right)^2}{\sum a^2(ab+2c^2)}=\frac{\left(\sum a^2\right)^2}{\sum a^3b+2\sum a^2b^2}.$$

Therefore, it suffices to show that

$$\left(\sum a^2\right)^2\geq 2\sum a^2b^2+\sum a^3b.$$

We get this inequality by summing the known inequality

$$\frac{2}{3}\left(\sum a^2\right)^2\geq 2\sum a^2b^2$$

and Vasc's inequality

$$\frac{1}{3}\left(\sum a^2\right)^2\geq\sum a^3b.$$

The equality holds for $a=b=c=1$.

(b) By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a^3}{a^2b+2c^3}\geq\frac{\left(\sum a^2\right)^2}{\sum a(a^2b+2c^3)}=\frac{\left(\sum a^2\right)^2}{\sum a^3b+2\sum ac^3}=\frac{\left(\sum a^2\right)^2}{3\sum a^3b}.$$

Therefore, it suffices to show that

$$\left(\sum a^2\right)^2\geq 3\sum a^3b,$$

which is just Vasc's inequality. The equality holds for $a=b=c=1$.

□

P 1.128. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a}{ab+1}+\frac{b}{bc+1}+\frac{c}{ca+1}\geq\frac{3}{2}.$$

Solution. We use the following hint

$$\frac{a}{ab+1}=a-\frac{a^2b}{ab+1},\quad \frac{b}{bc+1}=b-\frac{b^2c}{bc+1},\quad \frac{c}{ca+1}=c-\frac{c^2a}{ca+1},$$

which transforms the desired inequality into

$$\frac{a^2b}{ab+1}+\frac{b^2c}{bc+1}+\frac{c^2a}{ca+1}\leq\frac{3}{2}.$$

By the AM-GM inequality, we have

$$ab+1\geq 2\sqrt{ab},\quad bc+1\geq 2\sqrt{bc},\quad ca+1\geq 2\sqrt{ca}.$$

Consequently, it suffices to show that

$$\frac{a^2b}{2\sqrt{ab}}+\frac{b^2c}{2\sqrt{bc}}+\frac{c^2a}{2\sqrt{ca}}\leq\frac{3}{2},$$

which is equivalent to

$$a\sqrt{ab}+b\sqrt{bc}+c\sqrt{ca}\leq 3,$$

$$3(a\sqrt{ab}+b\sqrt{bc}+c\sqrt{ca})\leq(a+b+c)^2.$$

Replacing $\sqrt{a},\sqrt{b},\sqrt{c}$ by a,b,c, respectively, we get Vasc's inequality in P 1.125. The equality holds for $a=b=c=1$.

□

P 1.129. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a}{3a+b^2}+\frac{b}{3b+c^2}+\frac{c}{3c+a^2}\leq\frac{3}{2}.$$

(Vasile C., 2007)

Solution. Since

$$\frac{a}{3a+b^2}=\frac{1}{3}-\frac{b^2}{3(3a+b^2)},\quad \frac{b}{3b+c^2}=\frac{1}{3}-\frac{c^2}{3(3b+c^2)},\quad \frac{c}{3c+a^2}=\frac{1}{3}-\frac{a^2}{3(3c+a)},$$

the desired inequality can be rewritten as

$$\frac{b^2}{3a+b^2}+\frac{c^2}{3b+c^2}+\frac{a^2}{3c+a^2}\geq\frac{3}{2}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{b^2}{3a+b^2} \ge \frac{\left(\sum b^2\right)^2}{\sum b^2(3a+b^2)} = \frac{\left(\sum a^2\right)^2}{\sum a^4 + \left(\sum a\right)\left(\sum ab^2\right)}$$

$$= \frac{\left(\sum a^2\right)^2}{\sum a^4 + \sum a^2b^2 + abc\sum a + \sum ab^3} \ge \frac{\left(\sum a^2\right)^2}{\left(\sum a^2\right)^2 + \sum ab^3}.$$

Thus, it is enough to show that

$$\left(\sum a^2\right)^2 \ge 3\sum ab^3,$$

which is Vasc's inequality. The equality holds for $a = b = c = 1$.

□

P 1.130. *If a, b, c are positive real numbers such that $a + b + c = 3$, then*

$$\frac{a}{b^2+c} + \frac{b}{c^2+a} + \frac{c}{a^2+b} \ge \frac{3}{2}.$$

(Pham Kim Hung, 2007)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a}{b^2+c} \ge \frac{\left(\sum a^{3/2}\right)^2}{\sum a^2(b^2+c)} = \frac{\sum a^3 + 2\sum a^{3/2}b^{3/2}}{\sum a^2b^2 + \sum ab^2}.$$

Thus, it is enough to show that

$$2\sum a^3 + 4\sum a^{3/2}b^{3/2} \ge 3\sum a^2b^2 + 3\sum ab^2,$$

which is equivalent to the homogeneous inequality

$$2\left(\sum a\right)\left(\sum a^3\right) + 4\left(\sum a\right)\left(\sum a^{3/2}b^{3/2}\right) \ge 9\sum a^2b^2 + 3\left(\sum a\right)\left(\sum ab^2\right).$$

In order to get a symmetric inequality, we use Vasc's inequality. We have

$$\begin{aligned} 3\left(\sum a\right)\left(\sum ab^2\right) &= 3\sum a^2b^2 + 3abc\sum a + 3\sum ab^3 \\ &\le 3\sum a^2b^2 + 3abc\sum a + \left(\sum a^2\right)^2 \\ &= \sum a^4 + 5\sum a^2b^2 + 3abc\sum a. \end{aligned}$$

Therefore, it suffices to prove the symmetric inequality

$$2\left(\sum a\right)\left(\sum a^3\right) + 4\left(\sum a\right)\left(\sum a^{3/2}b^{3/2}\right) \ge 9\sum a^2b^2 + \sum a^4 + 5\sum a^2b^2 + 3abc\sum a,$$

which is equivalent to

$$\sum a^4+2\sum ab(a^2+b^2)+4abc\sum\sqrt{ab}+4A\ge 14\sum a^2b^2+3abc\sum a,$$

where

$$A=\sum(ab)^{3/2}(a+b).$$

Since

$$A\ge 2\sum a^2b^2,$$

it suffices to prove that

$$\sum a^4+2\sum ab(a^2+b^2)+4abc\sum\sqrt{ab}\ge 6\sum a^2b^2+3abc\sum a.$$

According to Schur's inequality of degree four

$$\sum a^4\ge\sum ab(a^2+b^2)-abc\sum a,$$

it is enough to show that

$$3\sum ab(a^2+b^2)+4abc\sum\sqrt{ab}\ge 6\sum a^2b^2+4abc\sum a.$$

Write this inequality as

$$3\sum ab(a-b)^2\ge 2abc\sum\left(\sqrt{a}-\sqrt{b}\right)^2,$$

$$\sum ab\left(\sqrt{a}-\sqrt{b}\right)^2\left[3\left(\sqrt{a}+\sqrt{b}\right)^2-2c\right]\ge 0.$$

We will prove the stronger inequality

$$\sum ab\left(\sqrt{a}-\sqrt{b}\right)^2\left[\left(\sqrt{a}+\sqrt{b}\right)^2-c\right]\ge 0,$$

which is equivalent to

$$\sum\left(\frac{\sqrt{a}-\sqrt{b}}{\sqrt{c}}\right)^2\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right)\ge 0.$$

Substituting $x=\sqrt{a}$, $y=\sqrt{b}$, $z=\sqrt{c}$, the inequality becomes

$$\sum\left(\frac{x-y}{z}\right)^2(x+y-z)\ge 0.$$

Without loss of generality, assume that $x\ge y\ge z$. It suffices to show that

$$\left(\frac{y-z}{x}\right)^2(y+z-x)+\left(\frac{x-z}{y}\right)^2(z+x-y)\ge 0.$$

Since

$$\left(\frac{x-z}{y}\right)^2 \ge \left(\frac{y-z}{x}\right)^2,$$

we have

$$\left(\frac{y-z}{x}\right)^2 (y+z-x) + \left(\frac{x-z}{y}\right)^2 (z+x-y) \ge$$

$$\ge \left(\frac{y-z}{x}\right)^2 (y+z-x) + \left(\frac{y-z}{x}\right)^2 (z+x-y)$$

$$= 2z\left(\frac{y-z}{x}\right)^2 \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.131. *If a,b,c are positive real numbers such that $abc=1$, then*

$$\frac{a}{b^3+2} + \frac{b}{c^3+2} + \frac{c}{a^3+2} \ge 1.$$

Solution. Using the substitution

$$a = \frac{x}{y}, \quad b = \frac{z}{x}, \quad c = \frac{y}{z}, \qquad x,y,z > 0,$$

the inequality turns into

$$\sum \frac{x^4}{y(2x^3+z^3)} \ge 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{x^4}{y(2x^3+z^3)} \ge \frac{\left(\sum x^2\right)^2}{\sum y(2x^3+z^3)} = \frac{\left(\sum x^2\right)^2}{2\sum x^3y + \sum xy^3}.$$

Thus, it is enough to show that

$$\left(\sum x^2\right)^2 \ge 2\sum x^3y + \sum xy^3.$$

According to Vasc's inequality, we have

$$\left(\sum x^2\right)^2 \ge 3\sum x^3y$$

and

$$\left(\sum x^2\right)^2 \ge 3\sum xy^3.$$

Thus, the conclusion follows. The equality holds for $a=b=c=1$.

□

P 1.132. *Let a,b,c be positive real numbers such that*

$$a^m+b^m+c^m=3,$$

where $m>0$. Prove that

$$\frac{a^{m-1}}{b}+\frac{b^{m-1}}{c}+\frac{c^{m-1}}{a}\ge 3.$$

Solution. Making the substitution

$$x=a^{\frac{1}{k}},\quad y=b^{\frac{1}{k}},\quad z=c^{\frac{1}{k}},$$

where

$$k=\frac{2}{m},\quad k>0,$$

we need to show that $x^2+y^2+z^2=3$ yields

$$\frac{x^{2-k}}{y^k}+\frac{y^{2-k}}{z^k}+\frac{z^{2-k}}{x^k}\ge 3,$$

which is equivalent to

$$\frac{x^2}{(xy)^k}+\frac{y^2}{(yz)^k}+\frac{z^2}{(zx)^k}\ge 3.$$

Applying Jensen's inequality to the convex function $f(u)=\dfrac{1}{u^k}$, we get

$$\frac{x^2}{(xy)^k}+\frac{y^2}{(yz)^k}+\frac{z^2}{(zx)^k}\ge \frac{x^2+y^2+z^2}{\left(\dfrac{x^2\cdot xy+y^2\cdot yz+z^2\cdot zx}{x^2+y^2+z^2}\right)^k}$$

$$=\frac{3^{k+1}}{(x^3y+y^3z+z^3x)^k}.$$

Thus, it suffices to show that $x^3y+y^3z+z^3x\le 3$. This is just Vasc's inequality in P 1.125. The equality holds for $a=b=c=1$.

□

P 1.133. *If a,b,c are positive real numbers, then*

(a) $$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\ge 3\left(\frac{1}{3a+b}+\frac{1}{3b+c}+\frac{1}{3c+a}\right);$$

(b) $$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{a+3b}+\frac{1}{b+3c}+\frac{1}{c+3a}\ge 2\left(\frac{1}{3a+b}+\frac{1}{3b+c}+\frac{1}{3c+a}\right).$$

(Gabriel Dospinescu and Vasile Cîrtoaje, 2004)

Solution. We will prove that the following more general inequalities hold for $t \ge 0$:

$$\frac{t^{4a}}{4a}+\frac{t^{4b}}{4b}+\frac{t^{4c}}{4c}+\frac{t^{2a+2b}}{a+b}+\frac{t^{2b+2c}}{b+c}+\frac{t^{2c+2a}}{c+a}-3\left(\frac{t^{3a+b}}{3a+b}+\frac{t^{3b+c}}{3b+c}+\frac{t^{3c+a}}{3c+a}\right)\ge 0,$$

$$\frac{t^{4a}}{4a}+\frac{t^{4b}}{4b}+\frac{t^{4c}}{4c}+\frac{t^{a+3b}}{a+3b}+\frac{t^{b+3c}}{b+3c}+\frac{t^{c+3a}}{c+3a}-2\left(\frac{t^{3a+b}}{3a+b}+\frac{t^{3b+c}}{3b+c}+\frac{t^{3c+a}}{3c+a}\right)\ge 0.$$

For $t = 1$, we get the desired inequalities.

(a) Denoting the left hand side of the former inequality by $f(t)$, the inequality becomes $f(t) \ge f(0)$. This is true if $f'(t) \ge 0$ for $t > 0$. We have the derivative

$$tf'(t) = t^{4a}+t^{4b}+t^{4c}+2(t^{2a+2b}+t^{2b+2c}+t^{2c+2a})-3(t^{3a+b}+t^{3b+c}+t^{3c+a}).$$

Using the substitution $x = t^a$, $y = t^b$, $z = t^c$, the inequality $f'(t) \ge 0$ turns into

$$x^4+y^4+z^4+2(x^2y^2+y^2z^2+z^2x^2) \ge 3(x^3y+y^3z+z^3x),$$

which is Vasc's inequality in P 1.125. The equality holds for $a = b = c$.

(b) Similarly, we have the derivative

$$tf'(t) = t^{4a}+t^{4b}+t^{4c}+t^{a+3b}+t^{b+3c}+t^{c+3a}-2(t^{3a+b}+t^{3b+c}+t^{3c+a}).$$

Denoting $x = t^a$, $y = t^b$, $z = t^c$, the inequality $f'(t) \ge 0$ turns into

$$x^4+y^4+z^4+xy^3+yz^3+zx^3 \ge 2(x^3y+y^3z+z^3x),$$

which is the the inequality in P 1.126. The equality holds for $a = b = c$.

□

P 1.134. *If a, b, c are positive real numbers such that $a^6 + b^6 + c^6 = 3$, then*

$$\frac{a^5}{b}+\frac{b^5}{c}+\frac{c^5}{a} \ge 3.$$

(Tran Quoc Anh, 2007)

Solution. By Hölder's inequality, we have

$$\left(\frac{a^5}{b}+\frac{b^5}{c}+\frac{c^5}{a}\right)^3 \ge \frac{(a^6+b^6+c^6)^4}{a^9b^3+b^9c^3+c^9a^3} = \frac{81}{a^9b^3+b^9c^3+c^9a^3}.$$

Therefore, it suffices to show that

$$a^9b^3+b^9c^3+c^9a^3 \le 3.$$

This is equivalent to

$$3(a^9b^3+b^9c^3+c^9a^3) \le (a^6+b^6+c^6)^2,$$

which is Vasc's inequality (see P 1.125). The equality holds for $a = b = c$.

□

P 1.135. *If a,b,c are positive real numbers such that $a^2+b^2+c^2=3$, then*

$$\frac{a^3}{a+b^5}+\frac{b^3}{b+c^5}+\frac{c^3}{c+a^5}\ge\frac{3}{2}.$$

(Marin Bancos, 2010)

Solution. Write the inequality as

$$\sum\left(\frac{a^3}{a+b^5}-a^2\right)+\frac{3}{2}\ge 0,$$

$$\sum\frac{a^2b^5}{a+b^5}\le\frac{3}{2}.$$

Since

$$a+b^5\ge 2\sqrt{ab^5},$$

it suffices to show that

$$\sum ab^2\sqrt{ab}\le 3.$$

In addition, since $2\sqrt{ab}\le a+b$, it suffices to prove that

$$\sum a^2b^2+\sum ab^3\le 6.$$

This is true since

$$\sum a^2b^2\le\frac{1}{3}(a^2+b^2+c^2)^2=3,$$

and, according to Vasc's inequality,

$$\sum ab^3\le\frac{1}{3}(a^2+b^2+c^2)^2=3.$$

The equality holds for $a=b=c=1$.

□

P 1.136. *If a,b,c are real numbers such that $a^2+b^2+c^2=3$, then*

$$a^2b+b^2c+c^2a+9\ge 4(a+b+c).$$

(Vasile C., 2007)

First Solution *(by Nguyen Van Quy)*. Since

$$2a^2b=a^2(b^2+1)-a^2(b-1)^2,$$

we have

$$\begin{aligned}4\sum a^2b &= 2\sum a^2b^2 + 2\sum a^2 - 2\sum a^2(b-1)^2\\ &= \left(\sum a^2\right)^2 - \sum a^4 + 2\sum a^2 - 2\sum c^2(a-1)^2\\ &= 15 - \sum a^4 - 2\sum c^2(a-1)^2.\end{aligned}$$

Therefore, we can write the desired inequality as follows:

$$\left[15 - \sum a^4 - 2\sum c^2(a-1)^2\right] + 36 \ge 16\sum a,$$

$$\sum(17 - 16a - a^4) \ge 2\sum c^2(a-1)^2,$$

$$\sum(17 - 16a - a^4) + 10\sum(a^2-1) \ge 2\sum c^2(a-1)^2,$$

$$\sum(7 - 16a + 10a^2 - a^4) \ge 2\sum c^2(a-1)^2,$$

$$\sum(a-1)^2(7 - 2a - a^2) \ge 2\sum c^2(a-1)^2,$$

$$\sum(a-1)^2(7 - 2a - a^2 - 2c^2) \ge 0.$$

Since

$$7 - 2a - a^2 - 2c^2 = (a-1)^2 + 2(3 - a^2 - c^2) = (a-1)^2 + 2b^2 \ge 0,$$

the conclusion follows. The equality holds for $a = b = c = 1$.

Second Solution. Consider only the case where a, b, c are nonnegative and $a + b + c > 0$. Multiplying both sides by $a + b + c$, the inequality can be restated as

$$(a+b+c)(a^2b + b^2c + c^2a) + 9(a+b+c) \ge 4(a+b+c)^2.$$

Using the known inequality $\sum a^2b^2 \ge \frac{1}{3}\left(\sum ab\right)^2$ and Vasc's inequality $\sum ab^3 \le \frac{1}{3}\left(\sum a^2\right)^2$, we have

$$\begin{aligned}\left(\sum a\right)\left(\sum a^2b\right) &= \sum a^3b + \sum a^2b^2 + abc\sum a\\ &= \left(\sum a^2\right)\left(\sum ab\right) + \sum a^2b^2 - \sum ab^3\\ &\ge \left(\sum a^2\right)\left(\sum ab\right) + \frac{1}{3}\left(\sum ab\right)^2 - \frac{1}{3}\left(\sum a^2\right)^2\\ &= 3\sum ab + \frac{1}{3}\left(\sum ab\right)^2 - 3.\end{aligned}$$

Therefore, it suffices to prove the symmetric inequality

$$3\sum ab + \frac{1}{3}\left(\sum ab\right)^2 - 3 + 9\sum a \ge 4\left(\sum a\right)^2.$$

Setting $\sum a = p$, which involves

$$\sum ab = \frac{p^2-3}{2},$$

the inequality becomes

$$\frac{3(p^2-3)}{2} + \frac{(p^2-3)^2}{12} - 3 + 9p \ge 4p^2,$$

$$(p-3)^2(p^2+6p-9) \ge 0.$$

The last inequality is true since

$$p^2+6p-9 > 6p-9 \ge 6\sqrt{a^2+b^2+c^2} - 9 = 6\sqrt{3} - 9 > 0.$$

□

P 1.137. *If a,b,c are real numbers such that $a^2+b^2+c^2=3$, then*

$$a^2b+b^2c+c^2a+3 \ge a+b+c+ab+bc+ca.$$

(Vasile C., 2007)

Solution. Write the inequality as follows:

$$\sum(1-ab) - \sum a(1-ab) \ge 0,$$

$$\sum(a^2+b^2+c^2-3ab) - \sum a(a^2+b^2+c^2-3ab) \ge 0,$$

$$3\left(\sum a^2 - \sum ab\right) - \sum a(a-b)^2 - \sum a(c^2-ab) \ge 0,$$

$$\frac{3}{2}\sum(a-b)^2 - \sum a(a-b)^2 \ge 0,$$

$$\sum(a-b)^2(3-2a) \ge 0.$$

Assume that

$$a = \max\{a,b,c\}.$$

For $3-2a \ge 0$, the inequality is clearly true. Consider now that $3-2a < 0$. Since

$$(a-b)^2 = [(a-c)+(c-b)]^2 \le 2[(a-c)^2+(c-b)^2],$$

it suffices to show that

$$2[(a-c)^2+(c-b)^2](3-2a) + (b-c)^2(3-2b) + (c-a)^2(3-2c) \ge 0,$$

which can be rewritten as

$$(a-c)^2(9-4a-2c)+(b-c)^2(9-4a-2b)\geq 0.$$

This inequality is true because $9>4a+2c$ and $9>4a+2b$. For instance, the last inequality is true if $81>4(2a+b)^2$; indeed, we have

$$\frac{81}{4}-(2a+b)^2>15-(2a+b)^2=5(a^2+b^2+c^2)-(2a+b)^2=(a-2b)^2+5c^2\geq 0.$$

The equality holds for $a=b=c=1$.

Remark. The inequality in P 1.137 is sharper than the inequality in P 1.136, namely

$$a^2b+b^2c+c^2a+9\geq 4(a+b+c).$$

This claim is true if

$$a+b+c+ab+bc+ca-3\geq 4(a+b+c)-9;$$

that is,

$$ab+bc+ca+6\geq 3(a+b+c),$$

which is equivalent to

$$(a+b+c-3)^2\geq 0.$$

□

P 1.138. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\frac{12}{a^2b+b^2c+c^2a}\leq 3+\frac{1}{abc}.$$

(Vasile Cîrtoaje and ShengLi Chen, 2009)

Solution. Let

$$p=a+b+c=3,\quad q=ab+bc+ca,\quad r=abc\leq 1.$$

Write the inequality as

$$2(a^2b+b^2c+c^2a)\geq\frac{24r}{3r+1}.$$

From

$$\begin{aligned}(a-b)^2(b-c)^2(c-a)^2&=-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3\\&=-27r^2+54(q-2)r+9q^2-4q^3,\end{aligned}$$

we get

$$(a-b)(b-c)(c-a) \le \sqrt{-27r^2+54(q-2)r+9q^2-4q^3},$$

hence

$$\begin{aligned}2(a^2b+b^2c+c^2a) &= \sum ab(a+b)-(a-b)(b-c)(c-a)\\ &= pq-3r-(a-b)(b-c)(c-a)\\ &\ge 3q-3r-\sqrt{-27r^2+54(q-2)r+9q^2-4q^3}.\end{aligned}$$

Therefore, it suffices to show that

$$3q-3r-\sqrt{-27r^2+54(q-2)r+9q^2-4q^3} \ge \frac{24r}{3r+1}.$$

which is equivalent to

$$3[(3r+1)q-3r^2-9r] \ge (3r+1)\sqrt{-27r^2+54(q-2)r+9q^2-4q^3}.$$

Before squaring this inequality, we need to show that $(3r+1)q-3r^2-9r \ge 0$. Using the known inequality $q^2 \ge 3pr$, we have

$$\begin{aligned}(3r+1)q-3r^2-9r &\ge 3(3r+1)\sqrt{r}-3r^2-9r\\ &= 3\sqrt{r}\,\left(1-\sqrt{r}\right)^3 \ge 0.\end{aligned}$$

By squaring, the desired inequality can be restated as

$$Aq^3+C \ge 3Bq,$$

where

$$A=4(3r+1)^2, \quad B=72r(3r+1)(r+1), \quad C=108r(r+1)(3r^2+12r+1).$$

By the AM-GM inequality,

$$Aq^3+C=Aq^3+\frac{C}{2}+\frac{C}{2} \ge 3\sqrt[3]{Aq^3\left(\frac{C}{2}\right)^2};$$

so, it is enough to show that

$$AC^2 \ge 4B^3,$$

which is equivalent to

$$(3r^2+12r+1)^2 \ge 32r(3r+1)(r+1).$$

Indeed,

$$(3r^2+12r+1)^2-32r(3r+1)(r+1)=(r-1)^2(3r-1)^2 \ge 0.$$

The equality holds for $a=b=c=1$, and also for $r=\frac{1}{3}$ and $q=\sqrt[3]{\frac{C}{2A}}=2$; that is, when a,b,c are the roots of the equation

$$x^3-3x^2+2x-\frac{1}{3}=0$$

such that $a\le b\le c$ or $b\le c\le a$ or $c\le a\le b$.

□

P 1.139. *If a,b,c are positive real numbers such that $a+b+c=3$, then*

$$\frac{24}{a^2b+b^2c+c^2a}+\frac{1}{abc}\ge 9.$$

(Vasile C., 2009)

Solution *(by Vo Quoc Ba Can)*. Let us denote

$$p=a+b+c=3,\quad q=ab+bc+ca,\quad r=abc.$$

Write the inequality as

$$24r\ge(9r-1)(a^2b+b^2c+c^2a),$$

and consider further the nontrivial case

$$r\ge\frac{1}{9}.$$

From

$$\begin{aligned}(a-b)^2(b-c)^2(c-a)^2&=-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3\\&=-27r^2+54(q-2)r+9q^2-4q^3,\end{aligned}$$

we get

$$-(a-b)(b-c)(c-a)\le\sqrt{-27r^2+54(q-2)r+9q^2-4q^3},$$

hence

$$\begin{aligned}2(a^2b+b^2c+c^2a)&=\sum ab(a+b)-(a-b)(b-c)(c-a)\\&=pq-3r-(a-b)(b-c)(c-a)\\&\le 3q-3r+\sqrt{-27r^2+54(q-2)r+9q^2-4q^3}.\end{aligned}$$

Therefore, it suffices to show that

$$48r\ge(9r-1)\left[3q-3r+\sqrt{-27r^2+54(q-2)r+9q^2-4q^3}\right],$$

which is true if

$$3[9r^2+15r-(9r-1)q] \ge (9r-1)\sqrt{-27r^2+54(q-2)r+9q^2-4q^3}.$$

We need first to show that $9r^2+15r-(9r-1)q \ge 0$. From Schur's inequality

$$p^3+9r \ge 4pq,$$

we get

$$q \le \frac{3(r+3)}{4},$$

hence

$$9r^2+15r-(9r-1)q \ge 9r^2+15r-\frac{3(r+3)(9r-1)}{4} = \frac{9(r-1)^2}{4} \ge 0.$$

By squaring the desired inequality, we get

$$Aq^3+C \ge 3Bq,$$

where

$$A=(9r-1)^2, \quad B=18r(9r-1)(3r+1), \quad C=27r(27r^3+99r^2+r+1).$$

Using the AM-GM inequality, we have

$$Aq^3+C = Aq^3+\frac{C}{2}+\frac{C}{2} \ge 3\sqrt[3]{Aq^3\left(\frac{C}{2}\right)^2};$$

thus, it is enough to show that

$$AC^2 \ge 4B^3,$$

which is equivalent to

$$(27r^3+99r^2+r+1)^2 \ge 32r(9r-1)(3r+1)^3,$$

$$729r^6-2430r^5+2943r^4-1476r^3+199r^2+34r+1 \ge 0,$$

$$(r-1)^2(27r^2-18r-1)^2 \ge 0.$$

The equality holds for $a=b=c=1$, and also for $r=\dfrac{3+2\sqrt{3}}{9}$ and $q=1+\sqrt{3}$; that is, when a,b,c are the roots of the equation

$$x^3-3x^2+(1+\sqrt{3})x-\frac{3+2\sqrt{3}}{9}=0$$

such that $a \ge b \ge c$ or $b \ge c \ge a$ or $c \ge a \ge b$.

□

P 1.140. *Let a, b, c be nonnegative real numbers such that*

$$2(a^2+b^2+c^2) = 5(ab+bc+ca).$$

Prove that

(a) $$8(a^4+b^4+c^4) \ge 17(a^3b+b^3c+c^3a);$$

(b) $$16(a^4+b^4+c^4) \ge 34(a^3b+b^3c+c^3a)+81abc(a+b+c).$$

(Vasile C., 2011)

Solution. (a) Let

$$x = a^2+b^2+c^2, \quad y = ab+bc+ca, \qquad 2x = 5y.$$

Since the equality holds for $a=2$, $b=1$, $c=0$ (when $abc=0$), we will use the inequality

$$a^2b^2+b^2c^2+c^2a^2 \le y^2$$

to get

$$a^4+b^4+c^4 = x^2-2(a^2b^2+b^2c^2+c^2a^2) \ge x^2-2y^2,$$

hence

$$a^4+b^4+c^4 \ge x^2-2y^2 = \frac{17}{144}(2x+y)^2.$$

Therefore, it suffices to prove that

$$(2x+y)^2 \ge 18(a^3b+b^3c+c^3a).$$

We will show that this inequality holds for all nonnegative real numbers a, b, c. Assume that $a = \max\{a, b, c\}$. There are two possible cases: $a \ge b \ge c$ and $a \ge c \ge b$.

Case 1: $a \ge b \ge c$. Using the AM-GM inequality gives

$$2(a^3b+b^3c+c^3a) \le 2ab(a^2+bc+c^2) \le \left[\frac{2ab+(a^2+bc+c^2)}{2}\right]^2.$$

Therefore, it suffices to show that

$$2x+y \ge \frac{3}{2}(2ab+a^2+bc+c^2),$$

which is equivalent to the obvious inequality

$$(a-2b)^2+c(2a-b+c) \ge 0.$$

Case 2: $a \ge c \ge b$. Since

$$ab^3+bc^3+ca^3-(a^3b+b^3c+c^3a) = (a+b+c)(a-b)(b-c)(c-a) \ge 0,$$

we have

$$2(a^3b+b^3c+c^3a) \le (a^3b+b^3c+c^3a)+(ab^3+bc^3+ca^3) \le xy.$$

Thus, it suffices to prove that

$$(2x+y)^2 \ge 9xy.$$

Since $x \ge y$, we get

$$(2x+y)^2-9xy=(x-y)(4x-y) \ge 0.$$

Thus, the proof is completed. The equality holds for $a=2b$ and $c=0$ (or any cyclic permutation).

(b) For $a=b=c=0$, the inequality is trivial. Otherwise, let us denote

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc,$$

and write the inequality as

$$16\sum a^4 \ge 17\sum ab(a^2+b^2)+17\left(\sum a^3b-\sum ab^3\right)+81abc\sum a.$$

Due to homogeneity, we may assume that $p=3$, which involves $q=2$. Since

$$abc\sum a=3r,$$

$$\begin{aligned}\sum a^4 &= \left(\sum a^2\right)^2-2\sum a^2b^2\\ &=(p^2-2q)^2-2q^2+4pr=17+12r,\end{aligned}$$

$$\begin{aligned}\sum ab(a^2+b^2) &= \left(\sum ab\right)\left(\sum a^2\right)-abc\sum a\\ &=q(p^2-2q)-pr=10-3r,\end{aligned}$$

$$\begin{aligned}\sum a^3b-\sum ab^3 &= -p(a-b)(b-c)(c-a)\\ &\le p\sqrt{(a-b)^2(b-c)^2(c-a)^2}\\ &=p\sqrt{p^2q^2-4q^3+2p(9q-2p^2)r-27r^2}\\ &=3\sqrt{4-27r^2},\end{aligned}$$

it suffices to prove that

$$16(17+12r) \ge 17(10-3r)+51\sqrt{4-27r^2}+243r,$$

which is equivalent to the obvious inequality

$$2 \ge \sqrt{4-27r^2}.$$

The equality holds for $a=2b$ and $c=0$ (or any cyclic permutation).

□

P 1.141. *Let a,b,c be nonnegative real numbers such that*

$$2(a^2+b^2+c^2)=5(ab+bc+ca).$$

Prove that

(a) $$2(a^3b+b^3c+c^3a)\ge a^2b^2+b^2c^2+c^2a^2+abc(a+b+c);$$

(b) $$11(a^4+b^4+c^4)\ge 17(a^3b+b^3c+c^3a)+129abc(a+b+c);$$

(c) $$a^3b+b^3c+c^3a\le \frac{14+\sqrt{102}}{8}(a^2b^2+b^2c^2+c^2a^2).$$

Solution. For $a=b=c=0$, the inequalities are trivial. Otherwise, let us denote

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc.$$

Due to homogeneity, we may assume that $p=3$, which involves $q=2$. From

$$\begin{aligned}\left|\sum a^3b-\sum ab^3\right| &= |-p(a-b)(b-c)(c-a)|\\ &= p\sqrt{(a-b)^2(b-c)^2(c-a)^2}\\ &= p\sqrt{p^2q^2-4q^3+2p(9q-2p^2)r-27r^2}\\ &= 3\sqrt{4-27r^2},\end{aligned}$$

it follows that

$$-3\sqrt{4-27r^2}\le \sum a^3b-\sum ab^3\le 3\sqrt{4-27r^2}.$$

In addition, we have

$$abc\sum a=3r,$$

$$\sum a^2b^2=q^2-2pr=4-6r,$$

$$\sum ab(a^2+b^2)=q(p^2-2q)-pr=10-3r,$$

$$\sum a^4=p^4-4p^2q+2q^2+4pr=17+12r.$$

(a) Write the inequality as

$$\sum ab(a^2+b^2)+\left(\sum a^3b-\sum ab^3\right)\ge \sum a^2b^2+abc\sum a.$$

It suffices to prove that

$$10-3r-3\sqrt{4-27r^2}\ge 4-6r+3r,$$

which is equivalent to the obvious inequality

$$2 \ge \sqrt{4-27r^2}.$$

The equality holds for $a = 0$ and $2b = c$ (or any cyclic permutation).

(b) Write the inequality as

$$22\sum a^4 \ge 17\sum ab(a^2+b^2) + 17\left(\sum a^3b - \sum ab^3\right) + 258abc\sum a.$$

It suffices to prove that

$$22(17+12r) \ge 17(10-3r) + 51\sqrt{4-27r^2} + 774r$$

for $0 \le r \le \dfrac{2}{3\sqrt{3}}$. Write this inequality as

$$4-9r \ge \sqrt{4-27r^2}.$$

We have $4-9r \ge 4-2\sqrt{3} > 0$. By squaring, the inequality becomes

$$(4-9r)^2 \ge 4-27r^2,$$

$$(3r-1)^2 \ge 0.$$

For $p = 3$, the equality holds when $q = 2$, $r = \dfrac{1}{3}$ and $(a-b)(b-c)(c-a) \le 0$. In general, the equality holds when a, b, c are proportional to the roots of the equation

$$3x^3 - 9x^2 + 6x - 1 = 0$$

and satisfy

$$(a-b)(b-c)(c-a) \le 0.$$

This occurs when (*Wolfgang Berndt*)

$$a\sin^2\frac{\pi}{9} = b\sin^2\frac{2\pi}{9} = c\sin^2\frac{4\pi}{9}.$$

(c) Write the inequality as

$$\sum ab(a^2+b^2) + \left(\sum a^3b - \sum ab^3\right) \le k(a^2b^2+b^2c^2+c^2a^2),$$

where

$$k = \frac{14+\sqrt{102}}{4}.$$

It suffices to prove that

$$10-3r+3\sqrt{4-27r^2} \le k(4-6r),$$

where $r \le \dfrac{2}{3\sqrt{3}}$. Write this inequality as

$$3\sqrt{4-27r^2} \le 4k-10-3(2k-1)r.$$

We have

$$4k-10-3(2k-1)r \ge 4k-10-\frac{2(2k-1)}{\sqrt{3}} = 4\left(1-\frac{1}{\sqrt{3}}\right)k-10+\frac{2}{\sqrt{3}} > 0.$$

By squaring, the inequality becomes

$$9(4-27r^2) \le [4k-10-3(2k-1)r]^2,$$

which is equivalent to

$$(r-k_1)^2 \ge 0,$$

where

$$k_1 = \frac{2}{129}\sqrt{\frac{787+72\sqrt{102}}{3}} \approx 0.3483.$$

For $p=3$, the equality holds when $q=2$, $r=k_1$ and $(a-b)(b-c)(c-a) \le 0$. In general, the equality holds when a,b,c are proportional to the roots of the equation

$$x^3-3x^2+2x-k_1=0$$

and satisfy

$$(a-b)(b-c)(c-a) \le 0.$$

□

P 1.142. *If a,b,c are real numbers such that*

$$a^3b+b^3c+c^3a \le 0,$$

then

$$a^2+b^2+c^2 \ge k(ab+bc+ca),$$

where

$$k = \frac{1+\sqrt{21+8\sqrt{7}}}{2} \approx 3.7468.$$

(Vasile C., 2012)

Solution. Let us denote

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc.$$

If $p=0$, then

$$3(ab+bc+ca) \le (a+b+c)^2 = 0,$$

hence
$$a^2+b^2+c^2 \ge 0 \ge k(ab+bc+ca).$$
Consider now that $p \neq 0$ and use the contradiction method. It suffices to prove that
$$a^2+b^2+c^2 < k(ab+bc+ca)$$
involves
$$a^3b+b^3c+c^3a > 0.$$
Since the statement remains unchanged by replacing a,b,c with $-a,-b,-c$, respectively, we may consider that $p>0$. In addition, due to homogeneity, we may assume that $p=1$. From the hypothesis $a^2+b^2+c^2 < k(ab+bc+ca)$, we get
$$q > \frac{1}{k+2}.$$
Write the desired inequality as
$$\sum ab(a^2+b^2)+\sum a^3b-\sum ab^3 > 0.$$
Since
$$\sum ab(a^2+b^2) = q(p^2-2q)-pr = q-2q^2-r$$
and
$$\sum a^3b-\sum ab^3 = -p(a-b)(b-c)(c-a) \ge -p\sqrt{(a-b)^2(b-c)^2(c-a)^2}$$
$$= -p\sqrt{p^2q^2-4q^3+2p(9q-2p^2)r-27r^2} = -\sqrt{q^2-4q^3+2(9q-2)r-27r^2},$$
it suffices to prove that
$$q-2q^2-r > \sqrt{q^2-4q^3+2(9q-2)r-27r^2}.$$
From $p^2 \ge 3q$, we get
$$\frac{1}{k+2} < q \le \frac{1}{3},$$
and from $q^2 \ge 3pr$, we get $r \le q^2/3$; therefore,
$$q-2q^2-r \ge q-2q^2-\frac{q^2}{3} = q\left(1-\frac{7q}{3}\right) > 0.$$
By squaring, the desired inequality can be restated as
$$(q-2q^2-r)^2 > q^2-4q^3+2(9q-2)r-27r^2,$$
$$7r^2+(1-5q+q^2)r+q^4 > 0.$$
This is true if the discriminant
$$D = (1-5q+q^2)^2-28q^4 = [1-5q+(1+2\sqrt{7})q^2][1-5q+(1-2\sqrt{7})q^2]$$

is negative. Since

$$1-5q+(1+2\sqrt{7})q^2=\left(1-\frac{5q}{2}\right)^2+\frac{8\sqrt{7}-21}{4}q^2>0,$$

we only need to show that $f(q)>0$, where

$$f(q)=(2\sqrt{7}-1)q^2+5q-1.$$

Since $q>\dfrac{1}{k+2}$, we have

$$f(q)>\frac{2\sqrt{7}-1}{(k+2)^2}+\frac{5}{k+2}-1=0.$$

For $p=1$, the equality holds when $(a-b)(b-c)(c-a)>0$ and

$$q=\frac{1}{k+2},\quad r=\frac{-q^2}{\sqrt{7}}=-\frac{1}{\sqrt{7}(k+2)^2}.$$

In general, the equality holds when a,b,c are proportional to the roots of the equation

$$w^3-w^2+\frac{1}{k+2}w+\frac{1}{\sqrt{7}(k+2)^2}=0$$

and satisfy $(a-b)(b-c)(c-a)>0$.

□

P 1.143. *If a,b,c are real numbers such that*

$$a^3b+b^3c+c^3a\geq 0,$$

then

$$a^2+b^2+c^2+k(ab+bc+ca)\geq 0,$$

where

$$k=\frac{-1+\sqrt{21+8\sqrt{7}}}{2}\approx 2.7468.$$

(Vasile C., 2012)

Solution. Let us denote

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc.$$

At least two of a,b,c have the same sign; let b and c be these numbers. If $p=0$, then the hypothesis $a^3b+b^3c+c^3a\geq 0$ can be written as

$$-(b+c)^3b+b^3c-c^3(b+c)\geq 0.$$

Clearly, this inequality is satisfied only for $a = b = c = 0$, when the desired inequality is trivial. Consider further that $p \neq 0$ and use the contradiction method. It suffices to prove that

$$a^2 + b^2 + c^2 + k(ab + bc + ca) < 0$$

involves

$$a^3b + b^3c + c^3a < 0.$$

Since the statement remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively, we may consider $p > 0$. In addition, due to homogeneity, we may assume $p = 1$. From the hypothesis $a^2 + b^2 + c^2 + k(ab + bc + ca) < 0$, we get

$$q < \frac{-1}{k-2} \approx -1.339.$$

Write the desired inequality as

$$\sum ab(a^2 + b^2) + \sum a^3b - \sum ab^3 < 0,$$

Since

$$\sum ab(a^2 + b^2) = q(p^2 - 2q) - pr = q - 2q^2 - r$$

and

$$\sum a^3b - \sum ab^3 = -p(a-b)(b-c)(c-a) \leq p\sqrt{(a-b)^2(b-c)^2(c-a)^2}$$

$$= p\sqrt{p^2q^2 - 4q^3 + 2p(9q - 2p^2)r - 27r^2} = \sqrt{q^2 - 4q^3 + 2(9q-2)r - 27r^2},$$

it suffices to prove that

$$\sqrt{q^2 - 4q^3 + 2(9q-2)r - 27r^2} < r + 2q^2 - q.$$

Since $q < -1$, we have

$$\frac{1-2q}{3} > 1,$$

hence

$$r^2 = a^2b^2c^2 \leq \left(\frac{a^2 + b^2 + c^2}{3}\right)^3 = \left(\frac{1-2q}{3}\right)^3 < \left(\frac{1-2q}{3}\right)^4,$$

which implies

$$r > -\left(\frac{1-2q}{3}\right)^2.$$

Therefore,

$$r + 2q^2 - q > -\left(\frac{1-2q}{3}\right)^2 + 2q^2 - q = \frac{(2q-1)(7q+1)}{9} > 0.$$

By squaring, the desired inequality becomes

$$q^2 - 4q^3 + 2(9q-2)r - 27r^2 < (r + 2q^2 - q)^2,$$

$$7r^2 + (1 - 5q + q^2)r + q^4 > 0.$$

This is true if the discriminant

$$D = (1-5q+q^2)^2 - 28q^4 = [1-5q+(1+2\sqrt{7})q^2][1-5q+(1-2\sqrt{7})q^2]$$

is negative. Since

$$1 - 5q + (1+2\sqrt{7})q^2 > 0,$$

we only need to show that $f(q) > 0$, where

$$f(q) = (2\sqrt{7}-1)q^2 + 5q - 1.$$

Since the derivative

$$f'(q) = 2(2\sqrt{7}-1)q + 5 < 2(2\sqrt{7}-1)(-1) + 5 = 7 - 4\sqrt{7} < 0,$$

$f(q)$ is strictly decreasing, hence

$$f(q) > f\left(\frac{-1}{k-2}\right) = 0.$$

For $p = 1$, the equality holds when $(a-b)(b-c)(c-a) < 0$ and

$$q = \frac{-1}{k-2}, \quad r = \frac{-q^2}{\sqrt{7}} = \frac{-1}{\sqrt{7}(k-2)^2}.$$

In general, the equality holds when a, b, c are proportional to the roots of the equation

$$w^3 - w^2 - \frac{1}{k-2}w + \frac{1}{\sqrt{7}(k-2)^2} = 0$$

and satisfy $(a-b)(b-c)(c-a) < 0$.

□

P 1.144. *If a, b, c are real numbers such that*

$$k(a^2+b^2+c^2) = ab + bc + ca, \qquad k \in \left(\frac{-1}{2}, 1\right),$$

then

$$\alpha_k \le \frac{a^3b + b^3c + c^3}{(a^2+b^2+c^2)^2} \le \beta_k,$$

where

$$27\alpha_k = 1+13k-5k^2-2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}},$$

$$27\beta_k = 1+13k-5k^2+2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}}.$$

(Vasile C., 2012)

Solution. Let us denote

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

The case $p = 0$ is not possible, because $p = 0$ and $k(a^2+b^2+c^2) = ab+bc+ca$ lead to

$$ab+bc+ca = 0,$$

$$a(b+c)+bc = 0,$$

$$-(b+c)^2+bc = 0,$$

$$b^2+bc+c^2 = 0,$$

which involves $a = b = c = 0$. Consider further that $p \neq 0$. Since the statement remains unchanged by replacing a, b, c with $-a, -b, -c$, respectively, it suffices to consider the case $p > 0$. In addition, due to homogeneity, we may assume $p = 1$, which implies

$$q = \frac{k}{1+2k}.$$

(a) Write the desired left inequality as

$$2\alpha_k(a^2+b^2+c^2)^2 \le \sum ab(a^2+b^2) + \left(\sum a^3b - \sum ab^3\right).$$

Since

$$\sum a^2 = p^2-2q = 1-2q,$$

$$\sum ab(a^2+b^2) = q(p^2-2q)-pr = q-2q^2-r,$$

$$\sum a^3b - \sum ab^3 = -p(a-b)(b-c)(c-a) \ge -p\sqrt{(a-b)^2(b-c)^2(c-a)^2}$$

$$= -p\sqrt{\frac{4(p^2-3q)^3-(2p^3-9pq+27r)^2}{27}} = -\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}},$$

it suffices to prove that

$$2\alpha_k(1-2q)^2 \le q-2q^2-r-\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}}.$$

Applying Lemma below for

$$\alpha = \frac{1}{\sqrt{27}}, \quad \beta = \frac{-1}{27}, \quad x = 2(1-3q)\sqrt{1-3q}, \quad y = 2-9q+27r,$$

we get

$$\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}} + r + \frac{2-9q}{27} \le \frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

with equality for

$$(1-3q)\sqrt{\frac{1-3q}{7}} - 2 + 9q - 27r = 0.$$

Thus, it suffices to show that

$$2\alpha_k(1-2q)^2 \le q - 2q^2 + \frac{2-9q}{27} - \frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

which is equivalent to

$$27\alpha_k \le 1 + 13k - 5k^2 - 2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}}.$$

For $p = 1$, the equality holds when $(a-b)(b-c)(c-a) \ge 0$, $q = k/(1+2k)$ and

$$27r = (1-3q)\sqrt{\frac{1-3q}{7}} - 2 + 9q = \frac{r_1}{1+2k},$$

where

$$r_1 = 5k - 2 + (1-k)\sqrt{\frac{1-k}{7(1+2k)}}.$$

Therefore, the equality holds when a, b, c are proportional to the roots of the equation

$$w^3 - w^2 + \frac{k}{1+2k}w - \frac{r_1}{27(1+2k)} = 0$$

and satisfy $(a-b)(b-c)(c-a) \ge 0$.

(b) Write the desired right inequality as

$$2\beta_k(a^2+b^2+c^2)^2 \ge \sum ab(a^2+b^2) + \left(\sum a^3b - \sum ab^3\right).$$

Since

$$\sum a^2 = p^2 - 2q = 1 - 2q,$$

$$\sum ab(a^2+b^2) = q(p^2-2q) - pr = q - 2q^2 - r,$$

$$\sum a^3b - \sum ab^3 = -p(a-b)(b-c)(c-a) \le p\sqrt{(a-b)^2(b-c)^2(c-a)^2}$$

$$= p\sqrt{\frac{4(p^2-3q)^3-(2p^3-9pq+27r)^2}{27}} = \sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}},$$

it suffices to prove that

$$2\beta_k(1-2q)^2 \ge q-2q^2-r+\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}}.$$

Applying Lemma below for

$$\alpha=\frac{1}{\sqrt{27}},\quad \beta=\frac{1}{27},\quad x=2(1-3q)\sqrt{1-3q},\quad y=2-9q+27r,$$

we get

$$\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}}-r-\frac{2-9q}{27} \le \frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

with equality for

$$(1-3q)\sqrt{\frac{1-3q}{7}}+2-9q+27r=0.$$

Thus, it suffices to show that

$$2\beta_k(1-2q)^2 \ge q-2q^2+\frac{2-9q}{27}+\frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

which is equivalent to

$$27\beta_k \ge 1+13k-5k^2+2(1-k)(1+2k)\sqrt{\frac{7(1-k)}{1+2k}}.$$

For $p=1$, the equality holds when $(a-b)(b-c)(c-a)\le 0$, $q=k/(1+2k)$ and

$$27r=9q-2-(1-3q)\sqrt{\frac{1-3q}{7}}=\frac{r_0}{1+2k},$$

where

$$r_0=5k-2-(1-k)\sqrt{\frac{1-k}{7(1+2k)}}.$$

Therefore, the equality holds when a,b,c are proportional to the roots of the equation

$$w^3-w^2+\frac{k}{1+2k}w-\frac{r_0}{27(1+2k)}=0$$

and satisfy $(a-b)(b-c)(c-a)\le 0$.

Lemma. *If α,β,x,y are real numbers such that*

$$\alpha\ge 0,\quad x\ge 0,\quad x^2\ge y^2,$$

then

$$\alpha\sqrt{x^2-y^2} \le x\sqrt{\alpha^2+\beta^2}+\beta y,$$

with equality if and only if

$$\beta x + y\sqrt{\alpha^2+\beta^2} = 0.$$

Proof. Since

$$x\sqrt{\alpha^2+\beta^2}+\beta y \ge |\beta|x+\beta y \ge |\beta||y|+\beta y \ge 0,$$

we can write the inequality as

$$\alpha^2(x^2-y^2) \le \left(x\sqrt{\alpha^2+\beta^2}+\beta y\right)^2,$$

which is equivalent to

$$\left(\beta x + y\sqrt{\alpha^2+\beta^2}\right)^2 \ge 0.$$

□

P 1.145. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$\frac{a^2}{4a+b^2}+\frac{b^2}{4b+c^2}+\frac{c^2}{4c+a^2} \ge \frac{3}{5}.$$

(Michael Rozenberg, 2008)

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2}{4a+b^2} \ge \frac{\left[\sum a(2a+c)\right]^2}{\sum(4a+b^2)(2a+c)^2} = \frac{\left(2\sum a^2+\sum ab\right)^2}{\sum(4a+b^2)(2a+c)^2}.$$

Therefore, it suffices to show that

$$5\left(2\sum a^2+\sum ab\right)^2 \ge 3\sum(4a+b^2)(2a+c)^2,$$

which is equivalent to the homogeneous inequalities

$$5\left(2\sum a^2+\sum ab\right)^2 \ge \sum[4a(a+b+c)+3b^2](2a+c)^2,$$

$$5\left(2\sum a^2+\sum ab\right)^2 \ge \sum(4a^2+3b^2+4ab+4ac)(4a^2+c^2+4ac),$$

$$2\sum a^4+5\sum a^2b^2 \ge abc\sum a+6\sum ab^3.$$

Using Vasc's inequality

$$3\sum ab^3 \le \left(\sum a^2\right)^2,$$

it is enough to prove the symmetric inequality

$$2\sum a^4+5\sum a^2b^2 \ge abc\sum a+2\left(\sum a^2\right)^2,$$

which is equivalent to the well-known inequality

$$\sum a^2b^2 \ge abc\sum a.$$

The equality holds for $a=b=c=1$.

□

P 1.146. *If a,b,c are positive real numbers, then*

$$\frac{a^2+bc}{a+b}+\frac{b^2+ca}{b+c}+\frac{c^2+ab}{c+a} \le \frac{(a+b+c)^3}{3(ab+bc+ca)}.$$

(Michael Rozenberg, 2013)

Solution *(by Manlio Marangelli)*. Write the inequality as

$$\sum\left(\frac{a^2+bc}{a+b}-a\right) \le \frac{(a+b+c)^3}{3(ab+bc+ca)}-(a+b+c),$$

$$\sum\frac{b(c-a)}{a+b} \le \frac{(a+b+c)^3}{3(ab+bc+ca)}-(a+b+c),$$

$$\frac{\sum b(c^2-a^2)(b+c)}{(a+b)(b+c)(c+a)} \le \frac{(a+b+c)^3}{3(ab+bc+ca)}-(a+b+c),$$

$$\frac{3\sum ab^3-3abc\sum a}{(a+b)(b+c)(c+a)} \le \frac{(a+b+c)^3}{ab+bc+ca}-3(a+b+c).$$

By the known Vasc's inequality

$$3\sum ab^3 \le \left(\sum a^2\right)^2,$$

it suffices to prove the symmetric inequality

$$\frac{\left(\sum a^2\right)^2-3abc\sum a}{(a+b)(b+c)(c+a)} \le \frac{(a+b+c)^3}{ab+bc+ca}-3(a+b+c).$$

Using the notation

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc,$$

this inequality can be written as

$$\frac{(p^2-2q)^2-3pr}{pq-r} \le \frac{p^3}{q} - 3p,$$

which is equivalent to

$$q^2(p^2-4q)-(p^2-6q)pr \ge 0.$$

Case 1: $p^2-6q \ge 0$. Since $3pr \le q^2$, we have

$$q^2(p^2-4q)-(p^2-6q)pr \ge q^2(p^2-4q) - \frac{q^2(p^2-6q)}{3} = \frac{2q^2(p^2-3q)}{3} \ge 0.$$

Case 2: $p^2-6q \le 0$. Using Schur's inequality of fourth degree

$$6pr \ge (p^2-q)(4q-p^2),$$

we get

$$q^2(p^2-4q)-(p^2-6q)pr \ge q^2(p^2-4q) - \frac{(p^2-6q)(p^2-q)(4q-p^2)}{6}$$
$$= \frac{(p^2-3q)(p^2-4q)^2}{6} \ge 0.$$

The equality holds for $a=b=c=1$.

□

P 1.147. *If a, b, c are positive real numbers such that $a+b+c=3$, then*

$$\sqrt{ab^2+bc^2}+\sqrt{bc^2+ca^2}+\sqrt{ca^2+ab^2} \le 3\sqrt{2}.$$

(Nguyen Van Quy, 2013)

Solution *(by Michael Rozenberg)*. By the Cauchy-Schwarz inequality, we have

$$\left(\sum \sqrt{ab^2+bc^2}\right)^2 \le \sum \frac{ab+c^2}{a+c} \cdot \sum b(a+c).$$

Therefore, it suffices to show that

$$\sum \frac{ab+c^2}{a+c} \le \frac{9}{ab+bc+ca},$$

which is equivalent to the homogeneous inequality

$$\sum \frac{ab+c^2}{a+c} \le \frac{(a+b+c)^3}{3(ab+bc+ca)},$$

which is the inequality from the preceding P 1.146. The equality holds for $a=b=c=1$.

□

P 1.148. *If a, b, c are positive real numbers such that $a^5 + b^5 + c^5 = 3$, then*

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge 3.$$

Solution. We will prove the inequality under the more general condition $a^m + b^m + c^m = 3$, where $0 < m \le 21/4$. First, write the inequality in the homogeneous form

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge 3\left(\frac{a^m + b^m + c^m}{3}\right)^{1/m}.$$

By the Power Mean inequality, we have

$$\left(\frac{a^m + b^m + c^m}{3}\right)^{1/m} \le \left(\frac{a^{21/4} + b^{21/4} + c^{21/4}}{3}\right)^{4/21}.$$

Thus, it suffices to show that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge 3\left(\frac{a^{21/4} + b^{21/4} + c^{21/4}}{3}\right)^{4/21}.$$

By the known Vasc's inequality in P 1.125, namely

$$(x^2 + y^2 + z^2)^2 \ge 3(x^3y + y^3z + z^3x), \quad x, y, z \in \mathbb{R},$$

we have

$$\left(\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a}\right)^2 \ge 3\left(\frac{a^3}{\sqrt{bc}} + \frac{b^3}{\sqrt{ca}} + \frac{c^3}{\sqrt{ab}}\right).$$

Therefore, it suffices to prove the symmetric inequality

$$\frac{a^3}{\sqrt{bc}} + \frac{b^3}{\sqrt{ca}} + \frac{c^3}{\sqrt{ab}} \ge 3\left(\frac{a^{21/4} + b^{21/4} + c^{21/4}}{3}\right)^{8/21},$$

which is equivalent to

$$\left(\frac{\frac{a^3}{\sqrt{bc}} + \frac{b^3}{\sqrt{ca}} + \frac{c^3}{\sqrt{ab}}}{3}\right)^{21/4} \ge 3\left(\frac{a^{21/4} + b^{21/4} + c^{21/4}}{3}\right)^2,$$

Setting

$$a = x^{2/7}, \quad b = y^{2/7}, \quad c = z^{2/7}, \quad x, y, z > 0,$$

the inequality becomes

$$\left(\frac{x + y + z}{3}\right)^{21/4} \ge 3(xyz)^{3/4}\left(\frac{x^{3/2} + y^{3/2} + z^{3/2}}{3}\right)^2.$$

By the Cauchy-Schwarz inequality, we have

$$(x+y+z)(x^2+y^2+z^2) \geq (x^{3/2}+y^{3/2}+z^{3/2})^2.$$

Thus, it is enough to prove that

$$\left(\frac{x+y+z}{3}\right)^{17/4} \geq \frac{1}{3}(xyz)^{3/4}(x^2+y^2+z^2).$$

Due to homogeneity, we may assume that $x+y+z=3$, when the inequality becomes

$$(xyz)^{3/4}(x^2+y^2+z^2) \leq 3.$$

Since

$$\frac{3}{4} > \frac{1}{\sqrt{2}},$$

this inequality follows from the inequality in P 2.89 from Volume 2:

$$(xyz)^k(x^2+y^2+z^2) \leq 3, \quad k \geq \frac{1}{\sqrt{2}}$$

The proof is completed. The equality holds for $a=b=c=1$.

□

P 1.149. *Let $P(a,b,c)$ be a cyclic homogeneous polynomial of degree three. The inequality*

$$P(a,b,c) \geq 0$$

holds for all $a,b,c \geq 0$ if and only if the following two conditions are fulfilled:

(a) $P(1,1,1) \geq 0$;

(b) $P(0,b,c) \geq 0$ *for all* $b,c \geq 0$.

(Pham Kim Hung, 2007)

Solution. The conditions (a) and (b) are clearly necessary. Therefore, we will prove further that these conditions are also sufficient to have $P(a,b,c) \geq 0$. The polynomial $P(a,b,c)$ has the general form

$$P(a,b,c) = A(a^3+b^3+c^3)+B(a^2b+b^2c+c^2a)+C(ab^2+bc^2+ca^2)+3Dabc.$$

Since

$$P(1,1,1) = 3(A+B+C+D), \quad P(0,1,1) = 2A+B+C, \quad P(0,0,1) = A,$$

the conditions (a) and (b) involves

$$A+B+C+D \geq 0, \quad 2A+B+C \geq 0, \quad A \geq 0.$$

Assume that $a = \min\{a, b, c\}$, and denote

$$b = a + p, \quad c = a + q, \quad p, q \geq 0.$$

For fixed p and q, define the function

$$f(a) = P(a, a + p, a + q), \quad a \geq 0.$$

Since

$$a' = b' = c' = 1,$$

we have the derivative

$$\begin{aligned} f'(a) &= 3A(a^2 + b^2 + c^2) + (B + C)(a + b + c)^2 + 3D(ab + bc + ca) \\ &= (3A + B + C)(a^2 + b^2 + c^2) + (2B + 2C + 3D)(ab + bc + ca) \\ &= (3A + B + C)(a^2 + b^2 + c^2 - ab - bc - ca) + 3(A + B + C + D)(ab + bc + ca). \end{aligned}$$

Because $f'(a) \geq 0$, f is increasing, hence $f(a) \geq f(0)$, which is equivalent to

$$P(a, b, c) \geq P(0, p, q) = P(0, b, c).$$

According to the condition (b), we have $P(0, b, c) \geq 0$, hence $P(a, b, c) \geq 0$.

Remark 1. From the proof of P 1.149, the following statement follows:

- *Let $P(a, b, c)$ be a cyclic homogeneous polynomial of degree three. The inequality*

$$P(a, b, c) \geq 0$$

holds for all nonnegative real numbers a, b, c satisfying

$$a \leq b \leq c$$

if and only if $P(1, 1, 1) \geq 0$ and $P(0, b, c) \geq 0$ for all $0 \leq b \leq c$.

Remark 2. From P 1.149, using the substitution

$$a = y + z, \quad b = z + x, \quad c = x + y, \qquad x, y, z \geq 0,$$

we get the following statement:

- *Let $P(a, b, c)$ be a cyclic homogeneous polynomial of degree three, where a, b, c are the lengths of the sides of a triangle. The inequality*

$$P(a, b, c) \geq 0$$

holds if and only if $P(1, 1, 1) \geq 0$ and $P(b + c, b, c) \geq 0$ for all $b, c \geq 0$.

□

P 1.150. *If a,b,c are nonnegative real numbers such that $a+b+c=3$, then*

$$8(a^2b+b^2c+c^2a)+9 \ge 11(ab+bc+ca).$$

Solution. Write the inequality in the homogeneous form $P(a,b,c) \ge 0$, where

$$P(a,b,c) = 24(a^2b+b^2c+c^2a)+(a+b+c)^3 - 11(a+b+c)(ab+bc+ca).$$

According to P 1.149, it suffices to show that $P(1,1,1) \ge 0$ and $P(0,b,c) \ge 0$ for all $b,c \ge 0$. We have

$$P(1,1,1) = 0$$

and

$$\begin{aligned} P(0,b,c) &= 24b^2c+(b+c)^3 - 11bc(b+c) \\ &= b^3+16b^2c-8bc^2+c^3 \\ &\ge 16b^2c-8bc^2+c^3 = c(4b-c)^2 \ge 0. \end{aligned}$$

The equality holds for $a=b=c=1$.

□

P 1.151. *If a,b,c are nonnegative real numbers such that $a+b+c=6$, then*

$$a^3+b^3+c^3+8(a^2b+b^2c+c^2a) \ge 166.$$

(Vasile C., 2010)

Solution. Write the inequality in the homogeneous form $P(a,b,c) \ge 0$, where

$$P(a,b,c) = a^3+b^3+c^3+8(a^2b+b^2c+c^2a) - 166\left(\frac{a+b+c}{6}\right)^3.$$

According to P 1.149, it suffices to show that $P(1,1,1) \ge 0$ and $P(0,b,c) \ge 0$ for all $b,c \ge 0$. We have

$$P(1,1,1) = 27 - \frac{83}{4} = \frac{25}{4} > 0$$

and

$$\begin{aligned} P(0,b,c) &= b^3+c^3+8b^2c - \frac{83}{108}(b+c)^3 \\ &= \frac{1}{108}(25b^3+615b^2c-249bc^2+25c^3) \\ &= \frac{1}{108}(5b-c)^2(b+25c) \ge 0. \end{aligned}$$

The equality holds for $a=0$, $b=1$, $c=5$ (or any cyclic permutation).

□

P 1.152. *If a, b, c are nonnegative real numbers, then*

$$a^3 + b^3 + c^3 - 3abc \geq \sqrt{9+6\sqrt{3}}\,(a-b)(b-c)(c-a).$$

First Solution. Write the inequality as $P(a,b,c) \geq 0$. According to P 1.149, it suffices to show that $P(1,1,1) \geq 0$ and $P(0,b,c) \geq 0$ for all $b, c \geq 0$. We have $P(1,1,1) = 0$ and

$$P(0,b,c) = b^3 + c^3 + \sqrt{9+6\sqrt{3}}\;bc(b-c).$$

The inequality $P(0,b,c) \geq 0$ is true if

$$(b^3+c^3)^2 \geq (9+6\sqrt{3}\,)b^2c^2(b-c)^2,$$

which is equivalent to

$$(b+c)^2(b^2-bc+c^2)^2 \geq (9+6\sqrt{3}\,)b^2c^2(b-c)^2.$$

For the non-trivial case $bc \neq 0$, denoting

$$x = \frac{b}{c} + \frac{c}{b} - 1,$$

we can write this inequality as

$$(x+3)x^2 \geq (9+6\sqrt{3}\,)(x-1),$$

$$(x-\sqrt{3}\,)^2(x+3+2\sqrt{3}\,) \geq 0.$$

The equality holds for $a = b = c$, and also for $a = 0$ and $b/c + c/b = 1 + \sqrt{3}$, $b < c$ (or any cyclic permutation).

Second Solution. Assume that $a = \min\{a,b,c\}$. Since the case $a \leq c \leq b$ is trivial, consider further that $a \leq b \leq c$. Write the inequality as

$$(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2] \geq 2\sqrt{9+6\sqrt{3}}\,(a-b)(b-c)(c-a).$$

Using the substitution $b = a+p$, $c = a+q$, where $q \geq p \geq 0$, the inequality becomes

$$(3a+p+q)(p^2-pq+q^2) \geq \sqrt{9+6\sqrt{3}}\;pq(q-p).$$

Since $p^2-pq+q^2 \geq 0$, it suffices to consider the case $a = 0$ (as in the first solution).

□

P 1.153. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+7 \geq \frac{17}{3}\left(\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}\right).$$

(Vasile C., 2007)

Solution. Write the inequality as $P(a,b,c) \geq 0$, where

$$\begin{aligned} P(a,b,c) &= \sum (3a-17b)(a+b)(a+c)+21(a+b)(b+c)(c+a) \\ &= 3(a^3+b^3+c^3)-10(a^2b+b^2c+c^2a)+7(ab^2+bc^2+ca^2). \end{aligned}$$

According to P 1.149, it suffices to show that $P(1,1,1) \geq 0$ and $P(0,b,c) \geq 0$ for all $b, c \geq 0$. We have $P(1,1,1)=0$ and

$$P(0,b,c)=3(b^3+c^3)-10b^2c+7bc^2.$$

Consider the nontrivial case $b, c > 0$. Setting $c=1$, we need to show that $f(b) \geq 0$, where

$$f(b)=3b^3-10b^2+7b+3.$$

Case 1: $b \geq 3$. We have

$$f(b) > 3b^3-10b^2+7b=(b-1)(3b-7)>0.$$

Case 2: $2 \leq b \leq 3$. We have

$$f(b) \geq 3b^3-10b^2+8b=b(b-2)(3b-4) \geq 0.$$

Case 3: $0 < b \leq 2$. We have

$$f(b) \geq 3b^3-10b^2+7b+1.5b=b(3b^2-10b+8.5)>3b(b-5/3)^2 \geq 0.$$

The equality holds for $a=b=c$.

□

P 1.154. *Let a, b, c be nonnegative real numbers, no two of which are zero. If $0 \leq k \leq 5$, then*

$$\frac{ka+b}{a+c}+\frac{kb+c}{b+a}+\frac{kc+a}{c+b} \geq \frac{3}{2}(k+1).$$

(Vasile C., 2007)

First Solution. Write the inequality as

$$\frac{b}{a+c}+\frac{c}{b+a}+\frac{a}{c+b}-\frac{3}{2}+k\left(\frac{a}{a+c}+\frac{b}{b+a}+\frac{c}{c+b}-\frac{3}{2}\right) \geq 0.$$

Since

$$\frac{b}{a+c}+\frac{c}{b+a}+\frac{a}{c+b}-\frac{3}{2}\geq 0,$$

it suffices to consider the case $k=5$, when the inequality can be written as follows:

$$\sum(5a+b)(b+a)(c+b)\geq 9(a+c)(b+a)(c+b),$$

$$2\sum ab^2+\sum a^3\geq 3\sum a^2b,$$

$$2\sum ab^2+\frac{4}{3}\sum a^3-\frac{1}{3}\sum b^3\geq 3\sum a^2b,$$

$$\sum(6ab^2+4a^3-b^3-9a^2b)\geq 0,$$

$$(a-b)^2(4a-b)+(b-c)^2(4b-c)+(c-a)^2(4c-a)\geq 0.$$

Assume that $a=\min\{a,b,c\}$, and use the substitution

$$b=a+p,\quad c=a+q,\quad p,q\geq 0.$$

The inequality becomes

$$p^2(3a-p)+(p-q)^2(3a+4p-q)+q^2(3a+4q)\geq 0,$$

$$2Aa+B\geq 0,$$

where

$$A=p^2-pq+q^2,\quad B=p^3-3p^2q+2pq^2+q^3.$$

Since $A\geq 0$, we only need to show that $B\geq 0$. For $q=0$, we have $B=p^3\geq 0$, while for $q>0$, the inequality $B\geq 0$ is equivalent to

$$1\geq x(x-1)(2-x),$$

where $x=p/q\geq 0$. For the non-trivial case $x\in[1,2]$, we get this inequality by multiplying the obvious inequalities

$$1\geq x-1$$

and

$$1\geq x(2-x).$$

The proof is completed. The equality holds for $a=b=c$.

Second Solution. We can write the inequality in the form $P(a,b,c)\geq 0$, where $P(a,b,c)$ is a cyclic homogeneous polynomial of degree three. According to P 1.149, it suffices to show that the desired inequality holds for $a=b=c$, and also for $a=0$. If $a=0$, then the inequality becomes

$$x+k+\frac{1}{x}+\frac{k}{1+x}\geq\frac{3}{2}(k+1),$$

$$2(x-1)^2 + x \geq \frac{kx(x-1)}{x+1},$$

where

$$x = \frac{b}{c} > 0.$$

For $0 < x \leq 1$, we have

$$2(x-1)^2 + x > 0 \geq \frac{kx(x-1)}{x+1}.$$

For $1 \leq x \leq 5$, it suffices to consider the case $k = 5$, when the inequality is equivalent to

$$2(x-1)^2 + x \geq \frac{5x(x-1)}{x+1},$$

$$x^3 - 3x^2 + 2x + 1 \geq 0,$$

$$x(x-2)^2 + (x-1)^2 \geq 0.$$

Remark. As in the second solution, we can prove that the inequality in P 1.154 holds for

$$0 \leq k \leq k_0, \qquad k_0 = \sqrt{13 + 16\sqrt{2}} \approx 5.969.$$

For $a = 0$ and $k = k_0$, the inequality becomes

$$2(x-1)^2 + x \geq \frac{kx(x-1)}{x+1}, \qquad x = \frac{b}{c} > 0,$$

$$2x^3 - (k_0+1)x^2 + (k_0-1)x + 2 \geq 0,$$

$$(x - x_0)^2 \left(x + \frac{1}{x_0^2} \right) \geq 0,$$

where

$$x_0 = \frac{1 + \sqrt{2} + \sqrt{2\sqrt{2} - 1}}{2} \approx 1.883.$$

If $k = k_0$, then the equality holds for $a = b = c$, and also for $a = 0$ and $\frac{b}{c} + \frac{c}{b} = 1 + \sqrt{2}$ (or any cyclic permutation).

□

P 1.155. *Let a, b, c be nonnegative real numbers. Prove that*

(a) if $k \leq 1 - \frac{2}{5\sqrt{5}}$, then

$$\frac{ka+b}{2a+b+c} + \frac{kb+c}{a+2b+c} + \frac{kc+a}{a+b+2c} \geq \frac{3}{4}(k+1).$$

(b) if $k \ge 1 + \frac{2}{5\sqrt{5}}$, *then*

$$\frac{ka+b}{2a+b+c} + \frac{kb+c}{a+2b+c} + \frac{kc+a}{a+b+2c} \le \frac{3}{4}(k+1).$$

(Vasile C., 2007)

Solution. (a) Write the inequality in the form $P(a,b,c) \ge 0$, where $P(a,b,c)$ is a cyclic homogeneous polynomial of degree three. According to P 1.149, it suffices to show that the desired inequality holds for $a = b = c$, and also for $a = 0$. For $a = 0$, the inequality becomes

$$\frac{x}{x+1} + \frac{kx+1}{2x+1} + \frac{k}{x+2} \ge \frac{3}{4}(k+1),$$

$$(x+2)(2x^2 - x + 1) \ge k(x+1)(2x^2 - x + 2),$$

where

$$x = \frac{b}{c} \ge 0.$$

It suffices to consider the case $k = 1 - \frac{2}{5\sqrt{5}}$, when the inequality is equivalent to

$$(x - x_0)^2 \left(x + \frac{2}{5\sqrt{5}\, x_0^2} \right) \ge 0,$$

where

$$x_0 = \frac{3 - \sqrt{5}}{2}.$$

The equality holds for $a = b = c$. If $k = 1 - \frac{2}{5\sqrt{5}}$, then the equality holds also for $a = 0$ and $\frac{b}{c} + \frac{c}{b} = 3$ (or any cyclic permutation).

(b) According to P 1.149, it suffices to show that the desired inequality holds for $a = b = c$, and also for $a = 0$. If $a = 0$, then the inequality becomes

$$\frac{x}{x+1} + \frac{kx+1}{2x+1} + \frac{k}{x+2} \le \frac{3}{4}(k+1),$$

$$(x+2)(2x^2 - x + 1) \le k(x+1)(2x^2 - x + 2),$$

where

$$x = \frac{b}{c} \ge 0.$$

It suffices to consider the case $k = 1 + \frac{2}{5\sqrt{5}}$, when the inequality is equivalent to

$$(x - x_1)^2 \left(x + \frac{2}{5\sqrt{5}\, x_1^2} \right) \ge 0,$$

where

$$x_1 = \frac{3+\sqrt{5}}{2}.$$

The equality holds for $a = b = c$. If $k = 1 + \frac{2}{5\sqrt{5}}$, then the equality holds also for $a = 0$ and $\frac{b}{c} + \frac{c}{b} = 3$ (or any cyclic permutation).

□

P 1.156. *Let a, b, c be nonnegative real numbers, no two of which are zero. If $k \le \frac{23}{8}$, then*

$$\frac{ka+b}{2a+c} + \frac{kb+c}{2b+a} + \frac{kc+a}{2c+b} \ge k+1.$$

(Vasile C., 2007)

Solution. We can write the inequality in the form $P(a,b,c) \ge 0$, where $P(a,b,c)$ is a cyclic homogeneous polynomial of degree three. According to P 1.149, it suffices to show that the desired inequality holds for $a = b = c$, and also for $a = 0$. For $a = 0$, the inequality becomes

$$x + \frac{k}{2} + \frac{1}{2x} + \frac{k}{2+x} \ge k+1,$$

$$x^2 + (x-1)^2 \ge \frac{kx^2}{x+2},$$

where

$$x = \frac{b}{c} > 0.$$

It suffices to consider that $k = 23/8$, when the inequality is equivalent to

$$2x^2 - 2x + 1 \ge \frac{23x^2}{8(x+2)},$$

$$16x^3 - 7x^2 - 24x + 16 \ge 0,$$

$$16x(x-1)^2 + (5x-4)^2 \ge 0.$$

The equality holds for $a = b = c$.

Remark. For $k = 2$, we get the inequality in P 1.21.

□

P 1.157. *If a, b, c are positive real numbers such that $a \le b \le c$, then*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 3 \ge 2\left(\frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b}\right).$$

Solution. Write the inequality as follows:

$$\sum\left(\frac{a}{b}-1\right)\ge 2\sum\left(\frac{b+c}{c+a}-1\right),$$

$$\sum(a-b)\left(\frac{1}{b}+\frac{2}{c+a}\right)\ge 0,$$

$$(a-b)\left(\frac{1}{b}+\frac{2}{c+a}\right)+(b-c)\left(\frac{1}{c}+\frac{2}{a+b}\right)+[(c-b)+(b-a)]\left(\frac{1}{a}+\frac{2}{b+c}\right)\ge 0,$$

$$(b-a)\left(\frac{1}{a}+\frac{2}{b+c}-\frac{1}{b}-\frac{2}{c+a}\right)+(c-b)\left(\frac{1}{a}+\frac{2}{b+c}-\frac{1}{c}-\frac{2}{a+b}\right)\ge 0,$$

$$(b-a)^2\left[\frac{1}{ab}-\frac{2}{(b+c)(c+a)}\right]+(c-b)(c-a)\left[\frac{1}{ac}-\frac{2}{(b+c)(a+b)}\right]\ge 0.$$

The inequality is true since

$$\frac{1}{ab}-\frac{2}{(b+c)(c+a)}=\frac{c(a+b+c)-ab}{(b+c)(c+a)}>\frac{a(c-b)}{(b+c)(c+a)}\ge 0$$

and

$$\frac{1}{ac}-\frac{2}{(b+c)(a+b)}=\frac{b(a+b+c)-ac}{(b+c)(a+b)}>\frac{c(b-a))}{(b+c)(a+b)}\ge 0.$$

The equality holds for $a=b=c$.

□

P 1.158. *If $a\ge b\ge c\ge 0$, then*

$$\frac{3a+b}{2a+c}+\frac{3b+c}{2b+a}+\frac{3c+a}{2c+b}\ge 4.$$

(Vasile C., 2007)

First Solution. Write the inequality as follows:

$$\sum(3a+b)(2b+a)(2c+b)\ge 4(2a+c)(2b+a)(2c+b),$$

$$2\sum a^3+13\sum ab^2+7\sum a^2b+42abc\ge 4(4\sum ab^2+2\sum a^2b+9abc),$$

$$2\sum a^3+6abc\ge 3\sum ab^2+\sum a^2b,$$

$$2E(a,b,c)\ge F(a,b,c),$$

where

$$E(a,b,c)=\sum a^3+3abc-\sum ab^2-\sum a^2b,$$

$$F(a,b,c)=\sum ab^2-\sum a^2b.$$

The inequality is true since $E(a,b,c) \geq 0$ (by Schur's inequality of degree three) and

$$F(a,b,c) = (a-b)(b-c)(c-a) \leq 0.$$

The equality holds for $a = b = c$, and also for $a = b$ and $c = 0$.

Second Solution. Denote

$$x = a - b \geq 0, \qquad y = b - c \geq 0,$$

and write the inequality as follows

$$\sum \left(\frac{3a+b}{2a+c} - \frac{4}{3} \right) \geq 0,$$

$$\sum \frac{a+3b-4c}{2a+c} \geq 0,$$

$$\frac{a+3b-4c}{2a+c} + \frac{b+3c-4a}{2b+a} + \frac{c+3a-4b}{2c+b} \geq 0,$$

$$\frac{x+4y}{2a+c} - \frac{4x+3y}{2b+a} + \frac{3x-y}{2c+b} \geq 0,$$

$$xA + yB \geq 0,$$

where

$$\begin{aligned} A &= \frac{1}{2a+c} - \frac{4}{2b+a} + \frac{3}{2c+b} \\ &= \left(\frac{1}{2a+c} - \frac{1}{2b+a} \right) + 3\left(\frac{1}{2c+b} - \frac{1}{2b+a} \right) \\ &= \frac{-x+y}{(2a+c)(2b+a)} + \frac{3(x+2y)}{(2b+a)(2c+b)} \end{aligned}$$

and

$$\begin{aligned} B &= \frac{4}{2a+c} - \frac{3}{2b+a} - \frac{1}{2c+b} \\ &= 3\left(\frac{1}{2a+c} - \frac{1}{2b+a} \right) + \left(\frac{1}{2a+c} - \frac{1}{2c+b} \right) \\ &= \frac{3(-x+y)}{(2a+c)(2b+a)} - \frac{2x+y}{(2a+c)(2c+b)}. \end{aligned}$$

Thus, the inequality is equivalent to

$$x[(-x+y)(2c+b)+3(x+2y)(2a+c)+y[3(-x+y)(2c+b)-(2x+y)(2b+a)] \geq 0,$$

$$x^2(6a-b+c) + xy(10a-6b+2c) - y^2(a-b-6c) \geq 0,$$

It suffices to show that

$$xy(10a-6b+2c)-y^2(a-b-6c)\geq 0,$$

which is true is

$$x(10a-6b+2c)-y(a-b-6c)\geq 0.$$

We have

$$\begin{aligned}x(10a-6b+2c)-y(a-b-6c)&=x(10x+4y+6c)-y(x-6c)\\&=10x^2+3xy+6c(x+y)\geq 0.\end{aligned}$$

Third Solution. According to Remark 1 from P 1.149, it suffices to prove that the inequality holds for $c=0$ and $a\geq b$; that is, to show that

$$\frac{3}{2}+\frac{1}{2x}+\frac{3}{2+x}+x\geq 4,$$

where

$$x=\frac{a}{b}\geq 1.$$

The inequality is equivalent to

$$2x^3-x^2-3x+2\geq 0,$$

$$(x-1)(2x^2+x-2)\geq 0.$$

□

P 1.159. *Let a,b,c be nonnegative real numbers such that*

$$a\geq b\geq 1\geq c,\qquad a+b+c=3.$$

Prove that

$$\frac{1}{a^2+3}+\frac{1}{b^2+3}+\frac{1}{c^2+3}\leq\frac{3}{4}.$$

(Vasile C., 2005)

First Solution. Let

$$r=abc,\quad q=ab+bc+ca.$$

From

$$(a-1)(b-1)(c-1)\leq 0,$$

we get

$$r\leq q-2.$$

The desired inequality is equivalent to

$$3a^2b^2c^2+5(a^2b^2+b^2c^2+c^2a^2)+3(a^2+b^2+c^2)-27\geq 0,$$

$$3r^2 - 30r + 5q^2 - 6q \ge 0,$$

$$3(5-r)^2 + 5q^2 - 6q - 75 \ge 0.$$

Since

$$3q \le (a+b+c)^2 = 9$$

and

$$5 - r \ge 5 - (q-2) = 7 - q > 0,$$

it suffices to show that

$$3(7-q)^2 + 5q^2 - 6q - 75 \ge 0.$$

This is equivalent to the obvious inequality

$$(q-3)^2 \ge 0.$$

The proof is completed. The equality holds for $a = b = c = 1$.

Second Solution (by *Nguyen Van Quy*). Write the inequality as follows:

$$\left(\frac{1}{a^2+3} - \frac{3-a}{8}\right) + \left(\frac{1}{b^2+3} - \frac{3-b}{8}\right) + \left(\frac{1}{c^2+3} - \frac{3-c}{8}\right) \le 0,$$

$$\frac{(a-1)^3}{a^2+3} + \frac{(b-1)^3}{b^2+3} \le \frac{(1-c)^3}{c^2+3}.$$

Indeed, we have

$$\frac{(1-c)^3}{c^2+3} = \frac{(a-1+b-1)^3}{c^2+3} \ge \frac{(a-1)^3 + (b-1)^3}{c^2+3} \ge \frac{(a-1)^3}{a^2+3} + \frac{(b-1)^3}{b^2+3}.$$

Third Solution. Denoting

$$d = 2 - c,$$

we have

$$a + b = 1 + d, \quad d \ge a \ge b \ge 1.$$

We claim that

$$\frac{1}{c^2+3} + \frac{1}{d^2+3} \le \frac{1}{2}.$$

Indeed,

$$\frac{1}{2} - \frac{1}{c^2+3} - \frac{1}{d^2+3} = \frac{(cd-1)^2}{2(c^2+3)(d^2+3)} \ge 0.$$

Thus, it suffices to show that

$$\frac{1}{a^2+3} + \frac{1}{b^2+3} \le \frac{1}{d^2+3} + \frac{1}{4}.$$

Since

$$\frac{1}{a^2+3}-\frac{1}{d^2+3}=\frac{(d-a)(d+a)}{(a^2+3)(d^2+3)}=\frac{(b-1)(d+a)}{(a^2+3)(d^2+3)},$$

$$\frac{1}{4}-\frac{1}{b^2+3}=\frac{(b-1)(b+1)}{4(b^2+3)},$$

we need to prove that

$$\frac{d+a}{(a^2+3)(d^2+3)}\le\frac{b+1}{4(b^2+3)}.$$

We can get this inequality by multiplying the inequalities

$$\frac{d+a}{d^2+3}\le\frac{a+1}{4},$$

$$\frac{a+1}{a^2+3}\le\frac{b+1}{b^2+3}.$$

We have

$$\frac{a+1}{4}-\frac{d+a}{d^2+3}=\frac{(d-1)(ad+a+d-3)}{4(d^2+3)}\ge 0,$$

$$\frac{b+1}{b^2+3}-\frac{a+1}{a^2+3}=\frac{(a-b)(ab+a+b-3)}{(a^2+3)(b^2+3)}\ge 0.$$

□

P 1.160. *Let a,b,c be nonnegative real numbers such that*

$$a\ge 1\ge b\ge c,\quad a+b+c=3.$$

Prove that

$$\frac{1}{a^2+2}+\frac{1}{b^2+2}+\frac{1}{c^2+2}\ge 1.$$

(Vasile C., 2005)

First Solution. Let

$$r=abc,\quad q=ab+bc+ca.$$

From

$$(a-1)(b-1)(c-1)\ge 0,$$

we get

$$r\ge q-2.$$

Also, we have

$$r\le\frac{(a+b+c)^3}{27}=1.$$

$$q \leq \frac{1}{3}(a+b+c)^3 = 3.$$

The desired inequality is equivalent to

$$3 \geq a^2b^2c^2 + a^2b^2 + b^2c^2 + c^2a^2,$$

$$4 \geq r^2 - 6r + q^2,$$

$$(3-r)^2 + q^2 \leq 13.$$

Consider further two cases: $q \leq 2$ and $2 \leq q \leq 3$.

Case 1: $q \leq 2$. We have

$$(3-r)^2 + q^2 \leq 3^2 + 2^2 = 13.$$

Case 2: $2 \leq q \leq 3$. From $r \leq q-2$, we get

$$(3-r)^2 + q^2 \leq (5-q)^2 + q^2 = 2(q-3)(q-2) \leq 0.$$

The proof is completed. The equality holds for $a = b = c = 1$, as well as for $a = 2$, $b = 1$ and $c = 0$.

Second Solution. First, we can check that the desired inequality becomes an equality for $a = b = c = 1$, and also for $a = 2$, $b = 1$, $c = 0$. Consider then the inequality $f(x) \geq 0$, where

$$f(x) = \frac{1}{x^2+2} - A - Bx.$$

We have the derivative

$$f'(x) = \frac{-2x}{(x^2+2)^2} - B.$$

From the conditions $f(1) = 0$ and $f'(1) = 0$, we get $A = 5/9$ and $B = -2/9$. Also, from the conditions $f(2) = 0$ and $f'(2) = 0$, we get $A = 7/18$ and $B = -1/9$. Using these values of A and B, we obtain the relations

$$\frac{1}{x^2+2} - \frac{5-2x}{9} = \frac{(x-1)^2(2x-1)}{9(x^2+2)},$$

$$\frac{1}{x^2+2} - \frac{7-2x}{18} = \frac{(x-2)^2(2x+1)}{18(x^2+2)},$$

which involve

$$\frac{1}{x^2+2} \geq \frac{5-2x}{9}, \quad x \geq \frac{1}{2},$$

$$\frac{1}{x^2+2} \geq \frac{7-2x}{18}, \quad x \geq 0.$$

Consider further two cases: $c \geq 1/2$ and $c \leq 1/2$.

Case 1: $c \ge \frac{1}{2}$. By summing the inequalities

$$\frac{1}{a^2+2} \ge \frac{5-2a}{9}, \quad \frac{1}{b^2+2} \ge \frac{5-2b}{9}, \quad \frac{1}{c^2+2} \ge \frac{5-2c}{9},$$

we get

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \ge \frac{15-2(a+b+c)}{9} = 1.$$

Case 2: $c \le \frac{1}{2}$. We have

$$\frac{1}{a^2+2} \ge \frac{7-2a}{18}.$$

Consider now the similar inequalities

$$\frac{1}{b^2+2} \ge \frac{B-2b}{18},$$

$$\frac{1}{c^2+2} \ge \frac{C-2c}{18},$$

which are satisfied as equalities for $b=1$ and $c=0$ if $B=8$ and $C=9$:

$$\frac{1}{b^2+2} \ge \frac{8-2b}{18},$$

$$\frac{1}{c^2+2} \ge \frac{9-2c}{18}$$

Since

$$\frac{1}{b^2+2} - \frac{8-2b}{18} = \frac{(1-b)(1+3b-b^2)}{9(b^2+2)}$$

and

$$\frac{1}{c^2+2} - \frac{9-2c}{18} = \frac{c(1-2c)(4-c)}{18(c^2+2)},$$

these inequalities holds for $0 \le b \le 1$ and $0 \le c \le 1/2$. Therefore, we have

$$\frac{1}{a^2+2} + \frac{1}{b^2+2} + \frac{1}{c^2+2} \ge \frac{7-2a}{18} + \frac{8-2b}{18} + \frac{9-2c}{18} = 1.$$

□

P 1.161. *Let a,b,c be real numbers such that*

$$a \ge b \ge 1 \ge c \ge -5, \qquad a+b+c=3.$$

Prove that

$$\frac{6}{a^3+b^3+c^3} + 1 \ge \frac{8}{a^2+b^2+c^2}.$$

(Vasile C., 2015)

Solution. First, we will show that

$$a^3+b^3+c^3>0.$$

Indeed, for the nontrivial case $-5\le c\le -2$, we have

$$\begin{aligned}4(a^3+b^3+c^3)&\ge (a+b)^3+4c^3=(3-c)^3+4c^3\\&=3c^3+9c^2-27c+27\ge -15c^2+9c^2-27c+27\\&=3(-2c^2-9c+9)>3(-2c^2-9c+5)=3(c+5)(1-2c)>0.\end{aligned}$$

From

$$(a-1)(b-1)(c-1)\le 0,$$

we get

$$r\le q-2,$$

where $q=ab+bc+ca$ and $r=abc$. Write the desired inequality as follows:

$$\frac{2}{r+9-3q}+1\ge\frac{8}{9-2q}.$$

Since

$$r+9-3q\le(q-2)+9-3q=7-2q,$$

it suffices to show that

$$\frac{2}{7-2q}+1\ge\frac{8}{9-2q}.$$

This is equivalent to the obvious inequality

$$(2q-5)^2\ge 0.$$

The equality holds for $a=1+\frac{1}{\sqrt{2}}$, $b=1$, $c=1-\frac{1}{\sqrt{2}}$.

□

P 1.162. *If $a\ge 1\ge b\ge c>-3$ such that $ab+bc+ca=3$, then*

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2}\ge 1.$$

(Vasile C., 2015)

Solution. We will show first that $c>-1$ and $p>0$, where $p=a+b+c$. We have

$$p\ge 1+c+c=1+2c,$$

hence

$$p-c\ge c+1.$$

On the other hand, from

$$(a-1)(b-1) \le 0,$$

we find

$$ab-(a+b)+1 \le 0,$$

$$3-c(a+b)-(a+b)+1 \le 0,$$

$$4 \le (c+1)(a+b),$$

$$4 \le (c+1)(p-c),$$

hence

$$p(c+1) \ge c^2+c+4 > 0.$$

From $p(c+1) > 0$, it follows that $c > -1$ involves $p > 0$. To show that $c > -1$, we use the contradiction method. The case $c = -1$ contradicts the inequality $(c+1)(p-c) \ge 4$, and the case $c < -1$ leads to

$$p-c \le \frac{4}{c+1},$$

$$c+1 \le \frac{4}{c+1},$$

$$(c+1)^2 \ge 4,$$

hence $c \le -3$, which is false. Therefore, we have $c > -1$ and $p > 0$. According Lemma below, we can write the inequality as

$$p^3abc-27+(p^2-9)^2 \ge 0.$$

From $(a-1)(b-1)(c-1) \ge 0$, we get

$$abc \ge 4-p.$$

Thus,

$$p^3abc-27+(p^2-9)^2 \ge p^3(4-p)-27+(p^2-9)^2 = 2(2p+3)(p-3)^2 \ge 0.$$

The equality holds for $a=b=c=1$.

Lemma. *Let a,b,c be real numbers, $p = a+b+c$ and $q = ab+bc+ca$. If $q > 0$, then the inequality*

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2} \ge \frac{3}{ab+bc+ca}$$

is equivalent to

$$3(p^3abc-q^3)+q(p^2-3q)^2 \ge 0.$$

Proof. Write the inequality as

$$q\sum(x+ab-c^2)(x+ac-b^2) \ge 3\prod(x+bc-a^2),$$

where

$$x = a^2+b^2+c^2 = p^2-2q.$$

From

$$\sum(ab-c^2)(ac-b^2) = q^2-xq,$$

$$\sum(x+ab-c^2)(x+ac-b^2) = x^2+xq+q^2$$

and

$$\prod(bc-a^2) = q^3-p^3abc,$$

$$\prod(x+bc-a^2) = xq^2+q^3-p^3abc,$$

the conclusion follows.

□

P 1.163. *If $a \ge b \ge 1 \ge c \ge 0$ such that $a+b+c=3$, then*

$$\frac{1}{a^2+ab+b^2}+\frac{1}{b^2+bc+c^2}+\frac{1}{c^2+ca+a^2} \le \frac{3}{ab+bc+ca}.$$

(Vasile C., 2015)

Solution. By Lemma from the preceding P 1.162, we need to show that

$$3(p^3abc-q^3)+q(p^2-3q)^2 \le 0,$$

where $p=3$ and $q=ab+bc+ca$; that is

$$27abc-q^3+3q(3-q)^2 \le 0.$$

From $p^2 \ge 3q$, we get $q \le 3$, and from $(a-1)(b-1)(c-1) \le 0$, we get

$$abc \le q-2, \quad q \ge 2.$$

Thus,

$$27abc-q^3+3q(3-q)^2 \le 27(q-2)-q^3+3q(3-q)^2 = 2(q-3)^3 \le 0.$$

Thus, the proof is completed. The equality holds for $a=b=c=1$.

Remark. Actually, the inequality holds for

$$a \ge b \ge 1 \ge c \ge 1-\sqrt{3}.$$

To prove this, it suffices to show that $ab+bc+ca \ge 0$. Indeed, we have

$$\begin{aligned} ab+bc+ca &= (a-1)(b-1)-1+a+b+c(a+b) \ge -1+(1+c)(a+b) \\ &= -1+(1+c)(3-c) \ge 0. \end{aligned}$$

□

P 1.164. *If a, b, c are positive real numbers such that*

$$a \ge 1 \ge b \ge c, \qquad abc = 1,$$

then

$$\frac{1-a}{3+a^2} + \frac{1-b}{3+b^2} + \frac{1-c}{3+c^2} \ge 0.$$

(Vasile C., 2009)

First Solution. Denote the left side of the inequality by $E(a, b, c)$. We will show that

$$E(a, b, c) \ge E(ab, 1, c) \ge 0.$$

Let

$$a + b = s, \quad ab = p.$$

We have

$$p \ge abc = 1, \quad s \ge 2\sqrt{p} \ge 2.$$

Therefore,

$$\begin{aligned} E(a, b, c) - E(ab, 1, c) &= \frac{1-a}{3+a^2} + \frac{1-b}{3+b^2} + \frac{ab-1}{3+a^2b^2} \\ &= \frac{s^2-(3+p)s+2(3-p)}{3s^2+(p-3)^2} + \frac{p-1}{3+p^2} \\ &= \frac{(3+p)(s-p-1)(ps+p-3)}{(3+p^2)[3s^2+(p-3)^2]}. \end{aligned}$$

Since

$$s - p - 1 = (a-1)(1-b) \ge 0, \quad ps + p - 3 \ge 2p + p - 3 \ge 0,$$

it follows that

$$E(a, b, c) - E(ab, 1, c) \ge 0.$$

Also, we have

$$E(ab, 1, c) = E(1/c, 1, c) = \frac{(1-c)^4}{(3c^2+1)(3+c^2)} \ge 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. Let $p = a + b + c$ and $q = ab + bc + ca$. From

$$(a-1)(b-1)(c-1) \ge 0,$$

we get

$$p \ge q.$$

The desired inequality is true because it is equivalent to

$$\sum (1-a)(9+3b^2+3c^2+b^2c^2) \ge 0,$$

$$27+6\sum a^2+\sum b^2c^2-9p-3pq+9-q\geq 0,$$
$$27+6(p^2-2q)+(q^2-2p)-9p-3pq+9-q\geq 0,$$
$$6p^2+q^2-3pq-11p-13q+36\geq 0,$$
$$(p+q-6)^2+5p^2-5pq+p-q\geq 0,$$
$$(p+q-6)^2+(5p+1)(p-q)\geq 0.$$

□

P 1.165. *If a,b,c are positive real numbers such that*

$$a\geq 1\geq b\geq c,\qquad abc=1,$$

then

$$\frac{1}{\sqrt{3a+1}}+\frac{1}{\sqrt{3b+1}}+\frac{1}{\sqrt{3c+1}}\geq\frac{3}{2}.$$

(Vasile C., 2007)

Solution. Let

$$b_1=1/b,\qquad b_1\geq 1.$$

We claim that

$$\frac{1}{\sqrt{3b+1}}+\frac{1}{\sqrt{3b_1+1}}\geq\frac{1}{2}.$$

This inequality is equivalent to

$$\frac{1}{\sqrt{3b+1}}+\sqrt{\frac{b}{b+3}}\geq\frac{1}{2}.$$

Making the substitution

$$\frac{1}{\sqrt{3b+1}}=t,\quad \frac{1}{2}\leq t<1,$$

the inequality becomes

$$\sqrt{\frac{1-t^2}{1+8t^2}}\geq 1-t.$$

By squaring, we get

$$t(1-t)(1-2t)^2\geq 0,$$

which is clearly true. Similarly, we have

$$\frac{1}{\sqrt{3c+1}}+\frac{1}{\sqrt{3c_1+1}}\geq\frac{1}{2},$$

where

$$c_1 = 1/c, \qquad c_1 \ge 1.$$

Using these inequalities, it suffices to show that

$$\frac{1}{\sqrt{3a+1}} + \frac{1}{2} \ge \frac{1}{\sqrt{3b_1+1}} + \frac{1}{\sqrt{3c_1+1}},$$

which is equivalent to

$$\frac{1}{\sqrt{3b_1c_1+1}} + \frac{1}{2} \ge \frac{1}{\sqrt{3b_1+1}} + \frac{1}{\sqrt{3c_1+1}}.$$

According to P 2.88 in Volume 2, the conclusion follows. The equality holds for $a = b = c = 1$.

□

P 1.166. *If a, b, c are positive real numbers such that*

$$a \ge 1 \ge b \ge c, \qquad abc = 1,$$

then

$$\frac{1}{a^2+4ab+b^2} + \frac{1}{b^2+4bc+c^2} + \frac{1}{c^2+4ca+a^2} \ge \frac{1}{2}.$$

(Vasile C., 2015)

Solution. Write the inequality as

$$2E \ge F,$$

where

$$E = \sum (a^2+4ab+b^2)(a^2+4ac+c^2), \qquad F = \prod (b^2+4bc+c^2).$$

Using Lemma below for $k = 4$ and $r = 1$, we get

$$E = 18pr + p^4 - 3q^2 = 18p + p^4 - 3q^2,$$

$$F = 27r^2 + 2p^3r + p^2q^2 + 2q^3 = 27 + 2p^3 + p^2q^2 + 2q^3,$$

hence

$$2E - F = 2p^4 - 2p^3 + 36p - 27 - (p^2+6)q^2 - 2q^3.$$

From $(a-1)(b-1)(c-1) \ge 0$, we get

$$p \ge q.$$

Thus,

$$2E - F \geq 2p^4 - 2p^3 + 36p - 27 - (p^2+6)p^2 - 2p^3$$
$$= p^4 - 4p^3 - 6p^2 + 36p - 27 = (p-1)(p-3)^2(p+3) \geq 0.$$

Thus, the proof is completed. The equality holds for $a = b = c = 1$.

Lemma. *If a, b, c are real numbers,*

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc$$

and

$$E = \sum (a^2+kab+b^2)(a^2+kac+c^2), \quad F = \prod (b^2+kbc+c^2),$$

then

$$E = (k-1)(k+2)pr + p^4 + (k-4)p^2q + (5-2k)q^2,$$

$$F = (k-1)^3r^2 + [(k-2)p^2 + (k-1)(k-4)q]pr + p^2q^2 + (k-2)q^3.$$

Proof. Let

$$x = a^2+b^2+c^2 = p^2-2q.$$

Since

$$E = \sum (x+kab-c^2)(x+kac-b^2)$$
$$= x^2 + kxq + (k-1)(k+2)pr + q^2$$

and

$$F = \prod (x+kbc-a^2)$$
$$= x[(k-1)(k+2)pr+q^2] + (k-1)^3r^2 - k[kp^2-3(k-1)q]pr + kq^3,$$

the conclusion follows.

□

P 1.167. *Let $a \geq 1 \geq b \geq c \geq 0$ such that*

$$a+b+c = 3, \quad ab+bc+ca = q,$$

where $q \in [0,3]$ is a fixed number. Prove that the product $r = abc$ is maximal for $b = c$, and minimal for $b = 1$ or $c = 0$.

(Vasile C., 2015)

Solution. For $q = 3$, from $(a+b+c)^2 = 3(ab+bc+ca)$, which is equivalent to

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 0,$$

we get $a = b = c = 1$. Consider further that $q \in [0,3)$, when $a > 1 \ge b \ge c \ge 0$. We will show first that $c \in [c_1,\ c_2]$, where

$$c_1 = \begin{cases} 1 - \sqrt{3-q}, & 2 \le q < 3 \\ 0, & 0 \le q \le 2 \end{cases}$$

and

$$c_2 = 1 - \sqrt{1 - \frac{q}{3}}.$$

From

$$(a-1)(b-1) \le 0,$$

which is equivalent to

$$ab - (a+b) + 1 \le 0, \quad q - (a+b)(c+1) + 1 \le 0, \quad q - (3-c)(c+1) + 1 \le 0,$$

we get

$$c^2 - 2c + q - 2 \le 0,$$

hence $c \ge 1 - \sqrt{3-q}$. In the case $2 \le q < 3$, when $1 - \sqrt{3-q} \ge 0$, the equality $c = 1 - \sqrt{3-q}$ is possible because it implies

$$b = 1, \quad a = 1 + \sqrt{3-q} \ge 1.$$

In the case $0 \le q \le 2$, the equality $c = 0$ is possible because it implies $a + b = 3$ and $ab = q$, hence

$$a = \frac{3 + \sqrt{9-4q}}{2} \ge 1, \quad b = \frac{3 - \sqrt{9-4q}}{2} \in [0,1].$$

In conclusion, we have $c \ge c_1$ in all cases, with equality for $b = 1$ or $c = 0$. Also, from

$$(b-c)(a-c) = c^2 - 2c(a+b) + q = c^2 - 2c(3-c) + q = 3c^2 - 6c + q \ge 0,$$

we get $c \le c_2$, with equality for $b = c$. On the other hand, from

$$abc = c[q - (a+b)c] = c[q - (3-c)c],$$

we get

$$r(c) = c^3 - 3c^2 + qc.$$

Since

$$r'(c) = 3c^2 - 6c + q = 3c^2 - 2(a+b+c)c + q = (c-a)(c-b) \ge 0,$$

$r(c)$ is strictly increasing on $[c_1, c_2]$, and hence $r(c)$ is minimal for $c = c_1$, when $b = 1$ or $c = 0$, and is maximal for $c = c_2$, when $b = c$.

□

P 1.168. *Let p and q be fixed real numbers such that there exist three real numbers a,b,c satisfying*

$$a \ge 1 \ge b \ge c \ge 0, \qquad a+b+c=p, \qquad ab+bc+ca=q.$$

Prove that

(a) *the product $r=abc$ is maximal for $b=c$;*

(b) *the product $r=abc$ is minimal for $a=1$ or $b=1$ or $c=0$.*

(Vasile C., 2015)

Solution. (a) According to P 3.57 in Volume 1, under the weaker condition $a \ge b \ge c \ge 0$ instead of $a \ge 1 \ge b \ge c \ge 0$, the product $r=abc$ is maximal for $b=c$, when

$$a=\frac{p+2\sqrt{p^2-3q}}{3}, \quad b=c=\frac{p-\sqrt{p^2-3q}}{3}.$$

Thus, it suffices to show that

$$\frac{p+2\sqrt{p^2-3q}}{3} \ge 1 \ge \frac{p-\sqrt{p^2-3q}}{3}.$$

The left inequality is true if

$$4(p^2-3q) \ge (3-p)^2,$$

which is equivalent to

$$(p+1)^2 \ge 4(q+1);$$

indeed,

$$(p+1)^2-4(q+1)=(b-c)^2+(a-1)(a+3-2b-2c) \ge 0.$$

The right inequality is equivalent to

$$\sqrt{p^2-3q} \ge p-3.$$

This is true if $p^2-3q \ge (p-3)^2$ for $p \ge 3$; indeed,

$$\begin{aligned}\frac{p^2-3q-(p-3)^2}{3} &= 2p-q-3\\ &=(a-1)(1-b)+(1-c)(a+b-2)\\ &=(a-1)(1-b)+(1-c)[(1-c)+(p-3)] \ge 0.\end{aligned}$$

(b) We will show that abc is minimal for $a=1$ or $b=1$ if $p \le q+1$, and for $c=0$ if $p \ge q+1$.

Case 1: $p \le q+1$. From

$$(a-1)(b-1)(c-1) \ge 0,$$

we get

$$abc \geq ab + bc + ca - a - b - c + 1 = q - p + 1 \geq 0,$$

with equality for $a = 1$ or $b = 1$. If one of a, b is 1, then the other two of a, b, c are

$$x = \frac{p-1+\sqrt{D}}{2}, \quad c = \frac{p-1-\sqrt{D}}{2},$$

where

$$\begin{aligned} D &= (p+1)^2 - 4(q+1) \\ &= (b-c)^2 + (a-1)(a+3-2b-2c) \geq 0. \end{aligned}$$

We only need to show that $c \geq 0$, which is equivalent to

$$p - 1 \geq \sqrt{D},$$

$$p \leq q + 1.$$

Case 2: $p \geq q + 1$. We will show that abc is minimal for $c = 0$. For this, we only need to prove that there exist two real numbers a and b such that

$$a \geq 1 \geq b \geq 0, \quad a + b = p, \; ab = q.$$

Since

$$a = \frac{p+\sqrt{p^2-4q}}{2}, \quad b = \frac{p-\sqrt{p^2-4q}}{2},$$

where

$$p^2 - 4q \geq (q+1)^2 - 4q = (q-1)^2 \geq 0,$$

the inequality $a \geq 1$ is equivalent to

$$\sqrt{p^2-4q} \geq 2 - p,$$

while the inequality $b \leq 1$ is equivalent to

$$\sqrt{p^2-4q} \geq p - 2.$$

These inequalities are true if

$$p^2 - 4q \geq (p-2)^2,$$

which reduces to $p \geq q + 1$.

□

P 1.169. *Let p and q be fixed real numbers such that there exist three real numbers a,b,c satisfying*

$$a \ge b \ge c \ge 1, \qquad a+b+c=p, \qquad ab+bc+ca=q.$$

Prove that

(a) the product $r=abc$ is maximal for $b=c$;

(b) the product $r=abc$ is minimal for $a=b$ or $c=1$.

(Vasile C., 2015)

Solution. From $a \ge b \ge c \ge 1$, it follows that

$$p=a+b+c \ge 3.$$

(a) According to P 3.57 in Volume 1, under the weaker condition $a \ge b \ge c \ge 0$ instead of $a \ge b \ge c \ge 1$, the product $r=abc$ is maximal for $b=c$, when

$$a=\frac{p+2\sqrt{p^2-3q}}{3}, \quad b=c=\frac{p-\sqrt{p^2-3q}}{3}.$$

Thus, it suffices to show that

$$\frac{p-\sqrt{p^2-3q}}{3} \ge 1,$$

which is equivalent to

$$p-3 \ge \sqrt{p^2-3q},$$

$$(p-3)^2 \ge p^2-3q,$$

$$q+3 \ge 2p.$$

We have

$$q+3-2p=(a-1)(b-1)+(b-1)(c-1)+(c-1)(a-1) \ge 1.$$

(b) We will show that abc is minimal for $a=b$ if $p+1 \le 2\sqrt{q+1}$, and for $c=1$ if $p+1 \ge 2\sqrt{q+1}$.

Case 1: $p+1 \le 2\sqrt{q+1}$. According to P 2.53 in Volume 1, under the weaker condition $a \ge b \ge c$ instead of $a \ge b \ge c \ge 1$, the product $r=abc$ is minimal for $a=b$, when

$$a=b=\frac{p+\sqrt{p^2-3q}}{3}, \quad c=\frac{p-2\sqrt{p^2-3q}}{3}.$$

Thus, it suffices to show that

$$\frac{p-2\sqrt{p^2-3q}}{3} \ge 1,$$

which is equivalent to

$$p-3 \ge 2\sqrt{p^2-3q},$$

$$(p-3)^2 \ge 4(p^2-3q),$$

$$(p+1)^2 \le 4(q+1),$$

$$p+1 \le 2\sqrt{q+1}.$$

Case 2: $p+1 \ge 2\sqrt{q+1}$. From

$$(a-1)(b-1)(c-1) \ge 0,$$

we get

$$abc \ge ab+bc+ca-a-b-c+1 = q-p+1 \ge 0,$$

with equality for $c=1$. In addition, $c=1$ involves

$$a = \frac{p-1+\sqrt{D}}{2}, \quad b = \frac{p-1-\sqrt{D}}{2},$$

where

$$D = (p+1)^2 - 4(q+1) \ge 0.$$

To end the proof, it suffices to show that

$$\frac{p-1-\sqrt{D}}{2} \ge 1,$$

which is equivalent to

$$p-3 \ge \sqrt{D},$$

$$(p-3)^2 \ge (p+1)^2 - 4(q+1)),$$

$$q+3 \ge 2p,$$

$$(a-1)(b-1)+(b-1)(c-1)+(c-1)(a-1) \ge 0.$$

□

P 1.170. *Let $a \ge b \ge 1 \ge c \ge 0$ such that*

$$a+b+c=3, \qquad ab+bc+ca=q,$$

where $q \in [0,3]$ is a fixed number. Prove that the product $r=abc$ is maximal for $b=1$, and minimal for $a=b$ or $c=0$.

(Vasile C., 2015)

Solution. From
$$ab+bc+ca \le \frac{1}{3}(a+b+c)^2 = 3$$
and
$$q-3 = ab+(a+b)c-a-b-c = (a-1)(b-1)+(a+b-1)c-1 \ge -1,$$
it follows that $2 \le q \le 3$. Since $q=2$ involves $b=1$ and $c=0$, and $q=3$ involves $a=b=c=1$, we consider further that $q \in (2,3)$, when $a \ge b \ge 1 > c \ge 0$. We will show first that $c \in [c_1,\ c_2]$, where
$$c_1 = \begin{cases} 1-2\sqrt{1-q/3}, & 9/4 \le q < 3 \\ 0, & 2 < q \le 9/4 \end{cases}$$
and
$$c_2 = 1-\sqrt{3-q}.$$
From
$$\begin{aligned}(a-b)^2 &= (a+b)^2-4ab = (a+b)^2+4c(a+b)-4q \\ &= (3-c)^2+4c(3-c)-4q = -3c^2+6c+9-4q,\end{aligned}$$
it follows that
$$3c^2-6c+4q-9 \le 0,$$
hence $c \ge 1-2\sqrt{1-q/3}$. In the case $9/4 \le q < 3$, when $1-2\sqrt{1-q/3} \ge 0$, the equality $c = 1-2\sqrt{1-q/3}$ is possible because it implies
$$a = b = 1+\sqrt{1-q/3} \ge 1.$$
In the case $2 < q \le 9/4$, the equality $c=0$ is possible because it implies $a+b=3$ and $ab=q$, hence
$$a = \frac{3+\sqrt{9-4q}}{2}, \qquad b = \frac{3-\sqrt{9-4q}}{2} > 1.$$
In conclusion, we have $c \ge c_1$ in all cases, with equality for $a=b$ or $c=0$. Also, from
$$(a-1)(b-1) \ge 0,$$
which is equivalent to
$$ab-(a+b)+1 \ge 0, \quad q-(a+b)(c+1)+1 \ge 0, \quad q-(3-c)(c+1)+1 \ge 0,$$
we get
$$c^2-2c+q-2 \ge 0,$$

hence $c \le c_2$, with equality for $b = 1$. On the other hand, from

$$abc = c[q-(a+b)c] = c[q-(3-c)c],$$

we get

$$r(c) = c^3 - 3c^2 + qc.$$

Since

$$r'(c) = 3c^2 - 6c + q = 3c^2 - 2(a+b+c)c + q = (c-a)(c-b) \ge 0,$$

$r(c)$ is strictly increasing on $[c_1, c_2]$, and hence $r(c)$ is minimal for $c = c_1$, when $a = b$ or $c = 0$, and is maximal for $c = c_2$, when $b = 1$.

□

P 1.171. *Let p and q be fixed real numbers such that there exist three real numbers a, b, c satisfying*

$$a \ge b \ge 1 \ge c \ge 0, \qquad a+b+c = p, \qquad ab+bc+ca = q.$$

Prove that

(a) the product $r = abc$ is maximal for $b = 1$ or $c = 1$;

(b) the product $r = abc$ is minimal for $a = b$ or $c = 0$.

(Vasile C., 2015)

Solution. (a) From

$$(a-1)(b-1)(c-1) \le 0,$$

we get

$$abc \le q - p + 1,$$

with equality for $b = 1$ or $c = 1$. If one of $b,\ c$ is 1, then the other two of a, b, c are

$$a = x = \frac{p-1+\sqrt{D}}{2}, \quad y = \frac{p-1-\sqrt{D}}{2},$$

where

$$D = (p+1)^2 - 4(q+1).$$

Notice that

$$D = (a-b)^2 + (1-c)(2a+2b-c-3) \ge 0,$$
$$x \ge 1,$$
$$xy = q-p+1 = (a-1)(b-1) + c(a+b-1) \ge 0, \quad y \ge 0.$$

The inequality $x \ge 1$ is equivalent to $\sqrt{D} \ge 3-p$, which is true if $p \le 3$ involves

$$D \ge (3-p)^2.$$

Indeed,

$$\begin{aligned}\frac{D-(3-p)^2}{4} &= 2p-q-3\\ &= (b-1)(1-c)+(a-1)(2-b-c)\\ &= (b-1)(1-c)+(a-1)[(a-1)+(3-p)] \ge 0.\end{aligned}$$

Also, we have $y \le 1$ for $p \le 3$ or $p \ge (q+3)/2$, and $y \ge 1$ for $3 \le p \le (q+3)/2$. Therefore, there is a unique point (a,b,c) such that the product $r = abc$ is maximal:

$$(a,b,c) = \left(\frac{p-1+\sqrt{D}}{2}, 1, \frac{p-1-\sqrt{D}}{2}\right)$$

for $2 \le p \le 3$ or $p \ge \dfrac{q+3}{2}$;

$$(a,b,c) = \left(\frac{p-1+\sqrt{D}}{2}, \frac{p-1-\sqrt{D}}{2}, 1\right)$$

for $3 \le p \le \dfrac{q+3}{2}$.

(b) According to P 3.57 in Volume 1, under the weaker condition $a \ge b \ge c \ge 0$ instead of $a \ge b \ge 1 \ge c \ge 0$, the product $r = abc$ is minimal for $a = b$ (if $p^2 \le 4q$) or $c = 0$ (if $p^2 \ge 4q$).

For $a = b$, we have

$$a = b = \frac{p+\sqrt{p^2-3q}}{3}, \qquad c = \frac{p-2\sqrt{p^2-3q}}{3}.$$

Thus, it suffices to show that

$$\frac{p+\sqrt{p^2-3q}}{3} \ge 1,$$

$$\frac{p-2\sqrt{p^2-3q}}{3} \le 1.$$

The first inequality holds if $p \le 3$ involves

$$(p^2-3q) \ge (3-p)^2,$$

that is

$$2p-q-3 \ge 0.$$

We have

$$\begin{aligned}2p-q-3&=2(a+b)-ab-3-(a+b-2)c\\&\ge 2(a+b)-\frac{1}{4}(a+b)^2-3-(a+b-2)c\\&=\frac{(a+b-2)(6-a-b)-4(a+b-2)c}{4}\\&=\frac{(a+b-2)(6-a-b-4c)}{4}\\&=\frac{(a+b-2)[(3-p)+3(1-c)]}{4}\ge 0.\end{aligned}$$

The second inequalities holds if $p\ge 3$ implies

$$4(p^2-3q)\ge(p-3)^2,$$

which is equivalent to the obvious inequality

$$(a-b)^2+(1-c)(2a+2b-c-3)\ge 0.$$

For $c=0$, we have

$$a=\frac{p+\sqrt{p^2-4q}}{2},\quad b=\frac{p-\sqrt{p^2-4q}}{2},\quad c=0.$$

Thus, it suffices to show that

$$\frac{p-\sqrt{p^2-4q}}{2}\ge 1,$$

that is

$$p-2\ge\sqrt{p^2-4q}.$$

Since $p-2=(a-1)+(b-1)+c\ge 0$, we only need to show that

$$(p-2)^2\ge p^2-4q,$$

which is equivalent to

$$q+1-p\ge 0,$$

$$(a-1)(b-1)+(a+b-1)c\ge 0.$$

□

P 1.172. *Let p and q be fixed real numbers such that there exist three real numbers a,b,c satisfying*

$$1 \ge a \ge b \ge c \ge 0, \qquad a+b+c=p, \qquad ab+bc+ca=q.$$

Prove that

(a) *the product $r=abc$ is maximal for $b=c$ or $a=1$;*

(b) *the product $r=abc$ is minimal for $a=b$ or $c=0$.*

(Vasile C., 2015)

Solution. We have $p \le 3$ because

$$p-3=(a-1)+(b-1)+(c-1) \le 0.$$

(a) We will show that abc is maximal for $b=c$ if $p+1 \le 2\sqrt{q+1}$, and for $a=1$ if $p+1 \ge 2\sqrt{q+1}$.

Case 1: $p+1 \le 2\sqrt{q+1}$. According to P 3.57 in Volume 1, under the weaker condition $a \ge b \ge c \ge 0$ instead of $1 \ge a \ge b \ge c \ge 0$, the product $r=abc$ is maximal for $b=c$, when

$$a=\frac{p+2\sqrt{p^2-3q}}{3}, \quad b=c=\frac{p-\sqrt{p^2-3q}}{3}.$$

Thus, it suffices to show that

$$\frac{p-\sqrt{p^2-3q}}{3} \ge 0$$

and

$$\frac{p+2\sqrt{p^2-3q}}{3} \le 1.$$

The first inequality is clearly true, and the second inequality is equivalent to

$$3-p \ge 2\sqrt{p^2-3q},$$

$$(3-p)^2 \ge 4(p^2-3q),$$

$$(p+1)^2 \le 4(q+1),$$

$$p+1 \le 2\sqrt{q+1}.$$

Case 2: $p+1 \ge 2\sqrt{q+1}$. From

$$(a-1)(b-1)(c-1) \ge 0,$$

we get

$$abc \ge ab+bc+ca-a-b-c+1=q-p+1 \ge 0,$$

with equality for $a=1$. In addition, $a=1$ involves

$$b=\frac{p-1+\sqrt{D}}{2}, \quad c=\frac{p-1-\sqrt{D}}{2},$$

where

$$D=(p+1)^2-4(q+1)\geq 0.$$

To end the proof, it suffices to show that

$$\frac{p-1-\sqrt{D}}{2}\geq 0$$

and

$$\frac{p-1+\sqrt{D}}{2}\leq 1.$$

Write the first inequality as

$$p-1\geq \sqrt{D}.$$

Since

$$p\geq -1+2\sqrt{q+1}\geq -1+2=1,$$

the inequality is equivalent to

$$(p-1)^2\geq D,$$

$$1-p+q\geq 0,$$

$$(1-a)(1-b)(1-c)+abc\geq 0.$$

Write the second inequality as

$$3-p\geq \sqrt{D},$$

$$(3-p)^2\geq D,$$

$$q+3\geq 2p,$$

$$(1-a)(1-b)+(1-b)(1-c)+(1-c)(1-a)\geq 0.$$

(b) We will show that abc is minimal for $a=b$ if $p^2\leq 4q$, and for $c=0$ if $p^2\geq 4q$.

Case 1: $p^2\leq 4q$. According to P 2.53 in Volume 1, under the weaker condition $a\geq b\geq c$ instead of $1\geq a\geq b\geq c\geq 0$, the product $r=abc$ is minimal for $a=b$, when

$$a=b=\frac{p+\sqrt{p^2-3q}}{3}, \quad c=\frac{p-2\sqrt{p^2-3q}}{3}.$$

Thus, it suffices to show that

$$\frac{p-2\sqrt{p^2-3q}}{3}\geq 0$$

and

$$\frac{p+\sqrt{p^2-3q}}{3} \le 1.$$

Write the first inequality as

$$p \ge 2\sqrt{p^2-3q},$$

$$p^2 \ge 4(p^2-3q),$$

$$p^2 \le 4q.$$

Write now the second inequality as

$$3-p \ge \sqrt{p^2-3q},$$

$$(3-p)^2 \ge p^2-3q,$$

$$q+3 \ge 2p,$$

$$(1-a)(1-b)+(1-b)(1-c)+(1-c)(1-a) \ge 0.$$

Case 1: $p^2 \ge 4q$. From

$$0 \le p^2-4q = (a-b)^2 - c(a+b-c) \le (a-b)^2 - c^2 = (a-b-c)(a-b+c),$$

we get $a \ge b+c$, hence

$$p = a+b+c \le 2a \le 2.$$

For $c=0$, we have

$$a = \frac{p+\sqrt{p^2-4q}}{2}, \quad b = \frac{p-\sqrt{p^2-4q}}{2}, \quad c = 0.$$

Since $p-\sqrt{p^2-4q} \ge 0$, we only need to show that

$$\frac{p+\sqrt{p^2-4q}}{2} \le 1,$$

which is equivalent to

$$2-p \ge \sqrt{p^2-4q},$$

$$(2-p)^2 \ge p^2-4q,$$

$$1-p+q \ge 0,$$

$$(1-a)(1-b)(1-c)+abc \ge 0.$$

□

P 1.173. *If $a \ge 1 \ge b \ge c \ge 0$ such that $a+b+c=3$, then*

$$abc + \frac{9}{ab+bc+ca} \ge 4.$$

(Vasile C., 2015)

Solution. Let

$$q = ab + bc + ca.$$

First Solution. According to P 1.167, for fixed q, the product abc is minimal when $b=1$ or $c=0$. Therefore, it suffices to consider these cases. If $b=1$, then $a+c=2$, and the inequality becomes

$$ac + \frac{9}{2+ac} \ge 4,$$

$$(ac-1)^2 \ge 0.$$

For $c=0$, we need to show that $a+b=3$ involves $4ab \le 9$. Indeed,

$$4ab < (a+b)^2 = 9.$$

The equality holds for $a=b=c=1$.

Second Solution. From $(a-1)(b-1)(c-1) \ge 0$, we get

$$abc \ge q-2.$$

Therefore,

$$abc + \frac{9}{ab+bc+ca} - 4 \ge q - 2 + \frac{9}{q} - 4 = \frac{(q-3)^2}{q} \ge 0.$$

□

P 1.174. *If $a \ge 1 \ge b \ge c \ge 0$ such that $a+b+c=3$, then*

$$abc + \frac{2}{ab+bc+ca} \ge \frac{5}{a^2+b^2+c^2}.$$

(Vasile C., 2015)

Solution. Let

$$q = ab + bc + ca, \quad q \le 3.$$

First Solution. According to P 1.167, for fixed q, the product abc is minimal when $b=1$ or $c=0$. Therefore, it suffices to consider these cases. For $b=1$, when $a+c=2$, the inequality becomes

$$ac + \frac{2}{2+ac} \ge \frac{5}{5-2ac},$$

$$ac(1-ac)(1+2ac) \geq 0.$$

The last inequality is true since

$$4 = (a+c)^2 \geq 4ac.$$

For $c = 0$, we need to show that $a + b = 3$ involves

$$\frac{2}{ab} \geq \frac{5}{9-2ab},$$

that is $ab \leq 2$. Indeed,

$$ab - 2 = ab - a - b + 1 = (a-1)(b-1) \leq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 2$, $b = 1$ and $c = 0$.

Second Solution. Write the inequality as

$$abc + \frac{2}{q} \geq \frac{5}{9-2q}.$$

Case 1: $q \leq 2$. We have

$$abc + \frac{2}{q} - \frac{5}{9-2q} \geq \frac{2}{q} - \frac{5}{9-2q} = \frac{9(q-2)}{q(9-2q)} \geq 0.$$

Case 2: $2 \leq q \leq 3$. From $(a-1)(b-1)(c-1) \geq 0$, we get

$$abc \geq q-2,$$

hence

$$abc + \frac{2}{q} - \frac{5}{9-2q} \geq q - 2 + \frac{2}{q} - \frac{5}{9-2q} = \frac{(3-q)(q-2)(2q-3)}{q(9-2q)} \geq 0.$$

□

P 1.175. *If $a \geq b \geq 1 \geq c > 0$ such that $a + b + c = 3$, then*

$$\frac{1}{abc} + 2 \geq \frac{9}{ab+bc+ca}.$$

(Vasile C., 2015)

Solution. Let

$$q = ab + bc + ca.$$

First Solution. According to P 1.170, for fixed q, the product abc is maximal for $b = 1$. Therefore, it suffices to consider the case $b = 1$, when $a + c = 2$, and the inequality becomes

$$\frac{1}{ac} + 2 \ge \frac{9}{2 + ac},$$

$$(ac - 1)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. From $(a-1)(b-1)(c-1) \le 0$, we get

$$abc \le q - 2, \quad q > 2.$$

Thus, it suffices to show that

$$\frac{1}{q-2} + 2 \ge \frac{9}{q},$$

which is equivalent to

$$(q - 3)^2 \ge 0.$$

□

P 1.176. *If $a \ge b \ge 1 \ge c > 0$ such that $a + b + c = 3$, then*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 11 \ge 4(a^2 + b^2 + c^2).$$

(Vasile C., 2015)

Solution. Let

$$q = ab + bc + ca.$$

First Solution. Write the inequality as

$$\frac{q}{abc} + 8q \ge 25.$$

According to P 1.170, for fixed q, the product abc is maximal when $b = 1$. Therefore, it suffices to consider the case $b = 1$, when $a + c = 2$, and the inequality becomes

$$\frac{1}{ac} + 4ac \ge 4,$$

$$(2ac - 1)^2 \ge 0.$$

The equality holds for $a = 1 + \dfrac{1}{\sqrt{2}}$, $b = 1$ and $a = 1 - \dfrac{1}{\sqrt{2}}$.

Second Solution. From $(a-1)(b-1)(c-1) \leq 0$, we get

$$abc \leq q-2, \quad q > 2.$$

Thus, it suffices to show that

$$\frac{q}{q-2} + 11 \geq 4(9-2q),$$

which is equivalent to

$$(2q-5)^2 \geq 0.$$

□

P 1.177. *If $a \geq b \geq 1 \geq c > 0$ such that $a+b+c=3$, then*

$$\frac{1}{abc} + \frac{2}{a^2+b^2+c^2} \geq \frac{5}{ab+bc+ca}.$$

(Vasile C., 2015)

Solution. Let

$$q = ab+bc+ca.$$

First Solution. According to P 1.170, for fixed q, the product abc is maximal when $b=1$. Therefore, it suffices to consider the case $b=1$, when the inequality becomes

$$\frac{1}{ac} + \frac{2}{5-2ac} \geq \frac{5}{2+ac},$$

$$(ac-1)^2 \geq 0.$$

The equality holds for $a=b=c=1$.

Second Solution. From $(a-1)(b-1)(c-1) \leq 0$, we get

$$abc \leq q-2, \quad q > 2.$$

Thus, it suffices to show that

$$\frac{1}{q-2} + \frac{2}{9-2q} \geq \frac{5}{q},$$

which is equivalent to

$$(q-3)^2 \geq 0.$$

□

P 1.178. *If $a \ge b \ge 1 \ge c \ge 0$ such that $a+b+c=3$, then*

$$\frac{9}{a^3+b^3+c^3}+2 \le \frac{15}{a^2+b^2+c^2}.$$

(Vasile C., 2015)

Solution. Write the inequality as

$$\frac{3}{abc+9-3q}+2 \le \frac{15}{9-2q},$$

where

$$q = ab+bc+ca.$$

From

$$3q \le (a+b+c)^2 = 9$$

and

$$q = (1-a)(1-b)(1-c)+abc-1+a+b+c \ge -1+a+b+c = 2,$$

it follows that

$$2 \le q \le 3.$$

First Solution. Consider the following two cases.

Case 1: $2 \le q \le 9/4$. Since $abc \ge 0$, it suffices to prove that

$$\frac{1}{3-q}+2 \le \frac{15}{9-2q},$$

which is equivalent to the obvious inequality

$$(4q-9)(q-2) \le 0.$$

Case 2: $9/4 \le q \le 3$. By Schur's inequality of third degree

$$(a+b+c)^3+9abc \ge 4(a+b+c)(ab+bc+ca),$$

we get

$$3abc \ge 4q-9.$$

Therefore, it suffices to show that

$$\frac{9}{4q-9+3(9-3q)}+2 \le \frac{15}{9-2q},$$

which is equivalent to

$$\frac{9}{18-5q}+2 \le \frac{15}{9-2q},$$

$$4q^2 - 21q + 27 \leq 0,$$

$$(q-3)(4q-9) \leq 0.$$

The equality holds for $a = b = c = 1$, for $a = b = 3/2$ and $c = 0$, and also for $a = 2$, $b = 1$ and $c = 0$.

Second Solution. According to P 1.170, for fixed q, the product abc is minimal when $a = b$ or $c = 0$. Therefore, it suffices to consider these cases.

Case 1: $a = b \in [1, 3/2]$. The desired inequality is equivalent to

$$\frac{9}{2a^3 + (3-2a)^3} + 2 \leq \frac{15}{2a^2 + (3-2a)^2}.$$

$$(a-1)^2(3-2a)(9a - 2a^2 - 3) \geq 0,$$

which is true since

$$9a - 2a^2 - 3 > 3(3a - a^2 - 2) = 3(a-1)(2-a) \geq 0.$$

Case 2: $c = 0$. We have $2 \leq q \leq 9/4$, because

$$q = ab \leq \frac{1}{4}(a+b)^2 = \frac{9}{4}.$$

The desired inequality is equivalent to

$$\frac{1}{3-q} + 2 \leq \frac{15}{9-2q},$$

$$(4q-9)(q-2) \leq 0.$$

Clearly, the last inequality is true.

□

P 1.179. *If $a \geq b \geq 1 \geq c \geq 0$ such that $a + b + c = 3$, then*

$$\frac{36}{a^3 + b^3 + c^3} + 9 \leq \frac{65}{a^2 + b^2 + c^2}.$$

(Vasile C., 2015)

Solution. Write the inequality as

$$\frac{12}{abc + 9 - 3q} + 9 \leq \frac{65}{9-2q},$$

where

$$q = ab + bc + ca.$$

From

$$3q \le (a+b+c)^2 = 9$$

and

$$q = (1-a)(1-b)(1-c) + abc - 1 + a + b + c \ge -1 + a + b + c = 2,$$

it follows that

$$2 \le q \le 3.$$

First Solution. Consider the following two cases.
Case 1: $2 \le q \le 7/3$. Since $abc \ge 0$, it suffices to prove that

$$\frac{4}{3-q} + 9 \le \frac{65}{9-2q},$$

which is equivalent to the obvious inequality

$$(3q-7)(q-2) \le 0.$$

Case 2: $7/3 \le q \le 3$. By Schur's inequality of third degree

$$(a+b+c)^3 + 9abc \ge 4(a+b+c)(ab+bc+ca),$$

we get

$$3abc \ge 4q - 9.$$

Therefore, it suffices to show that

$$\frac{36}{4q-9+3(9-3q)} + 9 \le \frac{65}{9-2q},$$

which is equivalent to

$$\frac{198-45q}{18-5q} \le \frac{65}{9-2q}.$$

We will prove the sharper inequality

$$\frac{200-45q}{18-5q} \le \frac{65}{9-2q},$$

which is equivalent to

$$\frac{40-9q}{18-5q} \le \frac{13}{9-2q},$$

$$(q-3)(3q-7) \le 0.$$

The last inequality is clearly true. The equality holds for $a = 2$, $b = 1$ and $c = 0$.

Second Solution. According to the preceding P 1.178, it suffices to show that

$$4\left(\frac{15}{a^2+b^2+c^2} - 2\right) + 9 \le \frac{65}{a^2+b^2+c^2},$$

which is equivalent to

$$a^2+b^2+c^2 \geq 5,$$

$$ab+bc+ca \geq 2.$$

□

P 1.180. *If* $a \geq b \geq c \geq 0$ *and* $ab+bc+ca=2$ *, then*

$$\sqrt{a+ab}+\sqrt{b+bc}+\sqrt{c+ca} \geq 3.$$

(KaiRain, 2020)

Proof. Consider the main case $a \geq b \geq c$ and show that

$$\sqrt{a+ab}+\sqrt{b+bc}+\sqrt{c+ca} \geq 3.$$

For $c=0$, we need to show that $ab=2$ involves

$$\sqrt{a+ab}+\sqrt{b} \geq 3,$$

that is

$$\sqrt{a+2}+\sqrt{\frac{2}{a}} \geq 3.$$

Denoting $x=\sqrt{\frac{a}{2}}$, we need to show that

$$\sqrt{2x^2+2} \geq 3-\frac{1}{x}.$$

This is true if

$$2(x^2+1) \geq \left(3-\frac{1}{x}\right)^2$$

for $x \geq 1/3$, which is equivalent to the obvious inequality

$$(x-1)^2(2x^2+4x-1) \geq 0.$$

Using this result, it suffices to show that

$$\sqrt{a+ab}+\sqrt{b+bc}+\sqrt{c+ca} \geq \sqrt{a+2}+\sqrt{\frac{2}{a}},$$

that is equivalent to

$$\sqrt{c+ca} \geq \sqrt{a+2}-\sqrt{a+ab}+\sqrt{\frac{2}{a}}-\sqrt{b+bc},$$

$$\sqrt{c+ca} \ge \frac{2-ab}{\sqrt{a+2}+\sqrt{a+ab}} + \frac{2-ab-abc}{\sqrt{2a}+a\sqrt{b+bc}},$$

$$\sqrt{c+ca} \ge \frac{c(a+b)}{\sqrt{a+2}+\sqrt{a+ab}} + \frac{c(a+b-ab)}{\sqrt{2a}+a\sqrt{b+bc}}.$$

So, we need to show that

$$\sqrt{1+a} \ge \frac{\sqrt{c}(a+b)}{\sqrt{a+2}+\sqrt{a+ab}} + \frac{\sqrt{c}(a+b-ab)}{\sqrt{2a}+a\sqrt{b+bc}}.$$

We get this inequality by summing the inequalities

$$\frac{\sqrt{1+a}}{2} \ge \frac{\sqrt{c}(a+b)}{\sqrt{a+2}+\sqrt{a+ab}}, \qquad \frac{\sqrt{1+a}}{2} \ge \frac{\sqrt{c}(a+b-ab)}{\sqrt{2a}+a\sqrt{b+bc}}.$$

From $ab+bc+ca=2$, it follows $\frac{2}{3} \le ab \le 2$ and $b \le \sqrt{2}$. Since

$$\sqrt{a+ab} \le \sqrt{a+2}$$

and

$$a\sqrt{b} \le \sqrt{2a}, \qquad a\sqrt{b} \le a\sqrt{b+bc},$$

it suffice to prove the inequalities

$$\sqrt{1+a} \ge \frac{\sqrt{c}(a+b)}{\sqrt{a+ab}}, \qquad \sqrt{1+a} \ge \frac{\sqrt{c}(a+b-ab)}{a\sqrt{b}}.$$

By squaring, the first inequality becomes

$$a(1+a)(1+b) \ge c(a+b)^2,$$

$$a(1+a)(1+b) \ge (a+b)(2-ab).$$

Since $2a \ge a+b$, it suffices to show that

$$(1+a)(1+b) \ge 2(2-ab),$$

that is

$$a+b+3ab \ge 3.$$

Indeed, we have

$$a+b+3ab \ge 2\sqrt{ab}+3ab \ge 2\sqrt{\frac{2}{3}}+2 > 3.$$

Since $\sqrt{b} \ge \sqrt{c}$, the second inequality is true if

$$a\sqrt{1+a} \ge a+b-ab,$$

that is

$$a(\sqrt{1+a}-1) \geq b(1-a).$$

For the nontrivial case $a \leq 1$, it suffices to show that

$$a(\sqrt{1+a}-1) \geq a(1-a),$$

that is

$$\sqrt{1+a}+a \geq 2.$$

Since $3a^2 \geq ab+bc+ca=2$, we have

$$\sqrt{1+a}+a \geq \sqrt{1+\sqrt{\frac{2}{3}}}+\sqrt{\frac{2}{3}} > 2.$$

The inequality is an equality for $a=2$, $b=1$, $c=0$.

Remark. The following sharper inequality holds in the same conditions:

$$\sqrt{a+ab}+\sqrt{b}+\sqrt{c} \geq 3,$$

with equality for $a=2$, $b=1$, $c=0$.

For fixed b, according to the relation $ab+bc+ca=2$, we may consider that a is a function of c. Differentiating this equation, we get

$$a' = -\frac{a+b}{b+c},$$

$$a'' = \frac{(a+b+(b-c)a'}{(a+c)^2} = \frac{(a+b)(a-b+2c)}{(a+c)^3}.$$

Write the required inequality as $f(c) \geq 0$, where

$$f(c) = \sqrt{a+ab}+\sqrt{b}+\sqrt{c}-3, \qquad c \in [0,b].$$

We have

$$f'(c) = \frac{a'\sqrt{1+b}}{2\sqrt{a}} + \frac{1}{2\sqrt{c}},$$

$$f''(c) = \frac{(2aa''-(a')^2)\sqrt{1+b}}{4a^{3/2}} - \frac{1}{4c^{3/2}}$$

$$= \frac{(a+b)(a^2+3ac-3ab-bc)\sqrt{1+b}}{4a^{3/2}(a+c)^3} - \frac{1}{4c^{3/2}}.$$

Since

$$a^2+3ac-3ab-bc = a^2-3a(b-c)-bc < a^2,$$

we have

$$f''(c) < \frac{(a+b)\sqrt{a(1+b)}}{4(a+c)^3} - \frac{1}{4c^{3/2}}.$$

From $b^2 \le ab \le ab+bc+ca=2$, we get $b \le \sqrt{2}$, $\sqrt{1+b}<4$, hence

$$f''(c) < \frac{(a+b)\sqrt{a}}{(a+c)^3} - \frac{1}{4c^{3/2}} \le 2\left(\frac{\sqrt{a}}{a+c}\right)^3 - \frac{1}{4(\sqrt{c})^3} \le 0.$$

Since f is concave and $0 \le c \le b$, it is enough to show that $f(0) \ge 0$ (for $c=0$ and $ab=2$) and $f(b) \ge 0$ (for $c=b$ and $2ab+b^2=2$). We have

$$f(0) = \sqrt{\frac{2+2b}{b}} + \sqrt{b} - 3 = \frac{(1-\sqrt{b})^2(2+4\sqrt{b}-b)}{\sqrt{b(2+2b)} - b\sqrt{b} + 3b} \ge 0.$$

For $c=b$, when $2 = 2ab+b^2 \ge 3b^2$, hence $b \le \sqrt{\frac{2}{3}}$, we have

$$f(b) = \sqrt{\frac{(1+b)(2-b^2)}{2b}} + 2\sqrt{b} - 3 = \frac{A}{\sqrt{2b(1+b)(2-b^2)} - 4b\sqrt{b} + 6b},$$

where, for $x = \sqrt{b} \le \sqrt[4]{\frac{2}{3}} < 1$,

$$A = (1+x^2)(2-x^4) - 2x^2(3-2x)^2 = (1-x)(2+2x-14x^2+10x^3+x^4+x^5).$$

Since

$$2+2x-14x^2+10x^3+x^4+x^5 = 2-13x^2+13x^3+(1-x)^2x(2+3x+x^2)$$

$$> 2+13x^3-13x^2 = 2 + \frac{13x^3}{2} + \frac{13x^3}{2} - 13x^2$$

$$\ge 3\sqrt[3]{2 \cdot \frac{13x^3}{2} \cdot \frac{13x^3}{2}} - 13x^2 = \left(3\sqrt[3]{\frac{169}{2}} - 13\right)x^2 > 0,$$

we have $A>0$, hence $f(b)>0$.

□

P 1.181. *If $a \ge b \ge c$ are nonnegative numbers such that $ab+bc+ca=3$, then*

$$\sqrt{a+2ab} + \sqrt{b+2bc} + \sqrt{c+2ca} \ge 4.$$

(Vasile C., 2020)

Proof. We will prove the sharper inequality

$$\sqrt{a+2ab} + \sqrt{b+bc} + \sqrt{c+ca} \ge 4.$$

For $c = 0$, we need to show that $ab = 3$ involves

$$\sqrt{a+2ab}+\sqrt{b} \ge 4,$$

that is

$$\sqrt{a+6}+\sqrt{\frac{3}{a}} \ge 4.$$

It is easy to show that this inequality is true for all $a > 0$. Using this result, it suffices to show that

$$\sqrt{a+2ab}+\sqrt{b+bc}+\sqrt{c+ca} \ge \sqrt{a+6}+\sqrt{\frac{3}{a}},$$

that is equivalent to

$$\sqrt{c+ca} \ge \sqrt{a+6}-\sqrt{a+2ab}+\sqrt{\frac{3}{a}}-\sqrt{b+bc},$$

$$\sqrt{c+ca} \ge \frac{2(3-ab}{\sqrt{a+6}+\sqrt{a+2ab}}+\frac{3-ab-abc}{\sqrt{3a}+a\sqrt{b+bc}},$$

$$\sqrt{c+ca} \ge \frac{2c(a+b)}{\sqrt{a+6}+\sqrt{a+2ab}}+\frac{c(a+b-ab)}{\sqrt{3a}+a\sqrt{b+bc}}.$$

So, we need to show that

$$\sqrt{1+a} \ge \frac{2\sqrt{c}(a+b)}{\sqrt{a+6}+\sqrt{a+2ab}}+\frac{\sqrt{c}(a+b-ab)}{\sqrt{3a}+a\sqrt{b+bc}}.$$

We get this inequality by summing the inequalities

$$k\sqrt{1+a} \ge \frac{2\sqrt{c}(a+b)}{\sqrt{a+6}+\sqrt{a+2ab}}, \qquad (1-k)\sqrt{1+a} \ge \frac{\sqrt{c}(a+b-ab)}{\sqrt{3a}+a\sqrt{b+bc}},$$

where

$$k=\sqrt{\frac{2}{3}}.$$

From $ab+bc+ca=3$, it follows $1 \le ab \le 3$ and $b \le \sqrt{3}$. Since

$$\sqrt{a+2ab} \le \sqrt{a+6},$$

the first inequality is true if

$$k\sqrt{1+a} \ge \frac{\sqrt{c}(a+b)}{\sqrt{a+2ab}},$$

that is

$$2a(1+a)(1+2b) \ge 3c(a+b)^2,$$

$$2a(1+a)(1+2b) \ge 3(3-ab)(a+b).$$

Since $2a \ge a+b$, it suffices to show that

$$(1+a)(1+2b) \ge 3(3-ab),$$

that is

$$(5b+1)a+2b \ge 8.$$

For $a \ge b \ge 1$, this inequality is obvious. For $0 \le b \le 1$, from

$$b \ge c = \frac{3-ab}{a+b}$$

we get

$$a \ge \frac{3-b^2}{2b}.$$

Therefore,

$$(5b+1)a+2b-8 \ge \frac{(5b+1)(3-b^2)}{2b}+2b$$

$$= \frac{3-b+3b^2-5b^3}{2b} = \frac{(1-b)(3+2b+5b^2)}{2b} \ge 0.$$

Since $1-k > \dfrac{1}{4}$, the second inequality is true if

$$\sqrt{1+a} \ge \frac{4\sqrt{c}(a+b-ab)}{\sqrt{3a}+a\sqrt{b+bc}},$$

Consider the nontrivial case $a+b-ab \ge 0$, and claim that $\sqrt{3a} \ge a\sqrt{b+bc}$, which is equivalent to $3 \ge ab+abc$. Indeed, we have

$$3-ab-abc = 3-ab-\frac{ab(3-ab)}{a+b} = \frac{(3-ab)(a+b-ab)}{a+b} \ge 0.$$

Thus, it suffices to show that

$$\sqrt{1+a} \ge \frac{2\sqrt{c}(a+b-ab)}{a\sqrt{b+bc}}.$$

Since

$$\frac{a+b-ab}{a} \le 1,$$

it suffices to show that

$$\sqrt{1+a} \ge 2\sqrt{\frac{c}{b(1+c)}},$$

that is

$$b(1+a)(1+c) \ge 4c.$$

Since $ab \geq 1$, we have

$$b(1+a) \geq b+1 \geq c+1,$$

therefore,

$$b(1+a)(1+c)-4c \geq (1+c)^2-4c = (1-c)^2 \geq 0.$$

The inequality is an equality for $a=3,\ b=1,\ c=0$.

$\square$

P 1.182. *If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then*

$$\sqrt{a+3b}+\sqrt{b+3c}+\sqrt{c+3a} \geq 6.$$

Solution. Use the substitution

$$\sqrt{a+3b}=2x, \quad \sqrt{b+3c}=2y, \quad \sqrt{c+3a}=2z,$$

which yields

$$a=\frac{x^2-3y^2+9z^2}{7}, \quad a=\frac{y^2-3z^2+9x^2}{7}, \quad a=\frac{z^2-3x^2+9y^2}{7},$$

$$ab+bc+ca=\frac{-3(x^4+y^4+z^4)+10(x^2y^2+y^2z^2+z^2x^2)}{7}.$$

So, we need to show that

$$x+y+z \geq 3$$

for

$$3(x^4+y^4+z^4)+21=10(x^2y^2+y^2z^2+z^2x^2).$$

By the contradiction method, we need to prove that

$$x+y+z<3$$

involves

$$3(x^4+y^4+z^4)+21>10(x^2y^2+y^2z^2+z^2x^2).$$

It suffices to prove the homogeneous inequality $f(x,y,z) \geq 0$, where

$$f(x,y,z)=81(x^4+y^4+z^4)+7(x+y+z)^4-270(x^2y^2+y^2z^2+z^2x^2).$$

According to P 3.68 from Volume 1, it is enough to show that $f(0,y,z) \geq 0$ and $f(x,1,1) \geq 0$ for $x,y,z \geq 0$. We have

$$f(0,y,z)=81(y^4+z^4)+7(y+z)^4-270y^2z^2$$

$$\geq 162y^2z^2+112y^2z^2-270y^2z^2=4y^2z^2 \geq 0$$

and

$$f(x,1,1) = 81(x^4+2)+7(x+2)^4-540x^2 = 4(22x^4+14x^3-93x^2+56x+1)$$

$$= (x-1)^2(22x^2+58x+1) \ge 0.$$

The equality occurs for $a=b=c=1$.

□

P 1.183. *If a,b,c are the lengths of the sides of a triangle, then*

$$10\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) > 9\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right).$$

Solution. According to Remark 2 from the proof of P 1.149, it suffices to show that $P(1,1,1) \ge 0$ and $P(b+c,b,c) \ge 0$ for $b,c \ge 0$, where

$$P(a,b,c) = 10(ab^2+bc^2+ca^2) - 9(a^2b+b^2c+c^2a).$$

We have $P(1,1,1) = 3 > 0$ and

$$P(b+c,b,c) = b^3 - 7b^2c + 12bc^2 + c^3.$$

We need to show that

$$x^3 - 7x^2 + 12x + 1 > 0,$$

where $x = b/c$, $x > 0$. For $x \in (0,3] \cup [4,\infty)$, we have

$$x^3-7x^2+12x+1 > x^3-7x^2+12x = x(3-x)(4-x) \ge 0.$$

For $x \in (3,4)$, we have

$$x^3-7x^2+12x+1 > x^3-7x^2+12x+\frac{x}{4} = \frac{x(2x-7)^2}{4} \ge 0.$$

□

P 1.184. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{a}{3a+b-c}+\frac{b}{3b+c-a}+\frac{c}{3c+a-b} \ge 1.$$

Solution. Write the inequality as follows:

$$\sum\left(\frac{a}{3a+b-c}-\frac{1}{4}\right)\geq\frac{1}{4},$$

$$\sum\frac{a-b+c}{3a+b-c}\geq 1.$$

Applying the Cauchy-Schwarz inequality, we get

$$\sum\frac{a-b+c}{3a+b-c}\geq\frac{\left[\sum(a-b+c)\right]^2}{\sum(a-b+c)(3a+b-c)}=\frac{\left(\sum a\right)^2}{\sum a^2+2\sum ab}=1.$$

The equality holds for $a=b=c$.

□

P 1.185. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{a^2-b^2}{a^2+bc}+\frac{b^2-c^2}{b^2+ca}+\frac{c^2-a^2}{c^2+ab}\leq 0.$$

(Vasile C., 2007)

First Solution. Suppose that $a=\max\{a,b,c\}$. Since

$$c^2-a^2=-(a^2-b^2)-(b^2-c^2),$$

the inequality can be written as follows:

$$(a^2-b^2)\left(\frac{1}{a^2+bc}-\frac{1}{c^2+ab}\right)+(b^2-c^2)\left(\frac{1}{b^2+ca}-\frac{1}{c^2+ab}\right)\leq 0,$$

$$-\frac{(a^2-b^2)(a-c)(a-b+c)}{a^2+bc}-\frac{(b^2-c^2)(b-c)(b+c-a)}{a^2+bc}\leq 0.$$

The equality holds for an equilateral triangle, and also for a degenerate triangle having a side equal to zero.

Second Solution. The sequences

$$\{a^2,\quad b^2,\quad c^2\}$$

and

$$\left\{\frac{1}{a^2+bc},\quad\frac{1}{b^2+ca},\quad\frac{1}{c^2+ab}\right\}$$

are reversely ordered. Indeed, if $a\geq b\geq c$, then

$$\frac{1}{a^2+bc}\leq\frac{1}{b^2+ca}\leq\frac{1}{c^2+ab},$$

because

$$\frac{1}{b^2+ca}-\frac{1}{a^2+bc}=\frac{(a-b)(a+b-c)}{(b^2+ca)(a^2+bc)}\geq 0,$$

$$\frac{1}{c^2+ab}-\frac{1}{b^2+ca}=\frac{(b-c)(b+c-a)}{(c^2+ab)(b^2+ca)}\geq 0.$$

Then, by the rearrangement inequality, we have

$$a^2\cdot\frac{1}{a^2+bc}+b^2\cdot\frac{1}{b^2+ca}+c^2\cdot\frac{1}{c^2+ab}\leq$$

$$\leq b^2\cdot\frac{1}{a^2+bc}+c^2\cdot\frac{1}{b^2+ca}+a^2\cdot\frac{1}{c^2+ab},$$

which is the desired inequality.

□

P 1.186. *If a,b,c are the lengths of the sides of a triangle, then*

$$a^2(a+b)(b-c)+b^2(b+c)(c-a)+c^2(c+a)(a-b)\geq 0.$$

(Vasile C., 2006)

First Solution. Assume that

$$a=\max\{a,b,c\},$$

use the substitution

$$a=x+p+q,\quad b=x+p,\quad c=x+q,\qquad x,p,q\geq 0,$$

and write the inequality as

$$a^2b^2+b^2c^2+c^2a^2-abc(a+b+c)\geq ab^3+bc^3+ca^3-a^3b-b^3c-c^3a,$$

$$a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2\geq 2(a+b+c)(a-b)(b-c)(c-a),$$

$$(x+p+q)^2(p-q)^2+(x+p)^2p^2+(x+q)^2q^2\geq 2(3x+2p+2q)pq(q-p),$$

which is equivalent to

$$Ax^2+2Bx+C\geq 0,$$

where

$$A=p^2-pq+q^2\geq 0,$$

$$B=p^3+q(p-q)^2\geq 0,$$

$$C=(p^2+pq-q^2)^2\geq 0.$$

The equality holds for an equilateral triangle, and also for a degenerate triangle with

$$\frac{a}{2}=\frac{b}{1+\sqrt{5}}=\frac{c}{3+\sqrt{5}}$$

(or any cyclic permutation).

Second Solution. Using the substitution

$$x=\sqrt{\frac{ca}{b}},\quad y=\sqrt{\frac{ab}{c}},\quad z=\sqrt{\frac{bc}{a}},$$

we can write the inequality as follows:

$$b^2c^2+c^2a^2+a^2b^2\ge ab(b^2+c^2-a^2)+bc(c^2+a^2-b^2)+ca(a^2+b^2-c^2),$$

$$\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\ge 2b\cos A+2c\cos B+2a\cos C,$$

$$x^2+y^2+z^2\ge 2yz\cos A+2zx\cos B+2xy\cos C,$$

$$(x-y\cos C-z\cos B)^2+(y\sin C-z\sin B)^2\ge 0.$$

□

P 1.187. *If a,b,c are the lengths of the sides of a triangle, then*

$$a^2b+b^2c+c^2a\ge\sqrt{abc(a+b+c)(a^2+b^2+c^2)}.$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2005)

Solution. Without loss of generality, assume that b is between a and c; that is

$$(b-a)(b-c)\le 0.$$

First Solution. By the AM-GM inequality, we have

$$4abc(a+b+c)(a^2+b^2+c^2)\le[ac(a+b+c)+b(a^2+b^2+c^2)]^2.$$

Thus, we only need to show that

$$2(a^2b+b^2c+c^2a)\ge ac(a+b+c)+b(a^2+b^2+c^2),$$

which is equivalent to

$$b[a^2-(b-c)^2]-ac(a+b-c)\ge 0,$$

$$(a+b-c)(a-b)(b-c)\ge 0.$$

The equality holds for an equilateral triangle, and also for a degenerate triangle with

$$c = a + b, \qquad b^3 = a^2(a + b)$$

(or any cyclic permutation).

Second Solution. The desired inequality is equivalent to $D \ge 0$, where D is the discriminant of the quadratic function

$$f(x) = (a^2 + b^2 + c^2)x^2 - 2(a^2b + b^2c + c^2a)x + abc(a + b + c).$$

For the sake of contradiction, assume that $D < 0$ for some a, b, c. Then, $f(x) > 0$ for all real x. This is not true, because

$$f(b) = b(b - a)(b - c)(a + b - c) \le 0.$$

□

P 1.188. *If a, b, c are the lengths of the sides of a triangle, then*

$$a^2\left(\frac{b}{c} - 1\right) + b^2\left(\frac{c}{a} - 1\right) + c^2\left(\frac{a}{b} - 1\right) \ge 0.$$

(Vasile Cîrtoaje, Moldova TST, 2006)

First Solution. Using the substitution

$$a = \frac{1}{x}, \quad b = \frac{1}{y}, \quad c = \frac{1}{z},$$

the inequality becomes

$$E(x, y, z) \ge 0,$$

where

$$E(x, y, z) = yz^2(z - y) + zx^2(x - z) + xy^2(y - x).$$

Without loss of generality, assume that

$$a = \min\{a, b, c\}, \qquad x = \max\{x, y, z\}.$$

We will show that

$$E(x, y, z) \ge E(y, y, z) \ge 0.$$

We have

$$\begin{aligned} E(x, y, z) - E(y, y, z) &= z(x^3 - y^3) - z^2(x^2 - y^2) + y^3(x - y) - y^2(x^2 - y^2) \\ &= (x - y)(x - z)(xz + yz - y^2) \ge 0, \end{aligned}$$

because

$$xz + yz - y^2 \ge y(2z - y) = \frac{2b - c}{b^2c} = \frac{(b-a) + (a+b-c)}{b^2c} \ge 0.$$

Also,

$$E(y, y, z) = yz(y - z)^2 \ge 0.$$

The equality holds for $a = b = c$.

Second Solution. Write the inequality as $F(a, b, c) \ge 0$, where

$$F(a, b, c) = a^3b^2 + b^3c^2 + c^3a^2 - abc(a^2 + b^2 + c^2).$$

Since

$$\begin{aligned} 2E(a, b, c) &= \left(\sum a^3b^2 + \sum a^2b^3 - 2abc\sum a^2\right) - \left(\sum a^2b^3 - \sum a^3b^2\right) \\ &= \left(\sum a^3b^2 + \sum a^3c^2 - 2abc\sum a^2\right) - \left(\sum a^2b^3 - \sum a^2c^3\right) \\ &= \sum a^3(b-c)^2 - \sum a^2(b^3 - c^3) \end{aligned}$$

and

$$\sum a^2(b^3 - c^3) = \sum a^2(b - c)^3,$$

we get

$$E(a, b, c) = \sum a^3(b-c)^2 - \sum a^2(b-c)^3 = \sum a^2(b-c)^2(a - b + c) \ge 0.$$

Third Solution. By the Cauchy-Schwarz inequality, we have

$$\sum \frac{a^2b}{c} \ge \frac{\left(\sum a^2b\right)^2}{\sum a^2bc}.$$

Therefore, it suffices to show that

$$\left(\sum a^2b\right)^2 \ge abc(a + b + c)(a^2 + b^2 + c^2),$$

which is the inequality from the preceding P 1.187.

□

P 1.189. *If a, b, c are the lengths of the sides of a triangle, then*

(a) $$a^3b + b^3c + c^3a \ge a^2b^2 + b^2c^2 + c^2a^2;$$

(b) $$3(a^3b + b^3c + c^3a) \ge (ab + bc + ca)(a^2 + b^2 + c^2);$$

(c) $$\frac{a^3b + b^3c + c^3}{3} \ge \left(\frac{a + b + c}{3}\right)^4.$$

Solution. (a) ***First Solution.*** Write the inequality as

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\ge 0.$$

Using the substitution

$$a=y+z,\quad b=z+x,\quad c=x+y,\qquad x,y,z\ge 0,$$

the inequality turns into

$$xy^3+yz^3+zx^3\ge xyz(x+y+z),$$

which follows from the Cauchy-Schwarz inequality

$$(xy^3+yz^3+zx^3)(z+x+y)\ge xyz(y+z+x)^2.$$

The equality holds for an equilateral triangle, and also for a degenerate triangle with $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Multiplying by $a+b+c$, the inequality becomes as follows:

$$\sum a^4b+abc\sum a^2\ge\sum a^2b^3+abc\sum ab,$$

$$\sum b^4c+abc\sum a^2\ge\sum b^2c^3+abc\sum ab,$$

$$\sum\frac{b^3}{a}+\sum a^2\ge\sum\frac{bc^2}{a}+\sum ab,$$

$$\sum a^2\ge\sum\frac{b}{a}(c^2+a^2-b^2),$$

$$a^2+b^2+c^2\ge 2bc\cos B+2ca\cos C+2ab\cos A,$$

$$(a-b\cos A-c\cos C)^2+(b\sin A-c\sin C)^2\ge 0.$$

(b) Write the inequality as

$$\sum a^2b(a-b)+\sum b^2(a-b)(a-c)\ge 0.$$

Since $\sum a^2b(a-b)\ge 0$ (according to the inequality in (a)), it suffices to show that

$$\sum b^2(a-b)(a-c)\ge 0.$$

This is a particular case ($x=c$, $y=a$, $z=b$) of the following inequality

$$(x-y)(x-z)a^2+(y-z)(y-x)b^2+(z-x)(z-y)c^2\ge 0,$$

where x,y,z are real numbers. If two of x,y,z are equal, then the inequality is trivial. Otherwise, assume that $x>y>z$ and write the inequality as

$$\frac{a^2}{y-z}+\frac{c^2}{x-y}\ge\frac{b^2}{x-z}.$$

Applying the Cauchy-Schwarz inequality, we get

$$\frac{a^2}{y-z}+\frac{c^2}{x-y}\geq\frac{(a+c)^2}{(y-z)+(x-y)}=\frac{(a+c)^2}{x-z}\geq\frac{b^2}{x-z}.$$

The equality holds for $a=b=c$.

(c) According to the inequality (b), it suffices to show that

$$9(ab+bc+ca)(a^2+b^2+c^2)\geq(a+b+c)^4.$$

This is equivalent to

$$(A-B)(4B-A)\geq 0,$$

where

$$A=a^2+b^2+c^2,\qquad B=ab+bc+ca.$$

Since $A\geq B$ and

$$\begin{aligned}4B-A&>2(ab+bc+ca)-a^2-b^2-c^2\\&=a(2b+2c-a)-(b-c)^2\\&\geq a^2-(b-c)^2\\&=(a-b+c)(a+b-c)\geq 0.\end{aligned}$$

the conclusion follows. The equality holds for $a=b=c$.

□

P 1.190. *If a,b,c are the lengths of the sides of a triangle, then*

$$2\left(\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}\right)\geq\frac{b^2}{a^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}+3.$$

Solution. Write the inequality as follows:

$$\sum\frac{a^2}{b^2}\geq 3+\sum\frac{b^2}{a^2}-\sum\frac{a^2}{b^2},$$

$$\sum\frac{b^2}{c^2}\geq 3+\sum\frac{c^2}{b^2}-\sum\frac{a^2}{b^2},$$

$$\sum\frac{b^2}{c^2}\geq\sum\left(1+\frac{c^2}{b^2}-\frac{a^2}{b^2}\right),$$

$$\sum\frac{b^2}{c^2}\geq 2\sum\frac{c}{b}\cos A.$$

Putting

$$x=\frac{b}{c},\quad y=\frac{c}{a},\quad z=\frac{a}{b},$$

we have $xyz = 1$ and

$$\frac{c}{b} = \frac{1}{x} = yz, \quad \frac{a}{c} = \frac{1}{y} = zx, \quad \frac{b}{a} = \frac{1}{z} = xy.$$

Therefore, we can write the inequality as

$$x^2 + y^2 + z^2 \geq 2yz\cos A + 2zx\cos B + 2xy\cos C,$$

which is equivalent to the obvious inequality

$$(x - y\cos C - z\cos B)^2 + (y\sin C - z\sin B)^2 \geq 0.$$

The equality occurs for $a = b = c$.

□

P 1.191. *If a, b, c are the lengths of the sides of a triangle such that $a < b < c$, then*

$$\frac{a^2}{a^2 - b^2} + \frac{b^2}{b^2 - c^2} + \frac{c^2}{c^2 - a^2} \leq 0.$$

(Vasile C., 2003)

Solution. Write the inequality as

$$\frac{a^2}{b^2 - a^2} + \frac{b^2}{c^2 - b^2} \geq \frac{c^2}{c^2 - a^2}.$$

Since $c \leq a + b$, it suffices to show that

$$\frac{a^2}{b^2 - a^2} + \frac{b^2}{c^2 - b^2} \geq \frac{(a+b)^2}{c^2 - a^2},$$

which is equivalent to

$$a^2\left(\frac{1}{b^2 - a^2} - \frac{1}{c^2 - a^2}\right) + b^2\left(\frac{1}{c^2 - b^2} - \frac{1}{c^2 - a^2}\right) \geq \frac{2ab}{c^2 - a^2},$$

$$\frac{a^2(c^2 - b^2)}{b^2 - a^2} + \frac{b^2(b^2 - a^2)}{c^2 - b^2} \geq 2ab,$$

$$\left(a\sqrt{\frac{c^2 - b^2}{b^2 - a^2}} - b\sqrt{\frac{b^2 - a^2}{c^2 - b^2}}\right)^2 \geq 0.$$

The equality occurs for a degenerate triangle with $c = a + b$ and $a = xb$, where $x \approx 0.53209$ is the positive root of the equation $x^3 + 3x^2 - 1 = 0$.

□

P 1.192. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\ge 2\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right).$$

(Manlio Marangelli, 2008)

First Solution. Assume that $c=\max\{a,b,c\}$. If $a\le b\le c$, then the inequality follows from P 1.157. Consider further that

$$b\le a\le c.$$

Write the inequality as follows:

$$\sum\left(\frac{a}{b}-1\right)\ge 2\sum\left(\frac{b+c}{c+a}-1\right),$$

$$\sum(a-b)\left(\frac{1}{b}+\frac{2}{c+a}\right)\ge 0,$$

$$(a-b)\left(\frac{1}{b}+\frac{2}{c+a}\right)+[(b-a)+(a-c)]\left(\frac{1}{c}+\frac{2}{a+b}\right)+(c-a)\left(\frac{1}{a}+\frac{2}{b+c}\right)\ge 0,$$

$$(a-b)\left(\frac{1}{b}+\frac{2}{c+a}-\frac{1}{c}-\frac{2}{a+b}\right)+(c-a)\left(\frac{1}{a}+\frac{2}{b+c}-\frac{1}{c}-\frac{2}{a+b}\right)\ge 0,$$

$$(a-b)(c-b)\left[\frac{1}{bc}-\frac{2}{(a+b)(a+c)}\right]+(c-a)^2\left[\frac{1}{ac}-\frac{2}{(a+b)(b+c)}\right]\ge 0.$$

Since

$$\frac{1}{bc}-\frac{2}{(a+b)(a+c)}=\frac{c(a-b)+a(a+b)}{bc(a+b)(a+c)}\ge\frac{a(a+b)}{bc(a+b)(a+c)}=\frac{a}{bc(a+c)}$$

and

$$\frac{1}{ac}-\frac{2}{(a+b)(b+c)}=\frac{-c(a-b)+b(a+b)}{ac(a+b)(b+c)}>\frac{-c(a-b)}{ac(a+b)(b+c)}=\frac{-(a-b)}{a(a+b)(b+c)},$$

it suffices to show that

$$\frac{(a-b)(c-b)a}{bc(a+c)}-\frac{(c-a)^2(a-b)}{a(a+b)(b+c)}\ge 0,$$

which is true if

$$\frac{(c-b)a}{bc(a+c)}\ge\frac{(c-a)^2}{a(a+b)(b+c)}.$$

We can get this by multiplying the inequalities

$$c-b\ge c-a,$$

$$\frac{1}{b} \ge \frac{1}{a},$$
$$\frac{1}{c} \ge \frac{1}{a+b},$$
$$\frac{a}{a+c} \ge \frac{c-a}{b+c}.$$

The last inequality is true since

$$\frac{a}{a+c} - \frac{c-a}{b+c} \ge \frac{a}{a+c} - \frac{b}{b+c} = \frac{c(a-b)}{(a+c)(b+c)} \ge 0.$$

The equality holds for $a = b = c$.

Second Solution (*by Vo Quoc Ba Can*). Since

$$\sum \frac{a+b}{b+c} = \sum\left(1 + \frac{a-c}{b+c}\right) = 3 + \sum \frac{a-c}{b+c},$$

we can write the desired inequality as

$$\sum \frac{a}{b} - 3 \ge 2\sum \frac{a-c}{b+c}.$$

Since

$$(ab+bc+ca)\left(\sum \frac{a}{b} - 3\right) = \sum a^2 - 2\sum ab + \sum \frac{a^2c}{b}$$

and

$$(ab+bc+ca)\sum \frac{a-c}{b+c} = [a(b+c)+bc]\sum \frac{a-c}{b+c}$$
$$= \sum a^2 - \sum ab + \sum \frac{bc(a-c)}{b+c},$$

the inequality is equivalent to

$$\sum \frac{a^2c}{b} + 2\sum \frac{bc(c-a)}{b+c} \ge \sum a^2.$$

Since

$$\sum \frac{a^2c}{b} \ge \sum a^2$$

(see the inequality in P 1.188), we only need to show that

$$\sum \frac{bc(c-a)}{b+c} \ge 0.$$

Write this inequality as follows:

$$\sum bc(c^2 - a^2)(a+b) \ge 0,$$

$$\sum(c^2-a^2)\left(1+\frac{b}{a}\right)\geq 0,$$

$$\sum(c^2-a^2)\frac{b}{a}\geq 0,$$

$$\sum\frac{bc^2}{a}\geq\sum ab.$$

According to P 1.188, we have

$$\sum\frac{bc^2}{a}\geq\sum a^2\geq\sum ab.$$

□

P 1.193. *Let a, b, c be the lengths of the sides of a triangle. If $k\geq 2$, then*

$$a^kb(a-b)+b^kc(b-c)+c^ka(c-a)\geq 0.$$

(Vasile C., 1986)

Solution *(by Darij Grinberg)*. For $k=2$, we get the known inequality (a) in P 1.189:

$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a)\geq 0.$$

We will prove the following more general statement: if f is an increasing nonnegative function defined on $[0,\infty)$, then

$$E(a,b,c)\geq 0,$$

where

$$E(a,b,c)=a^2bf(a)(a-b)+b^2cf(b)(b-c)+c^2af(c)(c-a).$$

For $f(x)=x^{k-2}$, $k\geq 2$, we get the original inequality. In order to prove the claimed generalization, assume that $a=\max\{a,b,c\}$. There are two cases to consider.

Case 1: $a\geq b\geq c$. Since

$$f(a)\geq f(b)\geq f(c)\geq 0,$$

we have

$$\begin{aligned}E(a,b,c)&\geq a^2bf(c)(a-b)+b^2cf(c)(b-c)+c^2af(c)(c-a)\\&=f(c)[a^2b(a-b)+b^2c(b-c)+c^2a(c-a)]\geq 0.\end{aligned}$$

Case 2: $a\geq c\geq b$. Since

$$f(a)\geq f(c)\geq f(b)\geq 0,$$

we have

$$\begin{aligned}E(a,b,c) &\geq a^2bf(a)(a-b)+b^2cf(a)(b-c)+c^2af(a)(c-a)\\ &= f(a)[a^2b(a-b)+b^2c(b-c)+c^2a(c-a)] \geq 0.\end{aligned}$$

The equality holds for $a=b=c$, and also for a degenerate triangle with $a=0$ and $b=c$ (or any cyclic permutation).

□

P 1.194. *Let a,b,c be the lengths of the sides of a triangle. If $k \geq 1$, then*

$$3(a^{k+1}b+b^{k+1}c+c^{k+1}a) \geq (a+b+c)(a^kb+b^kc+c^ka).$$

Solution. For $k=1$, the inequality is equivalent to

$$2(a^2b+b^2c+c^2a) \geq ab^2+bc^2+ca^2+3abc,$$

$$(2c-a)b^2+(2a^2-3ac-c^2)b-ac(a-2c) \geq 0.$$

Assuming that $a=\min\{a,b,c\}$ and making the substitution

$$b = x+\frac{a+c}{2},$$

this inequality becomes

$$(2c-a)x^2+\left(x+\frac{3a}{4}\right)(a-c)^2 \geq 0.$$

It is true since

$$4x+3a = a+4b-2c = 2(a+b-c)+(2b-a) > 0.$$

In order to prove the desired inequality for $k>1$, we rewrite it as

$$a^kb(2a-b-c)+b^kc(2b-c-a)+c^ka(2c-a-b) \geq 0.$$

We will prove that if f is an increasing nonnegative function defined on $[0,\infty)$, then $E(a,b,c) \geq 0$, where

$$E(a,b,c) = ab(2a-b-c)f(a)+bc(2b-c-a)f(b)+ca(2c-a-b)f(c).$$

For $f(x)=x^{k-1}$, $k \geq 1$, we get the original inequality. In order to prove this generalization, assume that $a=\max\{a,b,c\}$. There are two cases to consider.

Case 1: $a \ge b \ge c$. Since $f(a) \ge f(b) \ge f(c) \ge 0$, we have

$$\begin{aligned} E(a,b,c) &\ge ab(2a-b-c)f(b) + bc(2b-c-a)f(b) + ca(2c-a-b)f(c) \\ &= b[2(a-b)(a-c) + ab - c^2]f(b) + ca(2c-a-b)f(c) \\ &\ge b[2(a-b)(a-c) + ab - c^2]f(c) + ca(2c-a-b)f(c) \\ &= [2(a^2b + b^2c + c^2a) - ab^2 - bc^2 - ca^2 - 3abc]f(c) \ge 0. \end{aligned}$$

Case 2: $a \ge c \ge b$. Since $f(a) \ge f(c) \ge f(b) \ge 0$, we have

$$\begin{aligned} E(a,b,c) &\ge ab(2a-b-c)f(c) + bc(2b-c-a)f(b) + ca(2c-a-b)f(c) \\ &= a[(c-b)(2c-a) + b(a-b)]f(c) + bc(2b-c-a)f(b). \end{aligned}$$

Since

$$(c-b)(2c-a) + b(a-b) \ge (c-b)(b+c-a) + b(a-b) \ge 0,$$

we get

$$\begin{aligned} E(a,b,c) &\ge a[(c-b)(2c-a) + b(a-b)]f(b) + bc(2b-c-a)f(b) \\ &= [2(a^2b + b^2c + c^2a) - ab^2 - bc^2 - ca^2 - 3abc]f(b) \ge 0. \end{aligned}$$

The equality holds for $a = b = c$.

Remark. For $k = 1$, the inequality has the form

$$2\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \ge \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 3.$$

A sharper inequality is the following

$$3\left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \ge 2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) + 3.$$

Using the substitution

$$b = x + \frac{a+c}{2},$$

this inequality turns into

$$(3c-2a)x^2 + \left(x + a - \frac{c}{4}\right)(a-c)^2 \ge 0,$$

which is true since, on the assumption $a = \min\{a,b,c\}$, we have $3c - 2a > 0$ and

$$4x + 4a - c = 2a + 4b - 3c = 3(a+b-c) + (b-a) > 0.$$

□

P 1.195. *Let a,b,c,d be positive real numbers such that $a+b+c+d=4$. Prove that*

$$\frac{a}{3+b}+\frac{b}{3+c}+\frac{c}{3+d}+\frac{d}{3+a}\ge 1.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$\sum\frac{a}{3+b}\ge\frac{\left(\sum a\right)^2}{\sum a(3+b)}=\frac{16}{12+\sum ab}.$$

Therefore, it suffices to show that

$$ab+bc+cd+da\le 4.$$

Indeed,

$$ab+bc+cd+da=(a+c)(b+d)\le\left[\frac{(a+c)+(b+d)}{2}\right]^2=2.$$

The equality occurs for $a=b=c=d=1$.

□

P 1.196. *Let a,b,c,d be positive real numbers such that $a+b+c+d=4$. Prove that*

$$\frac{a}{1+b^2}+\frac{b}{1+c^2}+\frac{c}{1+d^2}+\frac{d}{1+a^2}\ge 2.$$

Solution. Since

$$\frac{a}{1+b^2}=a-\frac{ab^2}{1+b^2},$$

the inequality is equivalent to

$$\frac{ab^2}{1+b^2}+\frac{bc^2}{1+c^2}+\frac{cd^2}{1+d^2}+\frac{da^2}{1+a^2}\le 2.$$

Since

$$\frac{ab^2}{1+b^2}\le\frac{ab^2}{2b}=\frac{ab}{2},$$

it suffices to show that

$$ab+bc+cd+da\le 4.$$

Indeed, we have

$$ab+bc+cd+da=(a+c)(b+d)\le\left[\frac{(a+c)+(b+d)}{2}\right]^2=2.$$

The equality occurs for $a=b=c=d=1$.

□

P 1.197. *If a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$a^2bc+b^2cd+c^2da+d^2ab \le 4.$$

(Song Yoon Kim, 2006)

Solution. Let (x, y, z, t) be a permutation of (a, b, c, d) such that

$$x \ge y \ge z \ge t,$$

hence

$$xyz \ge xyt \ge xzt \ge yzt.$$

By the rearrangement inequality, we have

$$\begin{aligned} a^2bc+b^2cd+c^2da+d^2ab &= a\cdot abc+b\cdot bcd+c\cdot cda+d\cdot dab \\ &\le x\cdot xyz+y\cdot xyt+z\cdot xzt+t\cdot yzt \\ &= (xy+zt)(xz+yt). \end{aligned}$$

Consequently, it suffices to show that $x+y+z+t=4$ involves

$$(xy+zt)(xz+yt) \le 4.$$

Indeed, by the AM-GM inequality, we have

$$(xy+zt)(xz+yt) \le \frac{1}{4}(xy+zt+xz+yt)^2 = \frac{1}{4}(x+t)^2(y+z)^2 \le 4,$$

because

$$(x+t)(y+z) \le \frac{1}{4}(x+t+y+z)^2 = 4.$$

The equality holds for $a=b=c=d=1$, and also for $a=2$, $b=c=1$ and $d=0$ (or any cyclic permutation).

□

P 1.198. *If a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$a(b+c)^2+b(c+d)^2+c(d+a)^2+d(a+b)^2 \le 16.$$

Solution *(by Vo Quoc Ba Can)*. Write the inequality as

$$(a+b+c+d)^3 \ge 4[a(b+c)^2+b(c+d)^2+c(d+a)^2+d(a+b)^2].$$

Since

$$(a+b+c+d)^2 \ge 4(a+b)(c+d),$$

we have

$$
\begin{aligned}
(a+b+c+d)^3 &\geq 4(a+b)(c+d)(a+b+c+d) \\
&= 4(c+d)(a+b)^2 + 4(a+b)(c+d)^2.
\end{aligned}
$$

Therefore, it suffices to show that

$$(c+d)(a+b)^2 + (a+b)(c+d)^2 \geq a(b+c)^2 + b(c+d)^2 + c(d+a)^2 + d(a+b)^2,$$

which is equivalent to

$$c(a+b)^2 + a(c+d)^2 \geq a(b+c)^2 + c(d+a)^2,$$

$$a[(c+d)^2 - (b+c)^2] + c[(a+b)^2 - (d+a)^2] \geq 0,$$

$$(b+d)(b-d)(c-a) \geq 0.$$

Similarly, due to cyclicity, the desired in equality is true if

$$(c+a)(c-a)(d-b) \geq 0.$$

Since one of the inequalities $(b-d)(c-a) \geq 0$ and $(c-a)(d-b) \geq 0$ is true, the conclusion follows. The equality holds for $a = c$ and $b = d$.

□

P 1.199. *If a, b, c, d are positive real numbers, then*

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+a} + \frac{d-a}{a+b} \geq 0.$$

Solution. We have

$$
\begin{aligned}
\frac{a-b}{b+c} + \frac{c-d}{d+a} + 2 &= \frac{a+c}{b+c} + \frac{a+c}{d+a} \\
&= (a+c)\left(\frac{1}{b+c} + \frac{1}{d+a}\right) \\
&\geq (a+c)\frac{4}{(b+c)+(d+a)} \\
&= \frac{4(a+c)}{a+b+c+d}.
\end{aligned}
$$

Similarly,

$$\frac{b-c}{c+d} + \frac{d-a}{a+b} + 2 \geq \frac{4(b+d)}{a+b+c+d}.$$

Adding these inequalities yields the desired inequality. The equality holds for $a = c$ and $b = d$.

Conjecture. *If a, b, c, d, e are positive real numbers, then*

$$\frac{a-b}{b+c} + \frac{b-c}{c+d} + \frac{c-d}{d+e} + \frac{d-e}{e+a} + \frac{e-a}{a+b} \geq 0.$$

□

P 1.200. *If a, b, c, d are positive real numbers, then*

$$\text{(a)} \qquad \frac{a-b}{a+2b+c}+\frac{b-c}{b+2c+d}+\frac{c-d}{c+2d+a}+\frac{d-a}{d+2a+b}\ge 0;$$

$$\text{(b)} \qquad \frac{a}{2a+b+c}+\frac{b}{2b+c+d}+\frac{c}{2c+d+a}+\frac{d}{2d+a+b}\le 1.$$

Solution. (a) Write the inequality as

$$\sum\left(\frac{a-b}{a+2b+c}+\frac{1}{2}\right)\ge 2,$$

$$\sum\frac{3a+c}{a+2b+c}\ge 4.$$

By the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\sum\frac{3a+c}{a+2b+c} &\ge \frac{\left[\sum(3a+c)\right]^2}{\sum(3a+c)(a+2b+c)}\\ &= \frac{16\left(\sum a\right)^2}{4\left(\sum a^2+2\sum ab+\sum ac\right)}\\ &= \frac{4\left(\sum a\right)^2}{\left(\sum a\right)^2}=4.\end{aligned}$$

The equality holds for $a=b=c=d$.

(b) Write the inequality as

$$\sum\left(\frac{1}{2}-\frac{a}{2a+b+c}\right)\ge 1,$$

$$\sum\frac{b+c}{2a+b+c}\ge 2.$$

By the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\sum\frac{b+c}{2a+b+c} &\ge \frac{\left[\sum(b+c)\right]^2}{\sum(b+c)(2a+b+c)}\\ &= \frac{4\left(\sum a\right)^2}{2\left(\sum a^2+2\sum ab+\sum ac\right)}\\ &= \frac{2\left(\sum a\right)^2}{\left(\sum a\right)^2}=2.\end{aligned}$$

The equality holds for $a = b = c = d$.

Conjecture 1. *If a, b, c, d, e are positive real numbers, then*

$$\frac{a-b}{a+2b+c}+\frac{b-c}{b+2c+d}+\frac{c-d}{c+2d+e}+\frac{d-e}{d+2e+a}+\frac{e-a}{e+2a+b}\geq 0.$$

Conjecture 2 (by *Ando*). *If $a_1, a_2, \dots, a_n$ $(n \geq 4)$ are positive real numbers, then*

$$\frac{a_1}{(n-2)a_1+a_2+a_3}+\frac{a_2}{(n-2)a_2+a_3+a_4}+\cdots+\frac{a_n}{(n-2)a_n+a_1+a_2}\leq 1.$$

□

P 1.201. *If a, b, c, d are positive real numbers such that $abcd = 1$, then*

$$\frac{1}{a(a+b)}+\frac{1}{b(b+c)}+\frac{1}{c(c+d)}+\frac{1}{d(d+a)}\geq 2.$$

(Vasile C., 2007)

Solution. Making the substitution

$$a=\sqrt{\frac{y}{x}},\quad b=\sqrt{\frac{z}{y}},\quad c=\sqrt{\frac{t}{z}},\quad d=\sqrt{\frac{x}{t}},$$

where x, y, z, t are positive real numbers, the inequality can be rewritten as

$$\frac{x}{y+\sqrt{xz}}+\frac{y}{z+\sqrt{yt}}+\frac{z}{t+\sqrt{zx}}+\frac{t}{x+\sqrt{ty}}\geq 2.$$

Since

$$2\sqrt{xz}\leq x+z,\qquad 2\sqrt{yt}\leq y+t,$$

it suffices to show that

$$\frac{x}{x+2y+z}+\frac{y}{y+2z+t}+\frac{z}{z+2t+x}+\frac{t}{t+2x+y}\geq 1.$$

By the Cauchy-Schwarz inequality, we have

$$\sum\frac{x}{z+2y+z}\geq\frac{\left(\sum x\right)^2}{\sum x(x+2y+z)}=\frac{\left(\sum x\right)^2}{\sum x^2+2\sum xy+\sum xz}=1.$$

The equality holds for $a=c=\dfrac{1}{b}=\dfrac{1}{d}$.

Conjecture 1. *If $a_1, a_2, \dots, a_n$ are positive real numbers such that $a_1a_2\cdots a_n = 1$, then*

$$\frac{1}{a_1^2+a_1a_2}+\frac{1}{a_2^2+a_2a_3}+\cdots+\frac{1}{a_n^2+a_na_1}\ge \frac{n}{2}.$$

Conjecture 2. *If $a_1, a_2, \dots, a_n$ are positive real numbers, then*

$$\frac{1}{a_1^2+a_1a_2}+\frac{1}{a_2^2+a_2a_3}+\cdots+\frac{1}{a_n^2+a_na_1}\ge \frac{n^2}{2(a_1a_2+a_2a_3+\cdots+a_na_1)}.$$

Remark 1. Using the substitution

$$a_1=\frac{x_2}{x_1},\ \ a_2=\frac{x_3}{x_2},\ \dots\ ,\ \ a_n=\frac{x_1}{x_n},$$

the inequality in Conjecture 1 becomes

$$\frac{x_1^2}{x_2^2+x_1x_3}+\frac{x_2^2}{x_3^2+x_2x_4}+\cdots+\frac{x_n^2}{x_1^2+x_nx_2}\ge \frac{n}{2},$$

where $x_1, x_2, \dots, x_n > 0$. This cyclic inequality is like Shapiro's inequality

$$\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\cdots+\frac{x_n}{x_1+x_2}\ge \frac{n}{2},$$

which is true for even $n \le 12$ and for odd $n \le 23$.

Remark 2. By the AM-GM inequality, we have

$$a_1a_2+a_2a_3+\cdots+a_na_1\ge n\sqrt[n]{a_1^2a_2^2\cdots a_n^2}.$$

Thus, the inequality in Conjecture 2 is weaker than the inequality in Conjecture 1. Therefore, if Conjecture 1 is true, then Conjecture 2 is also true.

□

P 1.202. *If a, b, c, d are positive real numbers, then*

$$\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+d)}+\frac{1}{d(1+a)}\ge \frac{16}{1+8\sqrt{abcd}}.$$

(Pham Kim Hung, 2007)

Solution. Let $p=\sqrt[4]{abcd}$. Putting

$$a=p\frac{x_2}{x_1},\quad b=p\frac{x_3}{x_2},\quad c=p\frac{x_4}{x_3},\quad d=p\frac{x_1}{x_4},$$

where x_1, x_2, x_3, x_4 are positive real numbers, the inequality turns into

$$\sum \frac{x_1}{x_2 + px_3} \geq \frac{16p}{1+8p^2}.$$

By the Cauchy-Schwarz inequality, we have

$$\sum \frac{x_1}{x_2 + px_3} \geq \frac{\left(\sum x_1\right)^2}{\sum x_1(x_2 + px_3)} = \frac{\left(\sum x_1\right)^2}{(x_1 + x_3)(x_2 + x_4) + 2p(x_1x_3 + x_2x_4)}.$$

Since

$$x_1x_3 + x_2x_4 \leq \left(\frac{x_1 + x_3}{2}\right)^2 + \left(\frac{x_2 + x_4}{2}\right)^2,$$

it suffices to show that

$$\frac{(A+B)^2}{2AB + p(A^2 + B^2)} \geq \frac{8p}{1+8p^2},$$

where

$$A = x_1 + x_3, \qquad B = x_2 + x_4.$$

This inequality is equivalent to

$$A^2 + B^2 + 2(8p^2 - 8p + 1)AB \geq 0,$$

which is true because

$$\begin{aligned} A^2 + B^2 + 2(8p^2 - 8p + 1)AB &\geq 2AB + 2(8p^2 - 8p + 1)AB \\ &= 4(2p-1)^2 AB \geq 0. \end{aligned}$$

The equality holds for $a = b = c = d = \dfrac{1}{2}$.

□

P 1.203. *If a, b, c, d are nonnegative real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$, then*

(a) $$3(a+b+c+d) \geq 2(ab+bc+cd+da)+4;$$

(b) $$a+b+c+d-4 \geq (2-\sqrt{2})(ab+bc+cd+da-4).$$

(Vasile C., 2006)

Solution. Let $p = a + b + c + d$. By the Cauchy-Schwarz inequality

$$(1 + 1 + 1 + 1)(a^2 + b^2 + c^2 + d^2) \geq (a + b + c + d)^2,$$

we get $p \leq 4$, and by the inequality

$$(a + b + c + d)^2 \geq a^2 + b^2 + c^2 + d^2,$$

we get $p \geq 2$. In addition, we have

$$ab + bc + cd + da = (a + c)(b + d) \leq \frac{(a + c + b + d)^2}{4} = \frac{p^2}{4}.$$

(a) It suffices to show that

$$3p \geq \frac{p^2}{2} + 4.$$

Indeed,

$$3p - \frac{p^2}{2} - 4 = \frac{(4 - p)(p - 2)}{2} \geq 0.$$

The equality holds for $a = b = c = d = 1$.

(b) It suffices to show that

$$p - 4 \geq (2 - \sqrt{2})\left(\frac{p^2}{4} - 4\right).$$

This inequality is equivalent to

$$(4 - p)(p - 2\sqrt{2}) \geq 0,$$

which is true for $p \geq 2\sqrt{2}$. So, it remains to consider the case $2 \leq p < 2\sqrt{2}$. Since

$$2(ab + bc + cd + da) \leq (a + b + c + d)^2 - (a^2 + b^2 + c^2 + d^2) = p^2 - 4,$$

it is enough to prove that

$$p - 4 \geq (2 - \sqrt{2}\,)\left(\frac{p^2 - 4}{2} - 4\right).$$

Write this inequality as

$$(2 + \sqrt{2}\,)(p - 4) \geq p^2 - 12,$$

$$(2\sqrt{2} - p)(p - 2 + \sqrt{2}\,) \geq 0.$$

The equality holds for $a = b = c = d = 1$, and also for $a = b = 0$ and $c = d = \sqrt{2}$ (or any cyclic permutation).

□

P 1.204. *Let a, b, c, d be positive real numbers.*

(a) If $a, b, c, d \geq 1$, then

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{d}\right)\left(d+\frac{1}{a}\right) \geq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right);$$

(b) If $abcd = 1$, then

$$\left(a+\frac{1}{b}\right)\left(b+\frac{1}{c}\right)\left(c+\frac{1}{d}\right)\left(d+\frac{1}{a}\right) \leq (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

(Vasile Cîrtoaje and Ji Chen, 2011)

Solution. Let

$$\begin{aligned} A =&(1+ab)(1+bc)(1+cd)(1+da) \\ =&1+\sum ab+\sum a^2bd+2abcd+abcd\sum ab+a^2b^2c^2d^2 \\ =&(1-abcd)^2+4abcd+(1+abcd)\sum ab+\sum a^2bd \\ =&(1-abcd)^2+4abcd+(1+abcd)(a+c)(b+d)+\sum a^2bd \end{aligned}$$

and

$$\begin{aligned} B =&(a+b+c+d)(abc+bcd+cda+dab) \\ =&4abcd+\sum a^2(bc+cd+db) \\ =&4abcd+\sum a^2c(b+d)+\sum a^2bd \\ =&4abcd+(ac+bd)(a+c)(b+d)+\sum a^2bd. \end{aligned}$$

Thus,

$$\begin{aligned} A-B &= (1-abcd)^2+(1+abcd)(a+c)(b+d)-(ac+bd)(a+c)(b+d) \\ &= (1-abcd)^2+(1-ac)(1-bd)(a+c)(b+d). \end{aligned}$$

(a) The inequality $A \geq B$ is clearly true for $a, b, c, d \geq 1$. The equality holds for $a = b = c = d = 1$.

(b) For $abcd = 1$, we have

$$B-A = \frac{1}{ac}(1-ac)^2(a+c)(b+d) \geq 0.$$

The equality holds for $ac = bd = 1$.

□

P 1.205. *If a,b,c,d are positive real numbers, then*

$$\left(1+\frac{a}{a+b}\right)^2+\left(1+\frac{b}{b+c}\right)^2+\left(1+\frac{c}{c+d}\right)^2+\left(1+\frac{d}{d+a}\right)^2>7.$$

(Vasile C., 2012)

First Solution. Assume that $d=\max\{a,b,c,d\}$. We get the desired inequality by summing the inequalities

$$\left(1+\frac{a}{a+b}\right)^2+\left(1+\frac{b}{b+c}\right)^2+\left(1+\frac{c}{c+a}\right)^2>6$$

and

$$\left(1+\frac{c}{c+d}\right)^2+\left(1+\frac{d}{d+a}\right)^2>1+\left(1+\frac{c}{c+a}\right)^2.$$

Let

$$x=\frac{a-b}{a+b},\quad y=\frac{b-c}{b+c},\quad z=\frac{c-a}{c+a}.$$

We have $-1<x,y,z<1$ and

$$x+y+z+xyz=0.$$

Since

$$\frac{a}{a+b}=\frac{x+1}{2},\quad \frac{b}{b+c}=\frac{y+1}{2},\quad \frac{c}{c+a}=\frac{z+1}{2},$$

we can write the first inequality as follows:

$$(x+3)^2+(y+3)^2+(z+3)^2>24,$$

$$x^2+y^2+z^2+6(x+y+z)+3>0,$$

$$x^2+y^2+z^2+3>6xyz.$$

By the AM-GM inequality, we have

$$x^2+y^2+z^2+3\ge 6\sqrt[6]{x^2y^2z^2}>6xyz.$$

Write now the second inequality as

$$\left(1+\frac{c}{c+d}\right)^2-1>\left(\frac{c}{c+a}-\frac{d}{d+a}\right)\left(2+\frac{c}{c+a}+\frac{d}{d+a}\right).$$

Since

$$\frac{c}{c+a}-\frac{d}{d+a}=\frac{a(c-d)}{(c+a)(d+a)}\le 0,$$

we have

$$\left(1+\frac{c}{c+d}\right)^2-1>0\ge\left(\frac{c}{c+a}-\frac{d}{d+a}\right)\left(2+\frac{c}{c+a}+\frac{d}{d+a}\right).$$

Second Solution. Using the inequality

$$(1+x)^2 > 1+3x^2, \quad 0<x<1,$$

we have

$$\left(1+\frac{a}{a+b}\right)^2+\left(1+\frac{b}{b+c}\right)^2+\left(1+\frac{c}{c+d}\right)^2+\left(1+\frac{d}{d+a}\right)^2 >$$

$$> 4+3\left[\left(\frac{a}{a+b}\right)^2+\left(\frac{b}{b+c}\right)^2+\left(\frac{c}{c+d}\right)^2+\left(\frac{d}{d+a}\right)^2\right].$$

Therefore, it suffices to prove that

$$\left(\frac{a}{a+b}\right)^2+\left(\frac{b}{b+c}\right)^2+\left(\frac{c}{c+d}\right)^2+\left(\frac{d}{d+a}\right)^2 \ge 1,$$

which is equivalent to the known inequality in P 1.191 from Volume 2:

$$\frac{1}{(1+x)^2}+\frac{1}{(1+y)^2}+\frac{1}{(1+z)^2}+\frac{1}{(1+t)^2} \ge 1,$$

where

$$x=\frac{a}{b}, \quad y=\frac{b}{c}, \quad z=\frac{c}{d}, \quad t=\frac{d}{a}, \qquad xyzt=1.$$

□

P 1.206. *If a,b,c,d are positive real numbers, then*

$$\frac{a^2-bd}{b+2c+d}+\frac{b^2-ca}{c+2d+a}+\frac{c^2-db}{d+2a+b}+\frac{d^2-ac}{a+2b+c} \ge 0.$$

(Vo Quoc Ba Can, 2009)

Solution. Write the inequality as follows:

$$\sum\left(\frac{4a^2-4bd}{b+2c+d}+b+d-2a\right) \ge 0,$$

$$\sum\frac{(b-d)^2+2(a-c)(2a-b-d)}{b+2c+d} \ge 0.$$

It suffices to show that

$$\sum\frac{(a-c)(2a-b-d)}{b+2c+d} \ge 0.$$

This inequality is equivalent to

$$(a-c)\left(\frac{2a-b-d}{b+2c+d}-\frac{2c-d-b}{d+2a+b}\right)+(b-d)\left(\frac{2b-c-a}{c+2d+a}-\frac{2d-a-c}{a+2b+c}\right)\ge 0,$$

which can be written as

$$\frac{(a-c)(a^2-c^2)}{(b+2c+d)(d+2a+b)}+\frac{(b-d)(b^2-d^2)}{(c+2d+a)(a+2b+c)}\ge 0.$$

The equality occurs for $a=c$ and $b=d$.

□

P 1.207. *If a,b,c,d are positive real numbers such that $a\le b\le c\le d$, then*

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+d}}+\sqrt{\frac{2d}{d+a}}\le 4.$$

(Vasile C., 2009)

Solution. According to the inequality in P 1.74, we have

$$\sqrt{\frac{2a}{a+b}}+\sqrt{\frac{2b}{b+c}}+\sqrt{\frac{2c}{c+a}}\le 3.$$

Therefore, it suffices to show that

$$\sqrt{\frac{2c}{c+d}}+\sqrt{\frac{2d}{d+a}}\le 1+\sqrt{\frac{2c}{c+a}}.$$

By squaring, this inequality becomes

$$\frac{2c}{c+d}+\frac{2d}{d+a}+2\sqrt{\frac{4cd}{(c+d)(d+a)}}\le 1+\frac{2c}{c+a}+2\sqrt{\frac{2c}{c+a}}.$$

We can get it by summing the inequalities

$$\frac{2c}{c+d}+\frac{2d}{d+a}\le 1+\frac{2c}{c+a},$$

$$2\sqrt{\frac{4cd}{(c+d)(d+a)}}\le 2\sqrt{\frac{2c}{c+a}}.$$

The former inequality is true since

$$\frac{2c}{c+d}+\frac{2d}{d+a}-1-\frac{2c}{c+a}=\frac{(a-d)(d-c)(c-a)}{(c+d)(d+a)(a+c)}\le 0,$$

while the second inequality reduces to

$$c(a-d)(d-c)\le 0.$$

The equality holds for $a=b=c=d$.

□

P 1.208. *Let a,b,c,d be nonnegative real numbers, and let*

$$x=\frac{a}{b+c}, \quad y=\frac{b}{c+d}, \quad z=\frac{c}{d+a}, \quad t=\frac{d}{a+b}.$$

Prove that

(a) $$\sqrt{xz}+\sqrt{yt}\le 1;$$

(b) $$x+y+z+t+4(xz+yt)\ge 4.$$

(Vasile C., 2004)

Solution. (a) Using the Cauchy-Schwarz inequality, we have

$$\sqrt{xz}+\sqrt{yt}=\frac{\sqrt{ac}}{\sqrt{(b+c)(d+a)}}+\frac{\sqrt{bd}}{\sqrt{(c+d)(a+b)}}$$
$$\le\frac{\sqrt{ac}}{\sqrt{ac}+\sqrt{bd}}+\frac{\sqrt{bd}}{\sqrt{ac}+\sqrt{bd}}=1.$$

The equality holds for $a=b=c=d$, for $a=c=0$, and for $b=d=0$

(b) Write the inequality as

$$A+B\ge 6,$$

where

$$A=x+z+4xz+1=\frac{(a+b)(c+d)+(a+c)^2+ab+2ac+cd}{(b+c)(d+a)}$$
$$=\frac{(a+b)(c+d)}{(b+c)(d+a)}+\frac{(a+c)^2}{(b+c)(d+a)}+\frac{a}{d+a}+\frac{c}{b+c},$$

$$B=y+t+4yt+1=\frac{(b+c)(d+a)}{(c+d)(a+b)}+\frac{(b+d)^2}{(c+d)(a+b)}+\frac{b}{a+b}+\frac{d}{c+d}.$$

Since

$$\frac{(a+b)(c+d)}{(b+c)(d+a)}+\frac{(b+c)(d+a)}{(c+d)(a+b)}\ge 2,$$

it suffices to show that

$$\frac{(a+c)^2}{(b+c)(d+a)}+\frac{(b+d)^2}{(c+d)(a+b)}+\sum\frac{a}{d+a}\ge 4.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{(a+c)^2}{(b+c)(d+a)}+\frac{(b+d)^2}{(c+d)(a+b)}\ge\frac{(a+b+c+d)^2}{C},$$

$$\sum\frac{a}{d+a}\ge\frac{(a+b+c+d)^2}{D},$$

where

$$C = (b+c)(d+a) + (c+d)(a+b),$$

$$D = \sum a(d+a) = a^2 + b^2 + c^2 + d^2 + ab + bc + cd + da,$$

$$C + D = (a+b+c+d)^2.$$

Thus, it is enough to show that

$$(C+D)\left(\frac{1}{C} + \frac{1}{D}\right) \ge 4,$$

which is clearly true. The equality holds for $a = b = c = d$.

□

P 1.209. *If a,b,c,d are nonnegative real numbers, then*

$$\left(1 + \frac{2a}{b+c}\right)\left(1 + \frac{2b}{c+d}\right)\left(1 + \frac{2c}{d+a}\right)\left(1 + \frac{2d}{a+b}\right) \ge 9.$$

(Vasile C., 2004)

Solution. We can rewrite the inequality as

$$\left(1 + \frac{a+c}{a+b}\right)\left(1 + \frac{a+c}{c+d}\right)\left(1 + \frac{b+d}{b+c}\right)\left(1 + \frac{b+d}{d+a}\right) \ge 9.$$

Using the Cauchy-Schwarz inequality and the AM-GM inequality yields

$$\left(1 + \frac{a+c}{a+b}\right)\left(1 + \frac{a+c}{c+d}\right) \ge \left[1 + \frac{a+c}{\sqrt{(a+b)(c+d)}}\right]^2 \ge \left(1 + \frac{2a+2c}{a+b+c+d}\right)^2,$$

$$\left(1 + \frac{b+d}{b+c}\right)\left(1 + \frac{b+d}{d+a}\right) \ge \left[1 + \frac{b+d}{\sqrt{(b+c)(d+a)}}\right]^2 \ge \left(1 + \frac{2b+2d}{a+b+c+d}\right)^2.$$

Thus, it suffices to show that

$$\left(1 + \frac{2a+2c}{a+b+c+d}\right)\left(1 + \frac{2b+2d}{a+b+c+d}\right) \ge 3.$$

This is equivalent to the obvious inequality

$$\frac{4(a+c)(b+d)}{(a+b+c+d)^2} \ge 0.$$

The equality holds for $a = c = 0$ and $b = d$, as well as for $b = d = 0$ and $a = c$.

□

P 1.210. *Let a, b, c, d be nonnegative real numbers. If $k > 0$, then*

$$\left(1+\frac{ka}{b+c}\right)\left(1+\frac{kb}{c+d}\right)\left(1+\frac{kc}{d+a}\right)\left(1+\frac{kd}{a+b}\right) \ge (1+k)^2.$$

(Vasile C., 2004)

Solution. Let us denote

$$x = \frac{a}{b+c}, \quad y = \frac{b}{c+d}, \quad z = \frac{c}{d+a}, \quad t = \frac{d}{a+b}.$$

Since

$$\prod(1+kx) \ge 1 + k(x+y+z+t) + k^2(xy+yz+zt+tx+xz+yt),$$

it suffices to show that

$$x+y+z+t \ge 2$$

and

$$xy+yz+zt+tx+xz+yt \ge 1.$$

The inequality $x+y+z+t \ge 2$ is the well-known Shapiro's inequality for 4 positive real numbers. This can be proved by the Cauchy-Schwarz inequality, as follows:

$$\frac{a}{b+c}+\frac{b}{c+d}+\frac{c}{d+a}+\frac{d}{a+b} \ge \frac{(a+b+c+d)^2}{a(b+c)+b(c+d)+c(d+a)+d(a+b)} \ge 2.$$

The right inequality reduces to the obvious inequality

$$(a-c)^2 + (b-d)^2 \ge 0.$$

To prove the inequality $xy+yz+zt+tx+xz+yt \ge 1$, we will use the inequalities

$$\frac{x+z}{2} \ge xz,$$

$$\frac{y+t}{2} \ge yt,$$

and the identity

$$xz(1+y+t)+yt(1+x+z) = 1.$$

If these are true, then

$$\begin{aligned} xy+yz+zt+tx+xz+yt &= \frac{x+z}{2}(y+t)+\frac{y+t}{2}(x+z)+xz+yt \\ &\ge xz(y+t)+yt(x+z)+xz+yt \\ &= xz(1+y+t)+yt(1+x+z) = 1. \end{aligned}$$

We have

$$\frac{x+z}{2} - xz = \frac{bc+da+(a-c)^2}{2(b+c)(d+a)} \geq 0$$

and

$$\frac{y+t}{2} - yt = \frac{ab+cd+(b-d)^2}{2(a+b)(c+d)} \geq 0.$$

To prove the identity above, we rewrite it as

$$\sum xyz + xz + yt = 1,$$

and see that

$$\sum xyz = \frac{\sum abc(a+b)}{A} = \frac{\sum a^2bc + \sum a^2bd}{A}$$

and

$$xz + yt = \frac{ac(a+b)(c+d) + bd(b+c)(d+a)}{A} = \frac{\sum a^2cd + (ac+bd)^2}{A},$$

where

$$A = \prod(a+b) = \sum a^2bc + \sum a^2bd + \sum a^2cd + (ac+bd)^2.$$

Thus, the proof is completed. The equality holds for $a = c = 0$ and $b = d$, as well as for $b = d = 0$ and $a = c$.

Remark. For $k = 2$, we get the inequality in P 1.209. For $k = 1$, we get the following known inequality

$$(a+b+c)(b+c+d)(c+d+a)(d+a+b) \geq 4(a+b)(b+c)(c+d)(d+a).$$

A proof of this inequality starts from the inequalities

$$(a+b+c)^2 \geq (2a+b)(2c+b)$$

and

$$(2a+b)(2b+a) \geq 2(a+b)^2.$$

We have

$$\begin{aligned}\prod(a+b+c)^2 &\geq \prod(2a+b) \cdot \prod(2c+b)\\ &= \prod(2a+b)(2b+a)\\ &\geq 2^4 \prod(a+b)^2,\end{aligned}$$

hence

$$\prod(a+b+c) \geq 4\prod(a+b).$$

□

P 1.211. *If a, b, c, d are positive real numbers such that $a+b+c+d=4$, then*

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{cd}+\frac{1}{da} \ge a^2+b^2+c^2+d^2.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$(a+c)(b+d) \ge abcd(a^2+b^2+c^2+d^2).$$

From $(a-c)^4 \ge 0$ and $(b-d)^4 \ge 0$, we get

$$(a+c)^4 \ge 8ac(a^2+c^2), \quad (b+d)^4 \ge 8bd(b^2+d^2),$$

hence

$$bd(a+c)^4+ac(b+d)^4 \ge 8abcd(a^2+b^2+c^2+d^2).$$

Therefore, it suffices to show that

$$8(a+c)(b+d) \ge bd(a+c)^4+ac(b+d)^4.$$

Since $4bd \le (b+d)^2$ and $4ac \le (a+c)^2$, we only need to show that

$$32(a+c)(b+d) \ge (b+d)^2(a+c)^4+(a+c)^2(b+d)^4.$$

This inequality is true if

$$32 \ge xy(x^2+y^2)$$

for all positive x, y satisfying $x+y=4$. Indeed,

$$8[32-xy(x^2+y^2)] = (x+y)^4-8xy(x^2+y^2) = (x-y)^4 \ge 0.$$

The equality occurs for $a=b=c=d=1$.

□

P 1.212. *If a, b, c, d are positive real numbers, then*

$$\frac{a^2}{(a+b+c)^2}+\frac{b^2}{(b+c+d)^2}+\frac{c^2}{(c+d+a)^2}+\frac{d^2}{(d+a+b)^2} \ge \frac{4}{9}.$$

(Pham Kim Hung, 2006)

First Solution. By Hölder's inequality, we have

$$\sum \frac{a^2}{(a+b+c)^2} \ge \frac{\left(\sum a^{4/3}\right)^3}{\left[\sum a(a+b+c)\right]^2}.$$

Since

$$\sum a(a+b+c) = (a+c)^2 + (b+d)^2 + (a+c)(b+d)$$

and

$$\sum a^{4/3} = \left(a^{4/3} + c^{4/3}\right) + \left(b^{4/3} + d^{4/3}\right) \geq 2\left(\frac{a+c}{2}\right)^{4/3} + 2\left(\frac{b+d}{2}\right)^{4/3},$$

it suffices to show that

$$9\left[(a+c)^{4/3} + (b+d)^{4/3}\right]^3 \geq 8[(a+c)^2 + (b+d)^2 + (a+c)(b+d)]^2.$$

Due to homogeneity, we may assume that $b+d=1$. Putting $a+c=t^3$, $t>0$, the inequality becomes

$$9(t^4+1)^3 \geq 8(t^6+1+t^3)^2,$$

$$9\left(t^2+\frac{1}{t^2}\right)^3 \geq 8\left(t^3+\frac{1}{t^3}+1\right)^2.$$

Setting

$$x = t + \frac{1}{t}, \qquad x \geq 2,$$

the inequality turns into

$$9(x^2-2)^3 \geq 8(x^3-3x+1)^2,$$

which is equivalent to

$$(x-2)^2(x^4+4x^3+6x^2-8x-20) \geq 0.$$

This is true since

$$x^4+4x^3+6x^2-8x-20 = x^4+4x^2(x-2)+4x(x-2)+10(x^2-2) > 0.$$

Thus, the proof is completed. The equality holds for $a=b=c=d$.

Second Solution. Due to homogeneity, we may assume that

$$a+b+c+d=1.$$

In this case, we write the inequality as

$$\left(\frac{a}{1-d}\right)^2 + \left(\frac{b}{1-a}\right)^2 + \left(\frac{c}{1-b}\right)^2 + \left(\frac{d}{1-c}\right)^2 \geq \frac{4}{9}.$$

Let (x,y,z,t) be a permutation of (a,b,c,d) such that

$$x \geq y \geq z \geq t.$$

Since

$$\frac{1}{(1-t)^2} \le \frac{1}{(1-z)^2} \le \frac{1}{(1-y)^2} \le \frac{1}{(1-x)^2},$$

by the rearrangement inequality, we have

$$\left(\frac{x}{1-t}\right)^2 + \left(\frac{y}{1-z}\right)^2 + \left(\frac{z}{1-y}\right)^2 + \left(\frac{t}{1-x}\right)^2 \le$$

$$\le \left(\frac{a}{1-d}\right)^2 + \left(\frac{b}{1-a}\right)^2 + \left(\frac{c}{1-b}\right)^2 + \left(\frac{d}{1-c}\right)^2.$$

Therefore, it suffices to show that $x+y+z+t=1$ involves

$$U + V \ge \frac{4}{9},$$

where

$$U = \left(\frac{x}{1-t}\right)^2 + \left(\frac{t}{1-x}\right)^2,$$

$$V = \left(\frac{y}{1-z}\right)^2 + \left(\frac{z}{1-y}\right)^2.$$

Let

$$s = x+t, \quad p = xt, \quad s \in (0,1),$$

Since

$$x^2+t^2 = s^2-2p, \quad x^3+t^3 = s^3-3ps, \quad x^4+t^4 = s^4-4ps^2+2p^2,$$

we get

$$U = \frac{x^2+t^2-2(x^3+t^3)+x^4+t^4}{(1-s+p)^2}$$

$$= \frac{2p^2-2(1-s)(1-2s)p+s^2(1-s)^2}{p^2+2(1-s)p+(1-s)^2},$$

$$(2-U)p^2-2(1-s)(1-2s+U)p+(1-s)^2(s^2-U)=0.$$

The quadratic trinomial in p has the discriminant

$$D = (1-s)^2[(1-2s+U)^2-(2-U)(s^2-U)].$$

From the necessary condition $D \ge 0$, we get

$$U \ge \frac{4s-1-2s^2}{(2-s)^2}.$$

Analogously,

$$V \ge \frac{4r-1-2r^2}{(2-r)^2},$$

where $r = y + z$. Taking into account that

$$s + r = 1,$$

we get

$$\begin{aligned}
U + V &\geq \frac{4s - 1 - 2s^2}{(2-s)^2} + \frac{4r - 1 - 2r^2}{(2-r)^2} \\
&= \frac{4s - 1 - 2s^2}{(1+r)^2} + \frac{4r - 1 - 2r^2}{(1+s)^2} \\
&= \frac{5(s^2 + r^2) - 2(s^4 + r^4)}{(2+sr)^2} \\
&= \frac{5(s^2 + r^2) - 2(s^2 + r^2)^2 + 4s^2r^2}{(2+sr)^2},
\end{aligned}$$

hence

$$\begin{aligned}
U + V - \frac{4}{9} &\geq \frac{5(s^2 + r^2) - 2(s^2 + r^2)^2 + 4s^2r^2}{(2+sr)^2} - \frac{4}{9} \\
&= \frac{5(s^2 + r^2) - 2(s^2 + r^2)^2}{(2+sr)^2} + \frac{2(1 - 4sr)^2 - 18}{9(2+sr)^2} \\
&\geq \frac{5(s^2 + r^2) - 2(s^2 + r^2)^2 - 2}{(2+sr)^2} \\
&= \frac{(2 - s^2 - r^2)(2s^2 + 2r^2 - 1)}{(2+sr)^2}.
\end{aligned}$$

Thus, we need to show that $(2 - s^2 - r^2)(2s^2 + 2r^2 - 1) \geq 0$. This is true since since

$$2 - s^2 - r^2 > 2 - (s + r)^2 = 1,$$

$$2s^2 + 2r^2 - 1 \geq (s + r)^2 - 1 = 0.$$

□

P 1.213. *If a, b, c, d are positive real numbers such that $a + b + c + d = 3$, then*

$$ab(b + c) + bc(c + d) + cd(d + a) + da(a + b) \leq 4.$$

(Pham Kim Hung, 2007)

Solution. Write the inequality as

$$\sum ab^2 + \sum abc \leq 4,$$

$$(ab^2 + cd^2 + bcd + dab) + (bc^2 + da^2 + abc + cda) \leq 4,$$

$$(b+d)(ab+cd)+(a+c)(bc+da) \le 4.$$

Without loss of generality, assume that $a+c \le b+d$. Since

$$(ab+cd)+(bc+da)=(a+c)(b+d),$$

we can rewrite the inequality as

$$(a+c)(b+d)^2+(a+c-b-d)(bc+da) \le 4.$$

Since $a+c-b-d \le 0$, it suffices to show that

$$(a+c)(b+d)^2 \le 4.$$

Indeed, by the AM-GM inequality, we have

$$(a+c)\left(\frac{b+d}{2}\right)\left(\frac{b+d}{2}\right) \le \frac{1}{27}\left(a+c+\frac{b+d}{2}+\frac{b+d}{2}\right)^3=1.$$

The equality holds for $a=b=0$, $c=1$ and $d=2$ (or any cyclic permutation).

□

P 1.214. *If $a \ge b \ge c \ge d \ge 0$ and $a+b+c+d=2$, then*

$$ab(b+c)+bc(c+d)+cd(d+a)+da(a+b) \le 1.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$\sum ab^2+\sum abc \le 1.$$

Since

$$\sum ab^2-\sum a^2b=(ab^2+bc^2+ca^2-a^2b-b^2c-c^2a)+(cd^2+da^2+ac^2-c^2d-d^2a-a^2c)$$
$$=(a-b)(b-c)(c-a)+(c-d)(d-a)(a-c) \le 0,$$

it suffices to show that

$$\sum ab^2+\sum a^2b+2\sum abc \le 2.$$

Indeed,

$$\begin{aligned}\sum ab^2+\sum a^2b+2\sum abc &= \sum(ab^2+a^2b+abc+abd)\\ &=(a+b+c+d)\sum ab\\ &=2(a+c)(b+d)\\ &\le 2\left[\frac{(a+c)+(b+d)}{2}\right]^2=2.\end{aligned}$$

The equality holds for $a=b=t$ and $c=d=1-t$, where $t \in \left[\frac{1}{2},1\right]$.

□

P 1.215. *Let a,b,c,d be nonnegative real numbers such that $a+b+c+d=4$. If $k \ge \frac{37}{27}$, then*

$$ab(b+kc)+bc(c+kd)+cd(d+ka)+da(a+kb) \le 4(1+k).$$

(Vasile C., 2007)

Solution. Write the inequality in the homogeneous form

$$ab(b+kc)+bc(c+kd)+cd(d+ka)+da(a+kb) \le \frac{(1+k)(a+b+c+d)^3}{16}.$$

Assume that $d=\min\{a,b,c,d\}$ and use the substitution

$$a=d+x, \quad b=d+y, \quad c=d+z,$$

where $x,y,z \ge 0$. The inequality can be restated as

$$4Ad+B \ge 0,$$

where

$$A=(3k-1)(x^2+y^2+z^2)-2(k+1)y(x+z)+(6-2k)xz,$$
$$B=(1+k)(x+y+z)^3-16(xy^2+yz^2+kxyz).$$

It suffices to show that $A \ge 0$ and $B \ge 0$. We have

$$\begin{aligned} A &= (3k-1)y^2+(3k-1)(x+z)^2-2(k+1)y(x+z)-8(k-1)xz \\ &\ge (3k-1)y^2+(3k-1)(x+z)^2-2(k+1)y(x+z)-2(k-1)(x+z)^2 \\ &= (3k-1)y^2+(k+1)(x+z)^2-2(k+1)y(x+z) \\ &\ge 2\sqrt{(3k-1)(k+1)}y(x+z)-2(k+1)y(x+z) \\ &= 2\sqrt{k+1}\left(\sqrt{3k-1}-\sqrt{k+1}\right)y(x+z) \ge 0. \end{aligned}$$

Since

$$(x+y+z)^3-16xyz \ge 0,$$

the inequality $B \ge 0$ holds for all $k \ge \frac{37}{27}$ if it holds for $k=\frac{37}{27}$. In this particular case, the inequality $B \ge 0$ can be written as

$$4\left(\frac{x+y+z}{3}\right)^3 \ge xy^2+yz^2+\frac{37}{27}xyz.$$

Actually, the following sharper inequality holds (see P 2.31)

$$4\left(\frac{x+y+z}{3}\right)^3 \ge xy^2+yz^2+\frac{3}{2}xyz.$$

Thus, the proof is completed. The equality holds for $a=b=c=d=1$. If $k=\frac{37}{27}$, then the equality holds also for $a=\frac{4}{3}$, $b=\frac{8}{3}$ and $c=d=0$ (or any cyclic permutation).

□

P 1.216. *If a, b, c, d are nonnegative real numbers such that $a+b+c+d=4$, then*

$$\sqrt{\frac{3a}{b+2}}+\sqrt{\frac{3b}{c+2}}+\sqrt{\frac{3c}{d+2}}+\sqrt{\frac{3d}{a+2}}\le 4.$$

(Vasile Cîrtoaje, 2020)

Solution. (after an idea of *Michael Rozenberg*) Let $(a1, a_2, a_3, a_4)$ be an increasing permutation of (a, b, c, d). Since the sequences

$$(a1, a_2, a_3, a_4) \qquad and \qquad \left(\frac{1}{a_4+2}, \frac{1}{a_3+2}, \frac{1}{a_2+2}, \frac{1}{a_1+2}\right)$$

are increasing, according to the rearrangement inequality, we have

$$\sqrt{\frac{3a}{b+2}}+\sqrt{\frac{3b}{c+2}}+\sqrt{\frac{3c}{d+2}}+\sqrt{\frac{3d}{a+2}}\le$$

$$\le \sqrt{\frac{3a_1}{a_4+2}}+\sqrt{\frac{3a_2}{a_3+2}}+\sqrt{\frac{3a_3}{a_2+2}}+\sqrt{\frac{3a_4}{a_1+2}}=A+B,$$

where

$$A=\sqrt{\frac{3a_1}{a_4+2}}+\sqrt{\frac{3a_4}{a_1+2}}, \qquad B=\sqrt{\frac{3a_2}{a_3+2}}++\sqrt{\frac{3a_3}{a_2+2}}.$$

We need to show that $A+B\le 2$. According to Lemma below, we have

$$A+B\le \frac{a_1+a_4+4}{3}+\frac{a_2+a_3+4}{3}=4.$$

The equality holds for $a=b=c=d=1$.

Lemma. If a, b are nonnegative real numbers, then

$$\sqrt{\frac{3a}{b+2}}+\sqrt{\frac{3b}{a+2}}\le \frac{a+b+4}{3}.$$

Proof. Use the substitution

$$x=\sqrt{\frac{3a}{b+2}}, \quad y=\sqrt{\frac{3b}{a+2}},$$

which yields $xy<3$ and

$$a=\frac{2x^2(y^2+3)}{9-x^2y^2}, \qquad b=\frac{2y^2(x^2+3)}{9-x^2y^2}, \qquad a+b=\frac{4x^2y^2+6(x^2+y^2)}{9-x^2y^2}.$$

Thus, we need to show that

$$3(x+y)\le \frac{4x^2y^2+6(x^2+y^2)}{9-x^2y^2}+4,$$

which is equivalent to

$$2(x+y)^2-(9-x^2y^2)(x+y)+12-4xy\ge 0,$$

$$\left(4x+4y-9+x^2y^2\right)^2+15-32xy+18x^2y^2-x^4y^4\ge 0,$$

$$\left(4x+4y-9+x^2y^2\right)^2+(1-xy)^2(3-xy)(5+xy)\ge 0.$$

The equality holds for $a=b=1$.

□

P 1.217. *Let a,b,c,d be positive real numbers such that $a\le b\le c\le d$. Prove that*

$$2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)\ge 4+\frac{a}{c}+\frac{c}{a}+\frac{b}{d}+\frac{d}{b}.$$

(Vasile C., 2012)

First Solution. Let

$$E(a,b,c,d)=2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\right)-4-\frac{a}{c}-\frac{c}{a}-\frac{b}{d}-\frac{d}{b}.$$

We show that

$$E(a,b,c,d)\ge E(b,b,c,d)\ge E(b,b,c,c).$$

We have

$$E(a,b,c,d)-E(b,b,c,d)=(b-a)\left(\frac{1}{c}+\frac{2d}{ab}-\frac{2}{b}-\frac{c}{ab}\right)\ge 0,$$

since

$$\frac{1}{c}+\frac{2d}{ab}-\frac{2}{b}-\frac{c}{ab}\ge\frac{1}{c}+\frac{2c}{ab}-\frac{2}{b}-\frac{c}{ab}$$

$$=\frac{1}{c}+\frac{c}{ab}-\frac{2}{b}\ge\frac{1}{c}+\frac{c}{b^2}-\frac{2}{b}=\frac{(b-c)^2}{b^2c}\ge 0.$$

Also,

$$E(b,b,c,d)-E(b,b,c,c)=(d-c)\left(\frac{1}{b}-\frac{2c-b}{cd}\right)\ge 0,$$

since

$$\frac{1}{b}-\frac{2c-b}{cd}\ge\frac{1}{b}-\frac{2c-b}{c^2}=\frac{(b-c)^2}{bc^2}\ge 0.$$

Because $E(b,b,c,c)=0$, the proof is completed. The equality holds for $a=b$ and $c=d$.

Second Solution. Using the substitution

$$x=\frac{a}{b},\quad y=\frac{b}{c},\quad z=\frac{c}{d},\quad 0<x,y,z\le 1,$$

the inequality becomes as follows:

$$2\left(x+y+z+\frac{1}{xyz}\right)\geq 4+xy+\frac{1}{xy}+yz+\frac{1}{yz},$$

$$y(2-x-z)+\frac{1}{y}\left(\frac{2}{xz}-\frac{1}{x}-\frac{1}{z}\right)-2(2-x-z)\geq 0,$$

$$(2-x-z)\left(y+\frac{1}{xyz}-2\right)\geq 0.$$

The last inequality is true since $2-x-y\geq 0$ and

$$y+\frac{1}{xyz}-2\geq y+\frac{1}{y}-2\geq 0.$$

□

P 1.218. *Let a,b,c,d be positive real numbers such that*

$$a\leq b\leq c\leq d,\qquad abcd=1.$$

Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}\geq ab+bc+cd+da.$$

(Vasile C., 2012)

***Solution*.** Write the inequality as follows:

$$a^2cd+b^2da+c^2ab+d^2bc\geq ab+bc+cd+da,$$

$$ac(ad+bc)+bd(ab+cd)\geq (ad+bc)+(ab+cd),$$

$$(ac-1)(ad+bc)+(bd-1)(ab+cd)\geq 0.$$

Since

$$ac-1=\frac{1}{bd}-1\geq 1-bd$$

and

$$bd\geq\sqrt{abcd}=1,$$

we have

$$\begin{aligned}(ac-1)(ad+bc)+(bd-1)(ab+cd)&\geq (1-bd)(ad+bc)+(bd-1)(ab+cd)\\&=(bd-1)(a-c)(b-d)\geq 0.\end{aligned}$$

The equality holds for $a=b=\dfrac{1}{c}=\dfrac{1}{d}\leq 1$.

□

P 1.219. *Let a,b,c,d be positive real numbers such that*

$$a \le b \le c \le d, \qquad abcd = 1.$$

Prove that

$$4 + \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \ge 2(a+b+c+d).$$

(Vasile C., 2012)

Solution. Making the substitution

$$x = \sqrt[4]{\frac{a}{b}}, \quad y = \sqrt{\frac{b}{c}}, \quad z = \sqrt[4]{\frac{c}{d}}, \quad 0 < x,y,z \le 1,$$

we need to show that $E(x,y,z) \ge 0$, where

$$E(x,y,z) = 4 + x^4 + z^4 + y^2 + \frac{1}{x^4y^2z^4} - 2\left(x^3yz + \frac{yz}{x} + \frac{z}{xy} + \frac{1}{xyz^3}\right).$$

We will show that

$$E(x,y,z) \ge E(x,1,z) \ge E(x,1,1) \ge 0. \qquad (*)$$

The left inequality is equivalent to

$$(1-y)E_1(x,y,z) \ge 0,$$

where

$$E_1(x,y,z) = -1 - y + \frac{1+y}{x^4y^2z^4} + 2\left(x^3z + \frac{z}{x}\right) - \frac{2}{y}\left(\frac{z}{x} + \frac{1}{xz^3}\right).$$

To prove it, we show that

$$E_1(x,y,z) \ge E_1(x,1,z) \ge 0.$$

We have

$$E_1(x,1,z) = 2(1-x^3z)\left(\frac{1}{x^4z^4} - 1\right) \ge 0.$$

Since

$$E_1(x,y,z - E_1(x,1,z) = (1-y)E_2(x,y,z),$$

where

$$E_2(x,y,z) = 1 + \frac{1+2y}{x^4y^2z^4} - \frac{2}{y}\left(\frac{z}{x} + \frac{1}{xz^3}\right),$$

we need to show $E_2(x,y,z) \ge 0$. Indeed,

$$\begin{aligned}
E_2(x,y,z) &= 1 + \frac{1}{x^4y^2z^4} - \frac{2}{y}\left(\frac{z}{x} + \frac{1}{xz^3} - \frac{1}{x^4z^4}\right) \\
&\ge \frac{2}{x^2yz^2} - \frac{2}{y}\left(\frac{z}{x} + \frac{1}{xz^3} - \frac{1}{x^4z^4}\right) \\
&= \frac{2}{xyz}\left(\frac{1}{xz} - z^2 - \frac{1}{z^2} + \frac{1}{x^3z^3}\right) \\
&\ge \frac{2}{xyz}\left(\frac{1}{z} - z^2 - \frac{1}{z^2} + \frac{1}{z^3}\right) \\
&= \frac{2}{xyz}\left(\frac{1-z^3}{z} + \frac{1-z}{z^3}\right) \ge 0.
\end{aligned}$$

The middle inequality in (*) is equivalent to

$$(1-z)F(x,z) \ge 0,$$

where

$$F(x,z) = (1+z+z^2+z^3)\left(\frac{1}{x^4z^4} - 1\right) + 2\left(x^3 + \frac{2}{x}\right) - \frac{1+z+z^2}{xz}.$$

It is true since

$$\begin{aligned}
F(x,z) &> \frac{1}{x^4z^4} - 1 + \frac{3}{x} - \frac{1+z+z^2}{xz} \\
&\ge \frac{1}{xz} - 1 + \frac{3}{x} - \frac{1+z+z^2}{xz} \\
&= \frac{2-x-z}{x} \ge 0.
\end{aligned}$$

The right inequality in (*) is also true since

$$\begin{aligned}
x^4E(x,1,1) &= x^8 - 2x^7 + 6x^4 - 6x^3 + 1 \\
&= (x-1)^2(x^6 - x^4 - 2x^3 + 3x^2 + 2x + 1) \\
&\ge (x-1)^2(x^6 - x^4 - 2x^3 + 2x^2) \\
&= x^2(x-1)^4(x^2 + 2x + 2) \ge 0.
\end{aligned}$$

The proof is completed. The equality holds for $a = b = c = d = 1$.

□

P 1.220. *Let $A = \{a_1, a_2, a_3, a_4\}$ be a set of real numbers such that*

$$a_1 + a_2 + a_3 + a_4 = 0.$$

Prove that there exists a permutation $B = \{a, b, c, d\}$ of A such that

$$a^2 + b^2 + c^2 + d^2 + 3(ab + bc + cd + da) \ge 0.$$

Solution. Write the desired inequality as

$$a^2+b^2+c^2+d^2+3(ab+bc+cd+da)\ge(a+b+c+d)^2,$$

$$ab+bc+cd+da\ge 2(ac+bd),$$

$$(ab+cd-ac-bd)+(bc+da-ac-bd)\ge 0.$$

$$(a-d)(b-c)+(a-b)(d-c)\ge 0.$$

Clearly, this inequality is true for $a\le b\le d\le c$. The equality occurs when A has three equal elements.

□

P 1.221. *If a,b,c,d are nonnegative real numbers such that*

$$a\ge b\ge 1\ge c\ge d,\qquad a+b+c+d=3,$$

then

$$a^2+b^2+c^2+d^2+10abcd\le 5.$$

(Vasile C., 2015)

First Solution. Let

$$E(a,b,c,d)=a^2+b^2+c^2+d^2+10abcd.$$

We will show that

$$E(a,b,c,d)\le E(a,b,x,x)\le 5,$$

where

$$x=(c+d)/2,\qquad a+b+2x=3.$$

The left inequality is true since

$$E(a,b,c,d)-E(a,b,x,x)=\frac{1}{2}(c-d)^2(1-5ab)\le 0.$$

The right inequality can be written as follows:

$$a^2+b^2+2x^2+10abx^2\le 5,$$

$$(a+b)^2+2x^2+2ab(5x^2-1)\le 5,$$

$$2s^2+(3-s)^2+ab[5(3-s)^2-4]\le 10,$$

where

$$s=a+b,\quad s\in[2,3].$$

Case 1: $5(3-s)^2-4\ge 0$. Since $ab\le s^2/4$, it suffices to show that

$$2s^2+(3-s)^2+\frac{1}{4}s^2[5(3-s)^2-4]\le 10,$$

which is equivalent to the obvious inequality

$$(s-1)(s-2)[5s(s-3)-2] \le 0.$$

Case 2: $5(3-s)^2-4 \le 0$. From $(a-1)(b-1) \ge 0$, we get $ab \ge s-1$. Therefore, it suffices to show that

$$2s^2+(3-s)^2+(s-1)[5(3-s)^2-4] \le 10,$$

which is equivalent to the obvious inequality

$$(s-2)(s-3)(5s-7) \le 0.$$

The equality holds for $a=b=1$, $c=d=1/2$, and for $a=2$, $b=1$, $c=d=0$.

Second Solution. From

$$(a-1)(b-1)(c-1)(d-1) \ge 0,$$

we have

$$-2+\sum_{sym} ab - \sum abc + abcd \ge 0.$$

Since

$$2\sum_{sym} ab = 9-a^2-b^2-c^2-d^2,$$

we get

$$-2+\frac{9-a^2-b^2-c^2-d^2}{2} - \sum abc + abcd \ge 0,$$

$$a^2+b^2+c^2+d^2 \le 5-2\sum abc + 2abcd.$$

Therefore, it suffices to show that

$$(5-2\sum abc+2abcd)+10abcd \le 5,$$

which is equivalent to

$$\sum abc \ge 6abcd.$$

For the non-trivial case $d \ne 0$, this inequality is equivalent to

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \ge 6.$$

Since

$$\frac{1}{a}+\frac{1}{b} \ge \frac{4}{a+b}$$

and

$$\frac{1}{c}+\frac{1}{d} \ge \frac{4}{c+d},$$

it suffices to show that

$$\frac{2}{a+b}+\frac{2}{c+d}\ge 3,$$

which is equivalent to

$$(a+b-1)(a+b-2)\ge 0.$$

□

P 1.222. *If a,b,c,d are nonnegative real numbers such that*

$$a\ge b\ge 1\ge c\ge d,\qquad a+b+c+d=6,$$

then

$$a^2+b^2+c^2+d^2+4abcd\le 26.$$

(Vasile C., 2015)

First Solution. Let

$$E(a,b,c,d)=a^2+b^2+c^2+d^2+10abcd.$$

We will show that

$$E(a,b,c,d)\le E(a,b,x,x)\le 5,$$

where

$$x=(c+d)/2,\qquad a+b+2x=3.$$

The left inequality is true since

$$E(a,b,c,d)-E(a,b,x,x)=\frac{1}{2}(c-d)^2(1-2ab)\le 0.$$

The right inequality can be written as follows:

$$a^2+b^2+2x^2+4abx^2\le 26,$$

$$(a+b)^2+2x^2+2ab(2x^2-1)\le 26,$$

$$2s^2+(6-s)^2+2ab[(6-s)^2-2]\le 52,$$

where

$$s=a+b,\quad s\in[4,6].$$

Case 1: $(6-s)^2-2\ge 0$. Since $ab\le s^2/4$, it suffices to show that

$$2s^2+(6-s)^2+\frac{1}{2}s^2[(6-s)^2-2]\le 52,$$

which is equivalent to the obvious inequality

$$(s-2)(s-4)[s(s-6)-4]\le 0.$$

Case 2: $(6-s)^2-2\le 0$. From $(a-1)(b-1)\ge 0$, we get $ab\ge s-1$. Therefore, it suffices to show that

$$2s^2+(6-s)^2+2(s-1)[(6-s)^2-2]\le 52,$$

which is equivalent to the obvious inequality

$$(s-2)(s-6)(2s-7)\le 0.$$

The equality holds for $a=b=2$, $c=d=1$, and for $a=5$, $b=1$, $c=d=0$.

Second Solution. From

$$(a-1)(b-1)(c-1)(d-1)\ge 0,$$

we have

$$-5+\sum_{sym}ab-\sum abc+abcd\ge 0.$$

Since

$$2\sum_{sym}ab=36-a^2-b^2-c^2-d^2,$$

we get

$$-5+\frac{36-a^2-b^2-c^2-d^2}{2}-\sum abc+abcd\ge 0,$$

$$a^2+b^2+c^2+d^2\le 26-2\sum abc+2abcd.$$

Therefore, it suffices to show that

$$(26-2\sum abc+2abcd)+4abcd\le 26,$$

which is equivalent to

$$\sum abc\ge 3abcd.$$

For the non-trivial case $d\ne 0$, this inequality is equivalent to

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\ge 3.$$

Since

$$\frac{1}{a}+\frac{1}{b}\ge\frac{4}{a+b}$$

and

$$\frac{1}{c}+\frac{1}{d}\ge\frac{4}{c+d},$$

it suffices to show that

$$\frac{4}{a+b}+\frac{4}{c+d}\ge 3,$$

which is equivalent to

$$(a+b-2)(a+b-4)\ge 0,$$

$$(a+b-2)(2-c-d)\ge 0.$$

□

P 1.223. *Let a, b, c, d be nonnegative real numbers such that*

$$a \ge b \ge 1 \ge c \ge d, \qquad a+b+c+d=p, \quad p \ge 2.$$

Prove that

$$\frac{p^2-4p+8}{2} \le a^2+b^2+c^2+d^2 \le p^2-2p+2.$$

Solution. Write the right inequality as follows:

$$(p-1)^2-a^2+(1-b^2)-c^2-d^2 \ge 0,$$

$$(p-1-a)(p-1+a)+(1-b)(1+b)-c^2-d^2 \ge 0.$$

Since $p-1-a=(b-1)+c+d \ge 0$ and

$$(p-1+a)-(1+b)=2(a-1)+c+d \ge 0,$$

it suffices to show that

$$(p-1-a)(1+b)+(1-b)(1+b)-c^2-d^2 \ge 0,$$

which is equivalent to

$$(c+d)(1+b)-c^2-d^2 \ge 0.$$

Indeed,

$$(c+d)(1+b) \ge c+d \ge c^2+d^2.$$

The right inequality is an equality for

$$(a,b,c,d)=(p-1,1,0,0).$$

Since $(a+b)^2 \le 2(a^2+b^2)$ and $(c+d)^2 \le 2(c^2+d^2)$, the left inequality is true if

$$p^2-4p+8 \le (a+b)^2+(c+d)^2,$$

which is equivalent to

$$[(a+b)+(c+d)]^2-4[(a+b)+(c+d)]+8 \le (a+b)^2+(c+d)^2,$$

$$(a+b)(c+d)-2(a+b)-2(c+d)+4 \le 0,$$

$$(a+b-2)(c+d-2) \le 0.$$

The left inequality is an equality for

$$(a,b,c,d)=\left(1,1,\frac{p-2}{2},\frac{p-2}{2}\right), \quad 2 \le p \le 4,$$

$$(a,b,c,d)=\left(\frac{p-2}{2},\frac{p-2}{2},1,1\right), \quad p \ge 4.$$

□

P 1.224. *Let $a \ge b \ge 1 \ge c \ge d \ge 0$ such that*

$$a+b+c+d=4, \qquad a^2+b^2+c^2+d^2=q,$$

where $q \in [4,10]$ is a fixed number. Prove that the product $r=abcd$ is maximal when $b=1$ and $c=d$.

(Vasile C., 2015)

Solution. The condition $q \ge 4$ follows from the Cauchy-Schwarz inequality

$$4(a^2+b^2+c^2+d^2) \ge (a+b+c+d)^2.$$

The condition $q \le 10$ follows from the inequality $(a-1)(b-1) \ge 0$, which is equivalent to

$$ab \ge s-1,$$

where

$$s=a+b, \quad s \in [2,4].$$

Indeed,

$$q \le (a+b)^2-2ab+(c+d)^2 \le s^2-2(s-1)+(4-s)^2$$
$$=2(s-1)(s-4)+10 \le 10.$$

Notice that $q=4$ for $a=b=c=d=1$, and $q=10$ for $a=3$, $b=1$, $c=d=0$.

We will show that for any fixed $q \in [4,10]$, we have

$$abcd \le f(d) \le f(d_1),$$

where

$$f(d)=d\left(d^2-3d+5-\frac{q}{2}\right),$$

$$d_1=1-\sqrt{\frac{q-4}{6}}, \qquad d_1 \in [0,1].$$

The left inequality $abcd \le f(d)$ is a consequence of the inequality

$$(a-1)(b-1)(c-1) \le 0,$$

which leads to

$$abc \le 1-(a+b+c)+(ab+bc+ca)$$
$$=1-(4-d)+\frac{1}{2}[(a+b+c)^2-(a^2+b^2+c^2)]$$
$$=-3+d+\frac{1}{2}[(4-d)^2-(q-d^2)]$$
$$=d^2-3d+5-\frac{q}{2},$$

hence

$$abcd \le f(d),$$

with equality for $b = 1$.

The right inequality $f(d) \le f(d_1)$ follows immediately from

$$f(d) - f(d_1) = (d - d_1)^2(d + 2d_1 - 3) \le 0.$$

This inequality is an equality for $d = d_1$. In conclusion, for any fixed $q \in [4, 10]$, we have

$$abcd \le f(d_1),$$

with equality for $b = 1$ and $d = d_1$. These equality conditions are equivalent to $b = 1$ and $c = d$. Indeed, from $b = 1$, $d = d_1$, $a + b + c + d = 3$ and $a^2 + b^2 + c^2 + d^2 = q$, we get

$$a = 1 + \sqrt{\frac{2(q-4)}{3}}, \quad b = 1, \quad c = d = 1 - \sqrt{\frac{q-4}{6}}.$$

□

P 1.225. *If a, b, c, d are nonnegative real numbers such that*

$$a \ge b \ge 1 \ge c \ge d, \qquad a + b + c + d = 4,$$

then

$$a^2 + b^2 + c^2 + d^2 + 6abcd \le 10.$$

(Vasile C., 2015)

First Solution. According to P 1.224, it suffices to prove the inequality for $b = 1$ and $c = d$. Thus, we need to show that $a^2 + 2c^2 + 6ac^2 \le 9$ for $a + 2c = 3$; that is,

$$c(1 - c)^2 \ge 0.$$

The equality holds for $a = b = c = d = 1$, and for $a = 3$, $b = 1$, $c = d = 0$.

Second Solution. Let

$$E(a, b, c, d) = a^2 + b^2 + c^2 + d^2 + 6abcd.$$

We will show that

$$E(a, b, c, d) \le E(a, b, x, x) \le 10,$$

where

$$x = (c + d)/2.$$

The left inequality is true since

$$E(a, b, c, d) - E(a, b, x, x) = \frac{1}{2}(c - d)^2(1 - 3ab) \le 0.$$

The right inequality can be written as follows:

$$a^2+b^2+2x^2+6abx^2 \le 10, \quad a+b+2x=4,$$

$$(a+b)^2+2x^2+2ab(3x^2-1) \le 10,$$

$$2s^2+(4-s)^2+ab[3(4-s)^2-4] \le 20,$$

where

$$s=a+b, \quad s \in [2,4].$$

Case 1: $3(4-s)^2-4 \ge 0$. Since $ab \le s^2/4$, it suffices to show that

$$2s^2+(4-s)^2+\frac{1}{4}s^2[3(4-s)^2-4] \le 20,$$

which is equivalent to the obvious inequality

$$(s-2)^2[3s(s-4)-4] \le 0.$$

Case 2: $3(4-s)^2-4 \le 0$. From $(a-1)(b-1) \ge 0$, we get $ab \ge s-1$. Therefore, it suffices to show that

$$2s^2+(4-s)^2+(s-1)[3(4-s)^2-4] \le 20,$$

which is equivalent to the obvious inequality

$$(s-2)^2(s-4) \le 0.$$

Third Solution (by *Linqaszayi*). From

$$(a-1)(b-1)(c-1)(d-1) \ge 0,$$

we have

$$-3+\sum_{sym} ab - \sum abc + abcd \ge 0.$$

Since

$$2\sum_{sym} ab = 16-a^2-b^2-c^2-d^2,$$

we get

$$10-a^2-b^2-c^2-d^2 \ge 2\sum abc - 2abcd.$$

Therefore, it suffices to show that

$$2\sum abc - 2abcd \ge 6abcd,$$

which is equivalent to

$$\sum abc \ge 4abcd.$$

For the non-trivial case $d > 0$, this inequality is equivalent to the Cauchy-Schwarz inequality

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \geq 16.$$

Fourth Solution (by *Nguyen Van Quy*). Write the inequality as

$$a^2+(b+c+d)^2+6abcd-2(bc+cd+db) \leq 10,$$

$$3abcd-(bc+cd+db) \leq (a-1)(3-a).$$

By the AM-GM inequality or the Cauchy-Schwarz inequality, we have

$$bc+cd+db \geq \frac{9bcd}{b+c+d},$$

hence

$$3abcd-(bc+cd+db) \leq 3abcd-\frac{9bcd}{b+c+d} = \frac{3bcd(a-1)(3-a)}{b+c+d}.$$

Since

$$3-a \geq 4-a-b = c+d \geq 0,$$

it suffices to show that

$$\frac{3bcd}{b+c+d} \leq 1.$$

Indeed, using the AM-GM inequality and $b+c+d = 4-a \leq 3$, we get

$$\frac{3bcd}{b+c+d} \leq \frac{(b+c+d)^2}{9} \leq 1.$$

□

P 1.226. *If a,b,c,d are nonnegative real numbers such that*

$$a \geq b \geq 1 \geq c \geq d, \qquad a+b+c+d = 4,$$

then

$$a^2+b^2+c^2+d^2+6\sqrt{abcd} \leq 10.$$

(Vasile C., 2015)

First Solution. According to P 1.224, it suffices to prove the inequality for $b = 1$ and $c = d$. Thus, we need to show that $a+2c = 3$ implies $a^2+2c^2+6c\sqrt{a} \leq 9$; that is,

$$a^2+2c^2+6c\sqrt{a} \leq (a+2c)^2,$$

$$c(c+2a-3\sqrt{a}\,) \geq 0,$$

$$\frac{3c(\sqrt{a}-1)^2}{c+2a+3\sqrt{a}} \ge 0.$$

The equality holds for $a=b=c=d=1$, and for $a=3,\ b=1,\ c=d=0$.

Second Solution. Let

$$E(a,b,c,d)=a^2+b^2+c^2+d^2+6\sqrt{abcd}.$$

We will show that

$$E(a,b,c,d) \le E(a,b,x,x) \le 10,$$

where

$$x=\frac{c+d}{2}=\frac{4-a-b}{2}.$$

The left inequality can be reduces to the obvious form

$$\left(\sqrt{c}-\sqrt{d}\right)^2\left[6\sqrt{ab}-\left(\sqrt{c}+\sqrt{d}\right)^2\right] \ge 0,$$

while the right inequality is equivalent to

$$a^2+b^2+2x^2+6x\sqrt{ab} \le 10.$$

Since $2\sqrt{ab} \le a+b$, it suffices to show that

$$a^2+b^2+2x^2+3x(a+b) \le 10.$$

which can be rewritten as

$$2(a^2+b^2)+(4-a-b)^2+3(4-a-b)(a+b) \le 20,$$

$$2(a+b)^2-4ab+16-8(a+b)+(a+b)^2+12(a+b)-3(a+b)^2 \le 20,$$

$$4(a-1)(b-1) \ge 0.$$

□

P 1.227. *If a,b,c,d,e are positive real numbers, then*

$$\frac{a}{a+2b+2c}+\frac{b}{b+2c+2d}+\frac{c}{c+2d+2e}+\frac{d}{d+2e+2a}+\frac{e}{e+2a+2b} \ge 1.$$

Solution. The inequality follows by applying the Cauchy-Schwarz inequality:

$$\sum\frac{a}{a+2b+2c} \ge \frac{\left(\sum a\right)^2}{\sum a(a+2b+2c)}=\frac{\left(\sum a\right)^2}{\sum a^2+2\sum ab+2\sum ac}=1.$$

The equality holds for $a=b=c=d=e$.

□

P 1.228. *Let a, b, c, d, e be positive real numbers such that $a+b+c+d+e=5$. Prove that*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a}\le 1+\frac{4}{abcde}.$$

Solution. Let (x,y,z,t,u) be a permutation of (a,b,c,d,e) such that $x\ge y\ge z\ge t\ge u$. By the rearrangement inequality, we have

$$\begin{aligned}\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a}&\le\frac{x}{u}+\frac{y}{t}+\frac{z}{z}+\frac{t}{y}+\frac{u}{x}\\&=\left(\frac{x}{u}+\frac{u}{x}+2\right)+\left(\frac{y}{t}+\frac{t}{y}+2\right)-3\\&=4(p+q)-3,\end{aligned}$$

where

$$p=\frac{1}{4}\left(\frac{x}{u}+\frac{u}{x}+2\right)\ge 1,\quad q=\frac{1}{4}\left(\frac{y}{t}+\frac{t}{y}+2\right)\ge 1.$$

From $(p-1)(q-1)\ge 0$, we get

$$p+q\le 1+pq,$$

$$4(p+q)-3\le 1+4pq,$$

hence

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{e}+\frac{e}{a}\le 1+4pq.$$

Thus, it suffices to show that

$$pq\le\frac{1}{xyztu},$$

which is is equivalent to

$$z\left(\frac{x+u}{2}\right)^2\left(\frac{y+t}{2}\right)^2\le 1.$$

Indeed, by the AM-GM inequality, we get

$$z\left(\frac{x+u}{2}\right)^2\left(\frac{y+t}{2}\right)^2\le\left(\frac{z+\frac{x+u}{2}+\frac{x+u}{2}+\frac{y+t}{2}+\frac{y+t}{2}}{5}\right)^5=1.$$

The equality holds for $a=b=c=d=e=1$.

Remark. Similarly, we can prove the following generalization (*Michael Rozenberg*).

- *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1+a_2+\cdots+a_n=n$, then*

$$n-4+\frac{4}{a_1a_2\cdots a_n}\ge\frac{a_1}{a_2}+\frac{a_2}{a_3}+\cdots+\frac{a_n}{a_1}.$$

□

P 1.229. *If a, b, c, d, e are real numbers such that $a+b+c+d+e=0$, then*

$$\frac{-\sqrt{5}-1}{4} \le \frac{ab+bc+cd+de+ea}{a^2+b^2+c^2+d^2+e^2} \le \frac{\sqrt{5}-1}{4}.$$

Solution. From

$$(a+b+c+d+e)^2=0,$$

we get

$$\sum a^2+2\sum ab+2\sum ac=0.$$

Therefore, for any real k, we have

$$\sum a^2+(2k+2)\sum ab=\sum 2a(kb-c).$$

By the AM-GM inequality, we get

$$2a(kb-c) \le a^2+(kb-c)^2,$$

hence

$$\sum a^2+(2k+2)\sum ab \le \sum [a^2+(kb-c)^2]=(k^2+2)\sum a^2-2k\sum ab,$$

which is equivalent to

$$\sum a^2 \ge \frac{2(2k+1)}{k^2+1}\sum ab.$$

Choosing $k=\dfrac{-1-\sqrt{5}}{2}$ and $k=\dfrac{-1+\sqrt{5}}{2}$, we get the desired inequalities. The equality in both inequalities occurs when

$$a=kb-c, \quad b=kc-d, \quad c=kd-e, \quad d=ke-a, \quad e=ka-b;$$

that is, when

$$a=x, \quad b=y, \quad c=-x+ky, \quad d=-k(x+y), \quad e=kx-y,$$

where x and y are real numbers.

□

P 1.230. *Let a, b, c, d, e be positive real numbers such that*

$$a^2+b^2+c^2+d^2+e^2=5.$$

Prove that

$$\frac{a^2}{b+c+d}+\frac{b^2}{c+d+e}+\frac{c^2}{d+e+a}+\frac{d^2}{e+a+b}+\frac{e^2}{a+b+c} \ge \frac{5}{3}.$$

(Pham Van Thuan, 2005)

Solution. By the AM-GM Inequality, we get

$$2(b+c+d) \le (b^2+1)+(c^2+1)+(d^2+1) = 8-a^2-e^2.$$

Therefore, it suffices to show that

$$\sum \frac{a^2}{8-a^2-e^2} \ge \frac{5}{6}.$$

By the Cauchy-Schwarz Inequality, we have

$$\sum \frac{a^2}{8-a^2-e^2} \ge \frac{\left(\sum a^2\right)^2}{\sum a^2(8-a^2-e^2)} = \frac{25}{40-\sum a^4-\sum a^2e^2}$$
$$= \frac{50}{80-\sum (a^2+e^2)^2} \ge \frac{50}{80-\frac{1}{5}\left[\sum (a^2+e^2)\right]^2} = \frac{5}{6}.$$

The equality holds for $a=b=c=d=e=1$.

□

P 1.231. *Let a,b,c,d,e be nonnegative real numbers such that $a+b+c+d+e=5$. Prove that*

$$(a^2+b^2)(b^2+c^2)(c^2+d^2)(d^2+e^2)(e^2+a^2) \le \frac{729}{2}.$$

(Vasile C., 2007)

Solution. Write the inequality as

$$E(a,b,c,d,e) \le 0,$$

and, without loss of generality, assume that

$$e = \min\{a,b,c,d,e\}.$$

We claim that it suffices to prove the desired inequality for the case $e=0$. To prove this, it suffices to show that

$$E(a,b,c,d,e) \le E\left(a+\frac{e}{2}, b, c, d+\frac{e}{2}, 0\right), \tag{*}$$

which is equivalent to

$$(a^2+b^2)(c^2+d^2)(d^2+e^2)(e^2+a^2) \le$$
$$\le \left[\left(a+\frac{e}{2}\right)^2+b^2\right]\left[c^2+\left(d+\frac{e}{2}\right)^2\right]\left(d+\frac{e}{2}\right)^2\left(a+\frac{e}{2}\right)^2.$$

Since

$$a^2+b^2 \le \left(a+\frac{e}{2}\right)^2+b^2,$$

$$c^2+d^2 \le c^2+\left(d+\frac{e}{2}\right)^2,$$

$$d^2+e^2 \le d^2+de \le \left(d+\frac{e}{2}\right)^2,$$

$$e^2+a^2 \le ae+a^2 \le \left(a+\frac{e}{2}\right)^2,$$

the conclusion follows. Thus, we only need to show that

$$a+b+c+d=5$$

involves

$$E(a,b,c,d,0) \le 0,$$

where

$$E(a,b,c,d,0)=a^2d^2(a^2+b^2)(b^2+c^2)(c^2+d^2)-\frac{729}{2}.$$

Without loss of generality, assume that

$$c=\min\{b,c\}.$$

We claim that it suffices to prove the inequality $E(a,b,c,d,0) \le 0$ for the case $c=0$. To prove this, it suffices to show that

$$E(a,b,c,d,0) \le E\left(a,b+\frac{c}{2},0,d+\frac{c}{2},0\right), \tag{**}$$

which is equivalent to

$$d^2(a^2+b^2)(b^2+c^2)(c^2+d^2) \le \left(d+\frac{c}{2}\right)^2\left[a^2+\left(b+\frac{c}{2}\right)^2\right]\left(b+\frac{c}{2}\right)^2\left(d+\frac{c}{2}\right)^2.$$

This is true since

$$d^2(c^2+d^2) \le \left(d+\frac{c}{2}\right)^4,$$

$$a^2+b^2 \le a^2+\left(b+\frac{c}{2}\right)^2,$$

$$b^2+c^2 \le b^2+bc \le \left(b+\frac{c}{2}\right)^2.$$

Thus, we only need to show that

$$a+b+d=5$$

involves

$$E(a,b,0,d,0) \le 0,$$

where

$$E(a,b,0,d,0)=a^2b^2d^4(a^2+b^2)-\frac{729}{2}.$$

We will show that

$$E(a,b,0,d,0)\le E\left(\frac{a+b}{2},\frac{a+b}{2},0,d,0\right)\le 0. \tag{***}$$

The left inequality is true if

$$32a^2b^2(a^2+b^2)\le (a+b)^6.$$

Indeed, we have

$$(a+b)^6-32a^2b^2(a^2+b^2)\ge 4ab(a+b)^4-32a^2b^2(a^2+b^2)=4ab(a-b)^4\ge 0.$$

To prove the right inequality, denote

$$u=\frac{a+b}{2}.$$

We need to show that

$$2u+d=5$$

implies

$$E(u,u,0,d,0)\le 0;$$

that is,

$$u^6d^4\le\frac{729}{4},$$

$$u^3d^2\le\frac{27}{2}.$$

By the AM-GM inequality, we have

$$5=\frac{2u}{3}+\frac{2u}{3}+\frac{2u}{3}+\frac{d}{2}+\frac{d}{2}\ge 5\sqrt[5]{\left(\frac{2u}{3}\right)^3\left(\frac{t}{2}\right)^2},$$

from which the conclusion follows. The equality holds for $a=b=\frac{3}{2}$, $c=0$, $d=2$ and $e=0$ (or any cyclic permutation).

□

P 1.232. *If $a,b,c,d,e\in[1,5]$, then*

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+e}+\frac{d-e}{e+a}+\frac{e-a}{a+b}\ge 0.$$

(Vasile C., 2002)

Solution. Write the inequality as

$$\sum \left(\frac{a-b}{b+c} + \frac{2}{3} \right) \ge \frac{10}{3},$$

$$\sum \frac{3a-b+2c}{b+c} \ge 10.$$

Since

$$3a-b+2c \ge 3-5+2=0,$$

we may apply the Cauchy-Schwarz inequality to get

$$\sum \frac{3a-b+2c}{b+c} \ge \frac{\left[\sum(3a-b+2c)\right]^2}{\sum(b+c)(3a-b+2c)} = \frac{16\left(\sum a\right)^2}{\sum a^2+4\sum ab+3\sum ac}.$$

Therefore, it suffices to show that

$$8\left(\sum a\right)^2 \ge 5\sum a^2 + 20\sum ab + 15\sum ac.$$

Since

$$\left(\sum a\right)^2 = \sum a^2 + 2\sum ab + 2\sum ac,$$

this inequality is equivalent to

$$3\sum a^2 + \sum ac \ge 4\sum ab.$$

Indeed,

$$3\sum a^2 + \sum ac - 4\sum ab = \frac{1}{2}\sum(a-2b+c)^2 \ge 0.$$

The equality holds for $a=b=c=d=e$.

□

P 1.233. *If* $a,b,c,d,e,f \in [1,3]$, *then*

$$\frac{a-b}{b+c}+\frac{b-c}{c+d}+\frac{c-d}{d+e}+\frac{d-e}{e+f}+\frac{e-f}{f+a}+\frac{f-a}{a+b} \ge 0.$$

(Vasile C., 2002)

Solution. Write the inequality as

$$\sum \left(\frac{a-b}{b+c} + \frac{1}{2} \right) \ge 3,$$

$$\sum \frac{2a-b+c}{b+c} \ge 6.$$

Since

$$2a - b + c \geq 2 - 3 + 1 = 0,$$

we may apply the Cauchy-Schwarz inequality to get

$$\sum \frac{2a-b+c}{b+c} \geq \frac{\left[\sum(2a-b+c)\right]^2}{\sum(b+c)(2a-b+c)} = \frac{2\left(\sum a\right)^2}{\sum ab + \sum ac}.$$

Thus, we still have to show that

$$\left(\sum a\right)^2 \geq 3\left(\sum ab + \sum ac\right).$$

Let

$$x = a + d, \quad y = b + e, \quad z = c + f.$$

Since

$$\sum ab + \sum ac = xy + yz + zx,$$

we have

$$\left(\sum a\right)^2 - 3\left(\sum ab + \sum ac\right) = (x+y+z)^2 - 3(xy+yz+zx) \geq 0.$$

The equality holds for $a = c = e$ and $b = d = f$.

□

P 1.234. *If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are positive real numbers, then*

$$\sum_{i=1}^{n} \frac{a_i}{a_{i-1} + 2a_i + a_{i+1}} \leq \frac{n}{4},$$

where $a_0 = a_n$ and $a_{n+1} = a_1$.

(Vasile C., 2008)

Solution. Applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\sum_{i=1}^{n} \frac{a_i}{a_{i-1} + 2a_i + a_{i+1}} &= \sum_{i=1}^{n} \frac{a_i}{(a_{i-1}+a_i)+(a_i+a_{i+1})} \\
&\leq \frac{1}{4}\sum_{i=1}^{n} a_i\left(\frac{1}{a_{i-1}+a_i} + \frac{1}{a_i+a_{i+1}}\right) \\
&= \frac{1}{4}\left(\sum_{i=1}^{n}\frac{a_i}{a_{i-1}+a_i} + \sum_{i=1}^{n}\frac{a_i}{a_i+a_{i+1}}\right) \\
&= \frac{1}{4}\left(\sum_{i=1}^{n}\frac{a_{i+1}}{a_i+a_{i+1}} + \sum_{i=1}^{n}\frac{a_i}{a_i+a_{i+1}}\right) = \frac{n}{4}.
\end{aligned}$$

The equality holds for $a_1 = a_2 = \cdots = a_n$.

□

P 1.235. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be positive real numbers such that $a_1a_2\cdots a_n = 1$. Prove that*

$$\frac{1}{n-2+a_1+a_2}+\frac{1}{n-2+a_2+a_3}+\cdots+\frac{1}{n-2+a_n+a_1} \le 1.$$

(Vasile C., 2008)

First Solution. Let $r = \dfrac{n-2}{n}$. We can get the desired inequality by summing the following inequalities

$$\frac{n-2}{n-2+a_1+a_2} \le \frac{a_3^r+a_4^r+\cdots+a_n^r}{a_1^r+a_2^r+\cdots+a_n^r},$$

$$\frac{n-2}{n-2+a_2+a_3} \le \frac{a_1^r+a_4^r+\cdots+a_n^r}{a_1^r+a_2^r+\cdots+a_n^r},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{n-2}{n-2+a_n+a_1} \le \frac{a_2^r+a_3^r+\cdots+a_{n-1}^r}{a_1^r+a_2^r+\cdots+a_n^r}.$$

The first inequality is equivalent to

$$(a_1+a_2)(a_3^r+a_4^r+\cdots+a_n^r) \ge (n-2)(a_1^r+a_2^r).$$

By the AM-GM inequality, we have

$$a_3^r+a_4^r+\cdots+a_n^r \ge (n-2)(a_3a_4\cdots a_n)^{\frac{r}{n-2}} = \frac{n-2}{(a_1a_2)^{\frac{r}{n-2}}}.$$

Therefore, it suffices to show that

$$a_1+a_2 \ge (a_1a_2)^{\frac{r}{n-2}}(a_1^r+a_2^r),$$

or, equivalently,

$$a_1+a_2 \ge (a_1a_2)^{\frac{1}{n}}\left(a_1^{\frac{n-2}{n}}+a_2^{\frac{n-2}{n}}\right).$$

This is equivalent to the obvious inequality

$$\left(a_1^{\frac{n-1}{n}}-a_2^{\frac{n-1}{n}}\right)\left(a_1^{\frac{1}{n}}-a_2^{\frac{1}{n}}\right) \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n$.

Second Solution. Since

$$\frac{n-2}{n-2+a_1+a_2} = 1-\frac{a_1+a_2}{n-2+a_1+a_2},$$

we can write the desired inequality as

$$\sum_{i=1}^{n} \frac{a_i + a_{i+1}}{a_i + a_{i+1} + n - 2} \geq 2,$$

where $a_{n+1} = a_1$. Using the Cauchy-Schwarz inequality, we get

$$\sum_{i=1}^{n} \frac{a_i + a_{i+1}}{a_i + a_{i+1} + n - 2} \geq \frac{\left(\sum_{i=1}^{n} \sqrt{a_i + a_{i+1}}\right)^2}{\sum_{i=1}^{n}(a_i + a_{i+1} + n - 2)}$$

$$= \frac{2\sum_{i=1}^{n} a_i + 2\sum_{1 \leq i < j \leq n} \sqrt{(a_i + a_{i+1})(a_j + a_{j+1})}}{2\sum_{i=1}^{n} a_i + n(n-2)}.$$

Therefore, it suffices to prove that

$$\sum_{1 \leq i < j \leq n} \sqrt{(a_i + a_{i+1})(a_j + a_{j+1})} \geq \sum_{i=1}^{n} a_i + n(n-2).$$

Setting $a_{n+2} = a_2$, by the Cauchy-Schwarz inequality and the AM-GM inequality, we have

$$\sum_{1 \leq i < j \leq n} \sqrt{(a_i + a_{i+1})(a_j + a_{j+1})} =$$

$$= \sum_{i=1}^{n} \sqrt{(a_i + a_{i+1})(a_{i+1} + a_{i+2})} + \sum_{\substack{1 \leq i < j \leq n \\ j \neq i+1}} \sqrt{(a_i + a_{i+1})(a_j + a_{j+1})}$$

$$\geq \sum_{i=1}^{n} \left(a_{i+1} + \sqrt{a_i a_{i+2}}\right) + n(n-3)\sqrt[n]{a_1 a_2 \cdots a_n}$$

$$= \sum_{i=1}^{n} a_i + n(n-3) + \sum_{i=1}^{n} \sqrt{a_i a_{i+2}}$$

$$\geq \sum_{i=1}^{n} a_i + n(n-3) + n\sqrt[n]{a_1 a_2 \cdots a_n} = \sum_{i=1}^{n} a_i + n(n-2).$$

□

P 1.236. *If* $a_1, a_2, \ldots, a_n \geq 1$, *then*

$$\prod \left(a_1 + \frac{1}{a_2} + n - 2\right) \geq n^{n-2}(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right).$$

(Vasile C., 2011)

Solution. Write the inequality as $E(a_1,a_2,...,a_n) \ge 0$, and denote

$$A = \left(a_2 + \frac{1}{a_3} + n - 2\right)\left(a_3 + \frac{1}{a_4} + n - 2\right)\cdots\left(a_{n-1} + \frac{1}{a_n} + n - 2\right).$$

We will prove that

$$E(a_1,a_2,...,a_n) \ge E(1,a_2,...,a_n).$$

If this is true, then

$$E(a_1,a_2,...,a_n) \ge E(1,a_2,...,a_n) \ge E(1,1,a_3,...,a_n) \ge \cdots \ge E(1,1,...,1,a_n) = 0.$$

We have

$$E(a_1,a_2,...,a_n) - E(1,a_2,...,a_n) = (a_1 - 1)\left(B - \frac{C}{a_1}\right),$$

where

$$B = A(a_n + n - 2) - n^{n-2}\left(\frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}\right),$$

$$C = A\left(\frac{1}{a_2} + n - 2\right) - n^{n-2}(a_2 + a_3 + \cdots + a_n).$$

Since $a_1 - 1 \ge 0$, we need to show that

$$a_1 B - C \ge 0.$$

According to the AM-GM inequality, we have

$$A \ge \left(n\sqrt[n]{\frac{a_2}{a_3}}\right)\left(n\sqrt[n]{\frac{a_3}{a_4}}\right)\cdots\left(n\sqrt[n]{\frac{a_{n-1}}{a_n}}\right) = n^{n-2}\sqrt[n]{\frac{a_2}{a_n}},$$

$$a_n + n - 2 \ge (n-1)\sqrt[n-1]{a_n},$$

$$A(a_n + n - 2) \ge (n-1)n^{n-2}\sqrt[n]{a_2 a_n^{\frac{1}{n-1}}} \ge (n-1)n^{n-2},$$

therefore

$$B \ge n^{n-2}\left(n - 1 - \frac{1}{a_2} - \frac{1}{a_3} - \cdots - \frac{1}{a_n}\right) \ge 0$$

and

$$a_1 B - C \ge B - C = A\left(a_n - \frac{1}{a_2}\right) + n^{n-2}\left(a_2 - \frac{1}{a_2}\right) + \cdots + n^{n-2}\left(a_n - \frac{1}{a_n}\right) \ge 0.$$

The equality holds when $n-1$ of the numbers $a_1, a_2, ..., a_n$ are equal to 1.

□

P 1.237. *If $a_1, a_2, ..., a_n \geq 1$, then*

$$\left(a_1+\frac{1}{a_1}\right)\left(a_2+\frac{1}{a_2}\right)\cdots\left(a_n+\frac{1}{a_n}\right)+2^n \geq 2\left(1+\frac{a_1}{a_2}\right)\left(1+\frac{a_2}{a_3}\right)\cdots\left(1+\frac{a_n}{a_1}\right).$$

(Vasile C., 2011)

Solution. Write the inequality as $E(a_1, a_2, ..., a_n) \geq 0$, and denote

$$A=\left(a_2+\frac{1}{a_2}\right)\cdots\left(a_n+\frac{1}{a_n}\right),$$

$$B=\left(1+\frac{a_2}{a_3}\right)\cdots\left(1+\frac{a_{n-1}}{a_n}\right).$$

We will prove that

$$E(a_1, a_2, ..., a_n) \geq E(1, a_2, ..., a_n).$$

If this is true, then

$$E(a_1, a_2, ..., a_n) \geq E(1, a_2, ..., a_n) \geq E(1, 1, a_3, ..., a_n) \geq \cdots \geq E(1, 1, ..., 1, a_n) = 0.$$

We have

$$E(a_1, a_2, ..., a_n) - E(1, a_2, ..., a_n) = (a_1 - 1)\left(C - \frac{D}{a_1}\right),$$

where

$$C = A - \frac{2B}{a_2},$$

$$D = A - 2Ba_n.$$

Since $a_1 - 1 \geq 0$, we need to show that

$$a_1 C - D \geq 0.$$

First, we prove that $C \geq 0$; that is,

$$(a_2^2+1)\cdots(a_n^2+1) \geq 2(a_2+a_3)\cdots(a_{n-1}+a_n).$$

By squaring, this inequality becomes

$$(a_2^2+1)[(a_2^2+1)(a_3^2+1)]\cdots[(a_{n-1}^2+1)(a_n^2+1)](a_n^2+1) \geq$$

$$\geq 4(a_2+a_3)^2\cdots(a_{n-1}+a_n)^2.$$

By the Cauchy-Schwarz inequality, we have

$$(a_2^2+1)(a_3^2+1) \geq (a_2+a_3)^2, \ \ ... \ , \ (a_{n-1}^2+1)(a_n^2+1) \geq (a_{n-1}+a_n)^2.$$

Therefore, we still have to show that

$$(a_2^2+1)(a_n^2+1) \geq 4,$$

which is clearly true for $a_2 \ge 1$ and $a_n \ge 1$. Finally, we have

$$a_1C - D \ge C - D = 2B\left(a_n - \frac{1}{a_2}\right) \ge 0.$$

The equality holds when $n-1$ of $a_1, a_2, ..., a_n$ are equal to 1.

□

P 1.238. *Let k and n be positive integers, and let $a_1, a_2, ..., a_n$ be real numbers such that*

$$a_1 \le a_2 \le \cdots \le a_n.$$

Consider the inequality

$$(a_1 + a_2 + \cdots + a_n)^2 \ge n(a_1a_{k+1} + a_2a_{k+2} + \cdots + a_na_{n+k}),$$

where $a_{n+i} = a_i$ for any positive integer i. Prove this inequality for

(a) $n = 2k$;

(b) $n = 4k$.

(Vasile C., 2004)

Solution. (a) We need to prove that

$$(a_1 + a_2 + \cdots + a_{2k})^2 \ge 4k(a_1a_{k+1} + a_2a_{k+2} + \cdots + a_ka_{2k}).$$

If x is a real number such that

$$a_k \le x \le a_{k+1},$$

then

$$(x - a_1)(a_{k+1} - x) + (x - a_2)(a_{k+2} - x) + \cdots + (x - a_k)(a_{2k} - x) \ge 0.$$

Expanding and multiplying by $4k$, we get

$$4kx(a_1 + a_2 + \cdots + a_{2k}) \ge 4k^2x^2 + 4k(a_1a_{k+1} + a_2a_{k+2} + \cdots + a_ka_{2k}).$$

On the other hand, by the AM-GM inequality, we have

$$(a_1 + a_2 + \cdots + a_{2k})^2 + 4k^2x^2 \ge 4kx(a_1 + a_2 + \cdots + a_{2k}).$$

Adding these inequalities yields the desired inequality. The equality holds for

$$a_{j+1} = a_{j+2} = \cdots = a_{j+k} = \frac{a_1 + a_2 + \cdots + a_{2k}}{2k},$$

where $j \in \{1, 2, \cdots, k-1\}$.

(b) We need to show that

$$(a_1 + a_2 + \cdots + a_{4k})^2 \geq 4k(a_1 a_{k+1} + a_2 a_{k+2} + \cdots + a_{4k} a_k).$$

Using the substitution

$$b_i = a_i + a_{2k+i}, \quad i = 1, 2, ..., 2k,$$

this inequality becomes

$$(b_1 + b_2 + \cdots + b_{2k})^2 \geq 4k(b_1 b_{k+1} + b_2 b_{k+2} + \cdots + b_k b_{2k}),$$

which is just the inequality in (a). The equality holds for

$$\begin{cases} a_{j+1} = a_{j+2} = \cdots = a_{j+k} = a \\ a_{j+2k+1} = a_{j+2k+2} = \cdots = a_{j+3k} = b \\ a_1 + a_2 + \cdots + a_{4k} = 2k(a+b) \end{cases},$$

where $a \leq b$ are real numbers, and $j \in \{1, 2, \cdots, k-1\}$.

Remark. Actually, the inequality holds for any integer k satisfying $\frac{n}{4} \leq k \leq \frac{n}{2}$.

□

P 1.239. *If $a_1, a_2, \ldots, a_n$ are real numbers, then*

$$a_1(a_1 + a_2) + a_2(a_2 + a_3) + \cdots + a_n(a_n + a_1) \geq \frac{2}{n}(a_1 + a_2 + \cdots + a_n)^2.$$

Solution. Making the substitution

$$a = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

and

$$x_i = a_i - a, \quad i = 1, 2, ..., n,$$

we have

$$x_1 + x_2 + \cdots + x_n = 0$$

and

$$\sum a_1(a_1 + a_2) - \frac{2}{n}(a_1 + a_2 + \cdots + a_n)^2 = \sum (x_1 + a)(x_1 + x_2 + 2a) - 2na^2$$

$$= \sum x_1(x_1 + x_2) = \frac{1}{2} \sum (x_1 - x_2)^2 \geq 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n$ - if n is odd, and for $a_1 = a_3 = \cdots = a_{n-1}$ and $a_2 = a_4 = \cdots = a_n$ - if n is even.

□

P 1.240. *If* $a_1,\ a_2,\ \dots,\ a_n \in [1,\ 2]$, *then*

$$\sum_{i=1}^{n} \frac{3}{a_i + 2a_{i+1}} \ge \sum_{i=1}^{n} \frac{2}{a_i + a_{i+1}},$$

where $a_{n+1} = a_1$.

(Vasile C., 2005)

Solution. Rewrite the inequality as follows

$$\sum_{i=1}^{n} \frac{a_i - a_{i+1}}{(a_i + a_{i+1})(a_i + 2a_{i+1})} \ge 0,$$

$$\sum_{i=1}^{n} \left[\frac{k(a_i - a_{i+1})}{(a_i + a_{i+1})(a_i + 2a_{i+1})} + \frac{1}{a_i} - \frac{1}{a_{i+1}} \right] \ge 0, \qquad k > 0,$$

$$\sum_{i=1}^{n} \frac{(a_i - a_{i+1})[(k-3)a_i a_{i+1} - a_i^2 - 2a_{i+1}^2]}{a_i a_{i+1}(a_i + a_{i+1})(a_i + 2a_{i+1})} \ge 0,$$

Setting $k = 6$, the inequality becomes

$$\sum_{i=1}^{n} \frac{(a_i - a_{i+1})^2(2a_{i+1} - a_i)}{a_i a_{i+1}(a_i + a_{i+1})(a_i + 2a_{i+1})} \ge 0.$$

Since $1 \le a_i \le 2$, we have $2a_{i+1} - a_i \ge 0$ for all $i = 1, 2, \dots, n$. Thus, the inequality is proved. The equality holds for $a_1 = a_2 = \cdots = a_n$.

□

P 1.241. *Let* $a_1, a_2, \dots, a_n$ $(n \ge 3)$ *be real numbers such that* $a_1 + a_2 + \cdots + a_n = n$.

(a) If $a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n$, *then*

$$a_1^3 + a_2^3 + \cdots + a_n^3 + 2n \ge 3(a_1^2 + a_2^2 + \cdots + a_n^2);$$

(b) If $a_1 \le 1 \le a_2 \le \cdots \le a_n$, *then*

$$a_1^3 + a_2^3 + \cdots + a_n^3 + 2n \le 3(a_1^2 + a_2^2 + \cdots + a_n^2).$$

(Vasile C., 2007)

Solution. (a) Write the inequality as

$$\sum (a_1^3 - 3a_1^2 + 3a_1 - 1) \ge 0,$$

$$\sum (a_1 - 1)^3 \ge 0,$$

$$(a_1 - 1)^3 \ge (1 - a_2)^3 + \cdots + (1 - a_n)^3,$$

$$[(1-a_2)+\cdots+(1-a_n)]^3 \geq (1-a_2)^3+\cdots+(1-a_n)^3.$$

Clearly, the last inequality is true. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 2$, $a_2 = \cdots = a_{n-1} = 1$, $a_n = 0$.

(b) Similarly, write the inequality as

$$\sum(a_1^3 - 3a_1^2 + 3a_1 - 1) \leq 0,$$

$$\sum(1-a_1)^3 \geq 0,$$

$$(1-a_1)^3 \geq (a_2-1)^3+\cdots+(a_n-1)^3,$$

$$[(a_2-1)+\cdots+(a_n-1)]^3 \geq (a_2-1)^3+\cdots+(a_n-1)^3.$$

The last inequality is obviously true. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 0$, $a_2 = \cdots = a_{n-1} = 1$, $a_n = 2$.

□

P 1.242. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$.*

(a) If $a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n$, then

$$a_1^4 + a_2^4 + \cdots + a_n^4 + 5n \geq 6(a_1^2 + a_2^2 + \cdots + a_n^2);$$

(b) If $a_1 \leq 1 \leq a_2 \leq \cdots \leq a_n$, then

$$a_1^4 + a_2^4 + \cdots + a_n^4 + 6n \leq 7(a_1^2 + a_2^2 + \cdots + a_n^2).$$

(Vasile C., 2007)

Solution. (a) Write the inequality as

$$\sum(a_1^4 - 6a_1^2 + 8a_1 - 3) \geq 0,$$

$$\sum(a_1-1)^3(a_1+3) \geq 0,$$

$$(a_1-1)^3(a_1+3) \geq (1-a_2)^3(a_2+3)+\cdots+(1-a_n)^3(a_n+3).$$

Since

$$(a_1-1)^3 = [(1-a_2)+\cdots+(1-a_n)]^3 \geq (1-a_2)^3+\cdots+(1-a_n)^3,$$

it suffices to show that

$$[(1-a_2)^3+\cdots+(1-a_n)^3](a_1+3) \geq (1-a_2)^3(a_2+3)+\cdots+(1-a_n)^3(a_n+3),$$

which is equivalent to the obvious inequality

$$(1-a_2)^3(a_1-a_2)+\cdots+(1-a_n)^3(a_1-a_n)\ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

(b) Write the inequality as

$$\sum(a_1^4-7a_1^2+10a_1-4)\le 0,$$

$$\sum(a_1-1)^2(a_1^2+2a_1-4)\le 0,$$

$$(a_2-1)^2(a_2^2+2a_2-4)+\cdots+(a_n-1)^2(a_n^2+2a_n-4)\le(1-a_1)^2(4-2a_1-a_1^2).$$

Since

$$(1-a_1)^2=[(a_2-1)+\cdots+(a_n-1)]^2\ge(a_2-1)^2+\cdots+(a_n-1)^2,$$

it suffices to show that

$$(a_2-1)^2(a_2^2+2a_2-4)+\cdots+(a_n-1)^2(a_n^2+2a_n-4)\le[(a_2-1)^2+\cdots+(a_n-1)^2](4-2a_1-a_1^2),$$

which is equivalent to

$$(a_2-1)^2(a_1^2+a_2^2+2a_1+2a_2-8)+\cdots+(a_n-1)^2(a_1^2+a_n^2+2a_1+2a_n-8)\le 0.$$

This inequality is true if

$$a_1^2+a_n^2+2a_1+2a_n-8\le 0.$$

Since

$$a_1+a_n=n-(a_2+\cdots+a_{n-1})=2+(1-a_2)+\cdots+(1-a_{n-1})\le 2,$$

we have

$$a_1^2+a_n^2+2a_1+2a_n-8=(a_1+a_n+1)^2-9-2a_1a_n\le(a_1+a_n+1)^2-9\le 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=0$, $a_2=\cdots=a_{n-1}=1$, $a_n=2$.

Remark. The inequality in (a) remains valid for all real $a_1,a_2,\ldots,a_n$ such that

$$a_1+a_2+\cdots+a_n=n,\qquad a_1\ge 1\ge a_2\ge\cdots\ge a_n.$$

□

P 1.243. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = n,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 + 2n \ge 3(a_1 + a_2 + \cdots + a_n).$$

(Vasile C., 2008)

Solution. Write the inequality as follows:

$$(a_1 - 1)(a_1 - 2) + (a_2 - 1)(a_2 - 2) + \cdots + (a_n - 1)(a_n - 2) \ge 0,$$

$$(a_1 - 1)(a_1 - 2) \ge (1 - a_2)(a_2 - 2) + \cdots + (1 - a_n)(a_n - 2),$$

$$\left(1 - \frac{1}{a_1}\right)(a_1^2 - 2a_1) \ge \left(\frac{1}{a_2} - 1\right)(a_2^2 - 2a_2) + \cdots + \left(\frac{1}{a_n} - 1\right)(a_n^2 - 2a_n),$$

$$\left[\left(\frac{1}{a_2} - 1\right) + \cdots + \left(\frac{1}{a_n} - 1\right)\right](a_1^2 - 2a_1) \ge \left(\frac{1}{a_2} - 1\right)(a_2^2 - 2a_2) + \cdots + \left(\frac{1}{a_n} - 1\right)(a_n^2 - 2a_n),$$

$$\left(\frac{1}{a_2} - 1\right)(a_1^2 - 2a_1 - a_2^2 + 2a_2) + \cdots + \left(\frac{1}{a_n} - 1\right)(a_1^2 - 2a_1 - a_n^2 + 2a_n) \ge 0,$$

$$\left(\frac{1}{a_2} - 1\right)(a_1 - a_2)(a_1 + a_2 - 2) + \cdots + \left(\frac{1}{a_n} - 1\right)(a_1 - a_n)(a_1 + a_n - 2) \ge 0.$$

Clearly, it suffices to prove that $a_1 + a_n - 2 \ge 0$. Indeed,

$$a_1 + a_n - 2 = n - 2 - (a_2 + \cdots + a_{n-1}) = (1 - a_2) + \cdots + (1 - a_{n-1}) \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.244. *If $a_1, a_2, \ldots, a_n$ are real numbers such that*

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

(a)
$$\frac{a_1 + 1}{a_1^2 + 1} + \frac{a_2 + 1}{a_2^2 + 1} + \cdots + \frac{a_n + 1}{a_n^2 + 1} \le n;$$

(b)
$$\frac{1}{a_1^2 + 3} + \frac{1}{a_2^2 + 3} + \cdots + \frac{1}{a_1^2 + 3} \le \frac{n}{4}.$$

(Vasile C., 2009)

Solution. (a) Write the inequality as

$$\left(1-\frac{a_1+1}{a_1^2+1}\right)+\left(1-\frac{a_2+1}{a_2^2+1}\right)+\cdots+\left(1-\frac{a_n+1}{a_n^2+1}\right)\geq 0,$$

$$\frac{a_1(a_1-1)}{a_1^2+1}+\frac{a_2(a_2-1)}{a_2^2+1}+\cdots+\frac{a_n(a_n-1)}{a_n^2+1}\geq 0,$$

$$\frac{a_2(a_2-1)}{a_2^2+1}+\cdots+\frac{a_n(a_n-1)}{a_n^2+1}\geq\frac{a_1(1-a_1)}{a_1^2+1},$$

$$\frac{a_2(a_2-1)}{a_2^2+1}+\cdots+\frac{a_n(a_n-1)}{a_n^2+1}\geq\frac{a_1[(a_2-1)+\cdots+(a_n-1)]}{a_1^2+1},$$

$$(a_2-1)\left(\frac{a_2}{a_2^2+1}-\frac{a_1}{a_1^2+1}\right)+\cdots+(a_n-1)\left(\frac{a_n}{a_n^2+1}-\frac{a_1}{a_1^2+1}\right)\geq 0,$$

$$\frac{(a_2-1)(a_2-a_1)(1-a_1a_2)}{a_2^2+1}+\cdots+\frac{(a_n-1)(a_n-a_1)(1-a_1a_n)}{a_n^2+1}\geq 0.$$

For $a_1\geq 0$, it suffices to show that $1-a_1a_n\geq 0$. Indeed,

$$2\sqrt{a_1a_n}\leq a_1+a_n=2+(1-a_2)+\cdots+(1-a_{n-1})\leq 2.$$

For $a_1\leq 0$, the inequality is also true because

$$1-a_1a_2>0,\ \ \cdots,\ \ 1-a_1a_n>0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

(b) As in the case (a), we write the inequality as

$$\frac{(a_2-1)(a_2-a_1)(3-a_1a_2-a_1-a_2)}{a_2^2+3}+\cdots+\frac{(a_n-1)(a_n-a_1)(3-a_1a_n-a_1-a_n)}{a_2^2+3}\geq 0.$$

For $a_1\geq 0$, it suffices to show that $3-a_1a_n-a_1-a_n\geq 0$. From $(1-a_1)(a_n-1)\geq 0$, we get $3-a_1a_n\geq 4-a_1-a_n$, hence

$$\frac{1}{2}(3-a_1a_n-a_1-a_n)\geq 2-a_1-a_n=(a_2-1)+\cdots+(a_{n-1}-1)\geq 0.$$

For $a_1\leq 0$, the inequality is also true because

$$3-a_1a_2-a_1-a_2>2-a_1-a_2=(a_3-1)+\cdots+(a_n-1)\geq 0,$$

$$\cdots$$

$$3-a_1a_n-a_1-a_n>2-a_1-a_n=(a_2-1)+\cdots+(a_{n-1}-1)\geq 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 1.245. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that*

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{a_1^2-1}{(a_1+3)^2} + \frac{a_2^2-1}{(a_2+3)^2} + \cdots + \frac{a_n^2-1}{(a_n+3)^2} \ge 0.$$

(Vasile C., 2009)

Solution. Write the inequality as follows:

$$\frac{a_2^2-1}{(a_2+3)^2} + \cdots + \frac{a_n^2-1}{(a_n+3)^2} \ge \frac{1-a_1^2}{(a_1+3)^2},$$

$$\frac{a_2^2-1}{(a_2+3)^2} + \cdots + \frac{a_n^2-1}{(a_n+3)^2} \ge \frac{[(a_2-1)+\cdots+(a_n-1)](1+a_1)}{(a_1+3)^2},$$

$$(a_2-1)\left[\frac{a_2+1}{(a_2+3)^2} - \frac{a_1+1}{(a_1+3)^2}\right] + \cdots + (a_n-1)\left[\frac{a_n+1}{(a_n+3)^2} - \frac{a_1+1}{(a_1+3)^2}\right] \ge 0,$$

$$\frac{(a_2-1)(a_2-a_1)(3-a_1-a_2-a_1a_2)}{(a_1+3)^2(a_2+3)^2} + \cdots + \frac{(a_n-1)(a_n-a_1)(3-a_1-a_n-a_1a_n)}{(a_1+3)^2(a_n+3)^2} \ge 0.$$

It suffices to show that $3-a_1-a_n-a_1a_n \ge 0$. Since

$$3-a_1-a_n-a_1a_n \ge 3-a_1-a_n-\frac{1}{4}(a_1+a_n)^2 = \frac{1}{4}(2-a_1-a_n)(6+a_1+a_n) \ge 0,$$

we only need to show that $2-a_1-a_n \ge 0$. Indeed, we have

$$2-a_1-a_n = (a_2-1)+\cdots+(a_{n-1}-1) \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.246. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1}{3a_1^3+4} + \frac{1}{3a_2^3+4} + \cdots + \frac{1}{3a_n^3+4} \ge \frac{n}{7}.$$

(Vasile C., 2009)

Solution. Write the inequality as follows:

$$\left(\frac{1}{3a_2^3+4}-\frac{1}{7}\right)+\cdots+\left(\frac{1}{3a_n^3+4}-\frac{1}{7}\right)\ge\frac{1}{7}-\frac{1}{3a_1^3+4},$$

$$\frac{1-a_2^3}{3a_2^3+4}+\cdots+\frac{1-a_n^3}{3a_n^3+4}\ge\frac{a_1^3-1}{3a_1^3+4},$$

$$\frac{1-a_2^3}{3a_2^3+4}+\cdots+\frac{1-a_n^3}{3a_n^3+4}\ge\frac{[(1-a_2)+\cdots+(1-a_n)](1+a_1+a_1^2)}{3a_1^3+4},$$

$$(1-a_2)\left(\frac{1+a_2+a_2^2}{3a_2^3+4}-\frac{1+a_1+a_1^2}{3a_1^3+4}\right)+\cdots+(1-a_n)\left(\frac{1+a_n+a_n^2}{3a_n^3+4}-\frac{1+a_1+a_1^2}{3a_1^3+4}\right)\ge 0.$$

It suffices to show that

$$\frac{1+a_i+a_i^2}{3a_i^3+4}-\frac{1+a_1+a_1^2}{3a_1^3+4}\ge 0$$

for $i=2,\ldots,n$. Write these inequalities as

$$(a_1-a_i)E_i\ge 0,$$

where

$$\begin{aligned}E_i&=3a_1^2a_i^2+3a_1a_i(a_1+a_i)+3(a_1^2+a_1a_i+a_i^2)-4(a_1+a_i)-4\\&=(a_1+a_i)(3a_1+3a_i-4+3a_1a_i)+3a_1^2a_i^2-3a_1a_i-4.\end{aligned}$$

Since

$$a_1+a_i\ge a_1+a_n=2+(1-a_2)+\cdots+(1-a_{n-1})\ge 2,$$

we have

$$E_i\ge 2(6-4+3a_1a_i)+3a_1^2a_i^2-3a_1a_i-4=3a_1a_i+3a_1^2a_i^2\ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$, and also for $a_1=2$, $a_2=\cdots=a_{n-1}=1$, $a_n=0$.

□

P 1.247. *If $a_1,a_2,\ldots,a_n$ are nonnegative real numbers such that*

$$a_1\le 1\le a_2\le\cdots\le a_n,\qquad a_1+a_2+\cdots+a_n=n,$$

then

$$\sqrt{\frac{3a_1}{4-a_1}}+\sqrt{\frac{3a_2}{4-a_2}}+\cdots+\sqrt{\frac{3a_n}{4-a_n}}\le n.$$

(Vasile C., 2009)

Solution. Write the inequality as follows:

$$\left(\sqrt{\frac{3a_1}{4-a_1}}-1\right)+\left(\sqrt{\frac{3a_2}{4-a_2}}-1\right)+\cdots+\left(\sqrt{\frac{3a_n}{4-a_n}}-1\right)\le 0.$$

$$\frac{a_1-1}{4-a_1+\sqrt{3a_1(4-a_1)}}+\frac{a_2-1}{4-a_2+\sqrt{3a_2(4-a_2)}}+\cdots+\frac{a_n-1}{4-a_n+\sqrt{3a_n(4-a_n)}}\le 0,$$

$$\frac{a_2-1}{4-a_2+\sqrt{3a_2(4-a_2)}}+\cdots+\frac{a_n-1}{4-a_n+\sqrt{3a_n(4-a_n)}}\le\frac{(a_2-1)+\cdots+(a_n-1)}{4-a_1+\sqrt{3a_1(4-a_1)}},$$

$$(a_2-1)E_2+\cdots+(a_n-1)E_n\ge 0,$$

where

$$E_j=\frac{1}{4-a_1+\sqrt{3a_1(4-a_1)}}-\frac{1}{4-a_j+\sqrt{3a_j(4-a_j)}},\qquad j=2,\ldots,n.$$

It suffices to show that all $E_j\ge 0$. The inequality $E_j\ge 0$ is equivalent to

$$\sqrt{3a_j(4-a_j)}-\sqrt{3a_1(4-a_1)}\ge a_j-a_1,$$

$$\frac{3(a_j-a_1)(4-a_1-a_j)}{\sqrt{3a_j(4-a_j)}+\sqrt{3a_1(4-a_1)}}\ge a_j-a_1.$$

This is true if

$$\sqrt{3a_1(4-a_1)}+\sqrt{3a_j(4-a_j)}\le 3(4-a_1-a_j).$$

We have

$$a_1+a_j-2\le a_1+a_n-2=(1-a_2)+\cdots+(1-a_{n-1})\le 0.$$

Denote

$$x=a_1+a_j,\qquad x\le 2.$$

Since

$$\sqrt{3a_1(4-a_1)}+\sqrt{3a_j(4-a_j)}\le\sqrt{2[3a_1(4-a_1)+3a_j(4-a_j)]}\le\sqrt{24x-3x^2},$$

it suffices to show that

$$\sqrt{24x-3x^2}\le 3(4-x),$$

which is equivalent to the obvious inequality

$$(2-x)(6-x)\ge 0.$$

The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 1.248. *If* $a_1, a_2, \ldots, a_n$ *are nonnegative real numbers such that*

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = n,$$

then

$$\frac{1}{3-a_1} + \frac{1}{3-a_2} + \cdots + \frac{1}{3-a_n} \le \frac{n}{2}.$$

(Vasile C., 2009)

Solution. Write the inequality as follows:

$$\left(\frac{2}{3-a_1} - 1\right) + \left(\frac{2}{3-a_1} - 1\right) + \cdots + \left(\frac{2}{3-a_1} - 1\right) \le 0.$$

$$\frac{a_1-1}{3-a_1} + \frac{a_2-1}{3-a_2} + \cdots + \frac{a_n-1}{3-a_n} \le 0,$$

$$\frac{a_2-1}{3-a_2} + \cdots + \frac{a_n-1}{3-a_n} \le \frac{1-a_1}{3-a_1},$$

$$\frac{a_2^2-1}{(1+a_2)(3-a_2)} + \cdots + \frac{a_n^2-1}{(1+a_n)(3-a_n)} \le \frac{1-a_1^2}{(1+a_1)(3-a_1)},$$

$$\frac{a_2^2-1}{(1+a_2)(3-a_2)} + \cdots + \frac{a_n^2-1}{(1+a_n)(3-a_n)} \le \frac{(a_2^2-1) + \cdots + (a_n^2-1)}{(1+a_1)(3-a_1)},$$

$$(a_2^2-1)E_2 + \cdots + (a_n^2-1)E_n \le 0,$$

where

$$E_j = \frac{1}{(1+a_j)(3-a_j)} - \frac{1}{(1+a_1)(3-a_1)}, \qquad j = 2, \ldots, n.$$

It suffices to show that $E_j \le 0$, which is equivalent to

$$(a_j - a_1)(a_1 + a_j - 2) \le 0.$$

This is true because

$$a_1 + a_i - 2 \le a_1 + a_n - 2 = (1-a_2) + \cdots + (1-a_{n-1}) \le 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.249. *If* $a_1, a_2, \ldots, a_n$ *are real numbers such that*

$$a_1 \le 1 \le a_2 \le \cdots \le a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$(1+a_1^2)(1+a_2^2)\cdots(1+a_n^2) \ge 2^n.$$

(Vasile C., 2009)

Solution. Use the substitutions $a_1 = 1 - S$ and

$$a_2 = b_2 + 1,\ \ldots,\ a_n = b_n + 1,$$

where S and $b_2, \ldots, b_n$ are nonnegative real numbers such that

$$S = b_2 + \cdots + b_n.$$

We have

$$\frac{1}{2}(1 + a_1^2) = 1 - S + \frac{1}{2}S^2,$$

$$\frac{1}{2}(1 + a_i^2) = 1 + b_i + \frac{1}{2}b_i^2, \qquad i = 2, \ldots, n,$$

and, by Lemma below,

$$\frac{1}{2^{n-1}}(1 + a_2^2)\cdots(1 + a_n^2) = \left(1 + b_2 + \frac{1}{2}b_2^2\right)\cdots\left(1 + b_n + \frac{1}{2}b_n^2\right) \ge 1 + S + \frac{1}{2}S^2.$$

Therefore, it suffices to show that

$$\left(1 - S + \frac{1}{2}S^2\right)\left(1 + S + \frac{1}{2}S^2\right) \ge 1,$$

which is equivalent to $S^4 \ge 0$. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Lemma. If $c_1, c_2, \ldots, c_k$ are nonnegative real numbers such that $c_1 + c_2 + \cdots + c_k = S$, then

$$\left(1 + c_1 + \frac{1}{2}c_1^2\right)\left(1 + c_2 + \frac{1}{2}c_2^2\right)\cdots\left(1 + c_k + \frac{1}{2}c_k^2\right) \ge 1 + S + \frac{1}{2}S^2.$$

Proof. We have

$$\begin{aligned}\prod_{1\le i\le k}\left(1 + c_i + \frac{1}{2}c_i^2\right) &\ge 1 + \sum_{1\le i\le k}\left(c_i + \frac{1}{2}c_i^2\right) + \sum_{1\le i<j\le k}\left(c_i + \frac{1}{2}c_i^2\right)\left(c_j + \frac{1}{2}c_j^2\right)\\ &\ge 1 + \sum_{1\le i\le k}\left(c_i + \frac{1}{2}c_i^2\right) + \sum_{1\le i<j\le k} c_i c_j\\ &= 1 + S + \frac{1}{2}S^2.\end{aligned}$$

□

P 1.250. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \cdots + \frac{1}{a_n + 1} \ge \frac{n}{2}.$$

(Vasile C., 2009)

Solution. We use the induction method. For $n = 2$, the desired inequality is an identity. Let us denote

$$E_n(a_1, a_2, \ldots, a_n) = \frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} + \cdots + \frac{1}{a_n + 1} - \frac{n}{2}.$$

We will show that

$$E_n(a_1, a_2, a_3, \ldots, a_n) \ge E_n(a_1 a_2, 1, a_3, \ldots, a_{n-1}, a_n) \ge 0$$

for $n \ge 3$.

The right inequality can be written as

$$E_{n-1}(a_1 a_2, a_3, \ldots, a_{n-1}, a_n) \ge 0.$$

Since

$$a_1 a_2 = \frac{1}{a_3 \cdots a_{n-1} a_n} \ge 1$$

and

$$(a_1 a_2) a_3 \cdots a_{n-1} a_n = 1,$$

the right inequality follows by the induction hypothesis.

The left inequality is equivalent to

$$\frac{1}{a_1 + 1} + \frac{1}{a_2 + 1} \ge \frac{1}{a_1 a_2 + 1} + \frac{1}{2},$$

$$\frac{1 - a_2}{2(a_2 + 1)} \ge \frac{a_1(1 - a_2)}{(a_1 + 1)(a_1 a_2 + 1)},$$

which is true if

$$(a_1 + 1)(a_1 a_2 + 1) \ge 2a_1(a_2 + 1).$$

This inequality can be written in the obvious form

$$(a_1 - 1)(a_1 a_2 - 1) \ge 0.$$

The equality holds for $a_1 \ge 1 = a_2 = \cdots = a_{n-1} \ge a_n$.

□

P 1.251. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1 a_2 \cdots a_n = 1,$$

then

$$\frac{1}{(a_1 + 2)^2} + \frac{1}{(a_2 + 2)^2} + \cdots + \frac{1}{(a_n + 2)^2} \ge \frac{n}{9}.$$

(Vasile C., 2009)

Solution. We use the induction method. For $n = 2$, the desired inequality is equivalent to

$$(a_1 - 1)^4 \geq 0.$$

Let us denote

$$E_n(a_1, a_2, \ldots, a_n) = \frac{1}{(a_1+2)^2} + \frac{1}{(a_2+2)^2} + \cdots + \frac{1}{(a_n+2)^2} - \frac{n}{9}.$$

To end the proof, it suffices to show that

$$E_n(a_1, a_2, a_3, \ldots, a_{n-1}, a_n) \geq E_n(a_1, 1, a_3, \ldots, a_{n-1}, a_2a_n) \geq 0$$

for $n \geq 3$.

The right inequality can be written as

$$E_{n-1}(a_1, a_3, \ldots, a_{n-1}, a_2a_n) \geq 0.$$

Since

$$a_2a_n \leq a_n \leq a_{n-1}$$

and

$$a_1a_3 \ldots a_{n-1}(a_2a_n) = 1,$$

the inequality follows by the induction hypothesis.

The left inequality is equivalent to

$$\frac{1}{(a_2+2)^2} + \frac{1}{(a_n+2)^2} \geq \frac{1}{9} + \frac{1}{(a_2a_n+2)^2}.$$

Denoting

$$s = a_2 + a_n, \quad p = a_2a_n, \qquad s \leq 2, \quad p \leq 1,$$

the inequality becomes

$$\frac{s^2 + 4s + 8 - 2p}{(2s + 4 + p)^2} \geq \frac{p^2 + 4p + 13}{9(p+2)^2},$$

$$(1 + p - s)(As + B) \geq 0,$$

where

$$A = 16 - 20p - 5p^2, \qquad B = 80 - 32p - 29p^2 - p^3 > 0.$$

Since

$$1 + p - s = (1 - a_2)(1 - a_n) \geq 0,$$

we only need to show that $As + B \geq 0$. For the nontrivial case $A < 0$, we get

$$As + B \geq 2A + B = 112 - 72p - 39p^2 - p^3 = (1-p)(112 + 40p + p^2) \geq 0.$$

This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

Remark. Similarly, we can prove the following generalization:

- *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n, \qquad a_1 a_2 \cdots a_n = 1.$$

If $k \geq 1$, then

$$\frac{1}{(a_1+k)^k} + \frac{1}{(a_2+k)^k} + \cdots + \frac{1}{(a_n+k)^k} \geq \frac{n}{(1+k)^k}.$$

For $n = 2$, the desired inequality is true if $g(x) \geq 0$ for $x \geq 1$, where

$$g(x) = \frac{1}{(x+k)^k} + \frac{x^k}{(kx+1)^k} - \frac{2}{(1+k)^k},$$

$$\frac{g'(x)}{k} = \frac{x^{k-1}(x+k)^{k+1} - (kx+1)^{k+1}}{(x+k)^{k+1}(kx+1)^{k+1}}.$$

It suffices to show that $h(x) \geq 0$ for $x \geq 1$, where

$$h(x) = (k-1)\ln x + (k+1)\ln(x+k) - (k+1)\ln(kx+1),$$

$$h'(x) = \frac{k-1}{x} + \frac{k+1}{x+k} - \frac{k(k+1)}{kx+1} = \frac{k(k-1)(x-1)^2}{(x+k)(kx+1)}.$$

Since $h'(x) \geq 0$, $h(x)$ is increasing for $x \geq 1$, hence

$$h(x) \geq h(1) = 0.$$

Let

$$E_n(a_1, a_2, \ldots, a_n) = \frac{1}{(a_1+k)^k} + \frac{1}{(a_2+k)^k} + \cdots + \frac{1}{(a_n+k)^k} - \frac{n}{(1+k)^k}.$$

It suffices to show that

$$E_n(a_1, a_2, a_3, \ldots, a_{n-1}, a_n) \geq E_n(a_1, 1, a_3, \ldots, a_{n-1}, a_2 a_n) \geq 0.$$

The right inequality follows by the induction hypothesis, while the left inequality is equivalent to

$$f_1(a_2) + f_1(a_n) \geq f_1(1) + f_2(a_2 a_n),$$

where

$$f_1(x) = \frac{1}{(x+k)^k}.$$

Using the substitution

$$a_2 = e^a, \quad a_n = e^b,$$

the inequality becomes

$$f(a)+f(b) \ge f(0)+f(a+b),$$

where

$$f(x)=\frac{1}{(e^x+k)^k}.$$

From

$$f''(x)=\frac{k^2e^x(e^x-1)}{(e^x+k)^{k+2}},$$

it follows that f is concave on $(-\infty,0]$. Since

$$0 \ge a \ge b \ge a+b,$$

the inequality $f(a)+f(b) \ge f(0)+f(a+b)$ follows from Karamata's inequality.

□

P 1.252. *If $a_1,a_2,\ldots,a_n$ are positive real numbers such that*

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1a_2\cdots a_n=1,$$

then

$$a_1^n+a_2^n+\cdots+a_n^n-n \ge n^2\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n\right).$$

(Vasile C., 2009)

Solution. We use the induction method. For $n=2$, the desired inequality is equivalent to

$$(a_1-1)^4 \ge 0.$$

Let us denote

$$E_n(a_1,a_2,\ldots,a_n)=a_1^n+a_2^n+\cdots+a_n^n-n-n^2\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n\right).$$

We will show that

$$E_n(a_1,a_2,a_3,\ldots,a_{n-1},a_n) \ge E_n(a_1,1,a_3,\ldots,a_{n-1},a_2a_n) \ge 0.$$

The right inequality can be written as

$$E_{n-1}(a_1,a_3,\ldots,a_{n-1},a_2a_n) \ge 0.$$

Since

$$a_2a_n \le a_n \le a_{n-1}$$

and

$$a_1a_3\cdots a_{n-1}(a_2a_n)=1,$$

the inequality follows by the induction hypothesis.

The left inequality is equivalent to

$$a_2^n+a_n^n-1-a_2^na_n^n\ge n^2\left(\frac{1}{a_2}+\frac{1}{a_n}-1-\frac{1}{a_2a_n}\right),$$

$$n^2\left(\frac{1}{a_2}-1\right)\left(\frac{1}{a_n}-1\right)\ge(1-a_2^n)(1-a_n^n),$$

which is true if

$$\frac{n^2}{a_2a_n}\ge(1+a_2+\cdots+a_2^{n-1})(1+a_n+\cdots+a_n^{n-1}).$$

Since $a_2\le 1$ and $a_n\le 1$, this inequality is clearly true. The equality holds for $a_1=a_2=\cdots=a_n=1$.

□

P 1.253. *If* $a_1,a_2,\ldots,a_n$ $(n\ge 3)$ *are real numbers such that*

$$a_1+a_2+\cdots+a_n=n,\qquad a_1\ge a_2\ge 1\ge a_3\ge\cdots\ge a_n,$$

then

$$a_1^4+a_2^4+\cdots+a_n^4-n\ge\frac{14}{3}(a_1^2+a_2^2+\cdots+a_n^2-n).$$

(Vasile C., 2009)

Solution *(by Linqaszayi)*. Using the substitution

$$a_i=1+x_i,\ \ i=1,2,\ldots,n,$$

which implies

$$x_1\ge x_2\ge 0\ge x_3\ge\cdots\ge x_n,\qquad x_1+x_2+\cdots+x_n=0,$$

we need to show that $E(x_1,x_2,\ldots,x_n)\ge 0$, where

$$E(x_1,x_2,x_3,\ldots,x_n)=3\sum_{i=1}^n {x_i}^4+12\sum_{i=1}^n {x_i}^3+4\sum_{i=1}^n {x_i}^2.$$

We will prove that

$$E(x_1,x_2,\ldots,x_n)\ge E\left(\frac{x_1+x_2}{2},\frac{x_1+x_2}{2},x_3,\ldots,x_n\right)\ge 0.$$

The left inequality is true because

$$x_1^4 + x_2^4 \geq 2\left(\frac{x_1 + x_2}{2}\right)^4, \quad x_1^3 + x_2^3 \geq 2\left(\frac{x_1 + x_2}{2}\right)^3, \quad x_1^2 + x_2^2 \geq 2\left(\frac{x_1 + x_2}{2}\right)^2.$$

To prove the right inequality, we replace $x_3, \ldots, x_n$ with $-x_3, \ldots, -x_n$. So, we need to show that

$$x_1 + x_2 = x_3 + \cdots + x_n, \qquad x_1, x_2, x_3, \ldots, x_n \geq 0,$$

involves

$$E + 3(x_3^4 + \cdots + x_n^4) - 12(x_3^3 + \cdots + x_n^3) + 4(x_3^2 + \cdots + x_n^2) \geq 0,$$

where

$$E = 6\left(\frac{x_1 + x_2}{2}\right)^4 + 24\left(\frac{x_1 + x_2}{2}\right)^3 + 8\left(\frac{x_1 + x_2}{2}\right)^2 = 6A^4 + 24A^3 + 8A^2,$$

with

$$A = \frac{x_3 + \cdots + x_n}{2}.$$

Since

$$A^4 \geq \frac{x_3^4 + \cdots + x_n^4}{16}, \quad A^3 \geq \frac{x_3^3 + \cdots + x_n^3}{8}, \quad A^2 \geq \frac{x_3^2 + \cdots + x_n^2}{4},$$

we have

$$E \geq \frac{3}{8}(x_3^4 + \cdots + x_n^4) + 3(x_3^3 + \cdots + x_n^3) + 2(x_3^2 + \cdots + x_n^2).$$

Therefore, it suffices to show that

$$\left(\frac{3}{8} + 3\right)(x_3^4 + \cdots + x_n^4) + (3 - 12)(x_3^3 + \cdots + x_n^3) + (2 + 4)(x_3^2 + \cdots + x_n^2) \geq 0,$$

which is equivalent to the obvious inequality

$$x_3^2(3x_3 - 4)^2 + \cdots + x_n^2(3x_n^2 - 4)^2 \geq 0.$$

This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for

$$a_1 = a_2 = \frac{5}{3}, \quad a_3 = \cdots = a_{n-1} = 1, \quad a_n = \frac{-1}{3}.$$

□

P 1.254. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that*

$$a_1 \geq 1 \geq a_2 \geq \cdots \geq a_n, \quad a_1 a_2 \cdots a_n = 1.$$

Prove that

$$\frac{1 - a_1}{3 + a_1^2} + \frac{1 - a_2}{3 + a_2^2} + \cdots + \frac{1 - a_n}{3 + a_n^2} \geq 0.$$

(Vasile C., 2013)

Solution. We use the induction method. For $n = 2$, the desired inequality is equivalent to

$$(a_1 - 1)^4 \ge 0.$$

Let us denote

$$E_n(a_1, a_2, \ldots, a_n) = \frac{1 - a_1}{3 + a_1^2} + \frac{1 - a_2}{3 + a_2^2} + \cdots + \frac{1 - a_n}{3 + a_n^2}.$$

We will show that

$$E_n(a_1, \ldots, a_{n-2}, a_{n-1}, a_n) \ge E_n(a_1, \ldots, a_{n-2}, 1, a_{n-1}a_n) \ge 0.$$

The right inequality can be written as

$$E_{n-1}(a_1, a_2, \ldots, a_{n-2}, a_{n-1}a_n) \ge 0.$$

Since

$$a_1 \ge 1 \ge a_2 \ge \cdots \ge a_{n-2} \ge a_{n-1}a_n,$$

and

$$a_1 a_2 \cdots a_{n-2}(a_{n-1}a_n) = 1,$$

the inequality follows by the induction hypothesis.

The left inequality reduces to

$$\frac{1 - a_{n-1}}{3 + a_{n-1}^2} + \frac{1 - a_n}{3 + a_n^2} \ge \frac{1 - a_{n-1}a_n}{3 + a_{n-1}^2 a_n^2},$$

which is equivalent to the obvious inequality

$$(1 - a_{n-1})(1 - a_n)(3 + a_{n-1}a_n)(3 - a_{n-1}a_n - a_{n-1}^2 a_n - a_{n-1}a_n^2) \ge 0.$$

Thus, the proof is completed. The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.255. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ be nonnegative real numbers such that*

$$a_1 \ge \cdots \ge a_k \ge 1 \ge a_{k+1} \ge \cdots \ge a_n, \qquad 1 \le k \le n - 1,$$

and

$$a_1 + a_2 + \cdots + a_n = p.$$

Prove that

(a) if $p \ge k$, then

$$a_1^2 + a_2^2 + \cdots + a_n^2 \le (p - k + 1)^2 + k - 1;$$

(b) if $k \le p \le n$ *, then*

$$a_1^2 + a_2^2 + \cdots + a_n^2 \ge \frac{p^2 - 2kp + kn}{n-k};$$

(c) if $p \ge n$ *, then*

$$a_1^2 + a_2^2 + \cdots + a_n^2 \ge \frac{p^2 - 2(n-k)p + n(n-k)}{k}.$$

(Vasile C., 2015)

First Solution. (a) For $k = 1$, the inequality is equivalent to $a_1^2 + a_2^2 + \cdots + a_n^2 \le p^2$, which is clearly true. For $k \ge 2$, write the inequality as

$$[(p-k+1)^2 - a_1^2] + (1 - a_2^2) + \cdots + (1 - a_k^2) - a_{k+1}^2 - \cdots - a_n^2 \ge 0,$$

$$(p-k+1-a_1)(n-k+1+a_1) \ge (a_2-1)(a_2+1) + \cdots + (a_k-1)(a_k+1) + a_{k+1}^2 + \cdots + a_n^2.$$

Since

$$p - k + 1 - a_1 = (a_2 - 1) + \cdots + (a_k - 1) + a_{k+1} + \cdots + a_n \ge 0$$

and

$$(p - k + 1 + a_1) - (a_2 + 1) = (p - k + 1 - a_1) + (a_1 - a_2) + (a_1 - 1) \ge 0,$$

we have

$$(p - k + 1 - a_1)(p - k + 1 + a_1) \ge (p - k + 1 - a_1)(a_2 + 1).$$

In addition, we have

$$\begin{aligned}(a_2 - 1)(a_2 + 1) + \cdots + (a_k - 1)(a_k + 1) &\le (a_2 - 1)(a_2 + 1) + \cdots + (a_k - 1)(a_2 + 1)\\ &= (a_2 + \cdots + a_k - k + 1)(a_2 + 1).\end{aligned}$$

Thus, it suffices to show that

$$(p - k + 1 - a_1)(a_2 + 1) \ge (a_2 + \cdots + a_k - k + 1)(a_2 + 1) + a_{k+1}^2 + \cdots + a_n^2,$$

which is equivalent to

$$(a_{k+1} + \cdots + a_n)(a_2 + 1) \ge a_{k+1}^2 + \cdots + a_n^2.$$

Indeed, we have

$$(a_{k+1} + \cdots + a_n)(a_2 + 1) \ge a_{k+1} + \cdots + a_n \ge a_{k+1}^2 + \cdots + a_n^2.$$

The equality holds for

$$a_1 = p - k + 1, \qquad a_2 = \cdots = a_k = 1, \qquad a_{k+1} = \cdots = a_n = 0.$$

(b) Let

$$A = a_1 + \cdots + a_k, \quad B = a_{k+1} + \cdots + a_n, \qquad A \ge k, \quad A + B = p \le n.$$

We have

$$A^2 \le k(a_1^2 + \cdots + a_k^2), \quad B^2 \le (n-k)(a_{k+1}^2 + \cdots + a_n^2),$$

hence

$$\frac{A^2}{k} + \frac{B^2}{n-k} \le a_1^2 + a_2^2 + \cdots + a_n^2.$$

Thus, it suffices to show that

$$\frac{n-k}{k}A^2 + B^2 \ge p^2 - 2kp + kn,$$

which is equivalent to

$$\frac{n-k}{k}A^2 + B^2 \ge (A+B)^2 - 2k(A+B) + kn,$$

$$\frac{n-2k}{k}A^2 + 2kA - kn \ge 2kB(A-k),$$

$$(A-k)\left(\frac{n-2k}{k}A + n\right) \ge 2kB(A-k),$$

$$(A-k)\left(\frac{n-2k}{k}A + n - 2kB\right) \ge 0,$$

$$(A-k)\left[\frac{n}{k}(A-k) + 2(n-A-B)\right] \ge 0.$$

The equality holds for

$$a_1 = \cdots = a_k = 1, \qquad a_{k+1} = \cdots = a_n = \frac{p-k}{n-k}.$$

(c) Let

$$A = a_1 + \cdots + a_k, \quad B = a_{k+1} + \cdots + a_n, \qquad B \le n-k, \quad A + B = p \ge n.$$

We have

$$A^2 \le k(a_1^2 + \cdots + a_k^2), \quad B^2 \le (n-k)(a_{k+1}^2 + \cdots + a_n^2),$$

hence

$$\frac{A^2}{k} + \frac{B^2}{n-k} \le a_1^2 + a_2^2 + \cdots + a_n^2.$$

Thus, it suffices to show that

$$A^2 + \frac{k}{n-k}B^2 \ge p^2 - 2(n-k)p + (n-k)n,$$

which is equivalent to

$$A^2+\frac{k}{n-k}B^2\geq(A+B)^2-2(n-k)(A+B)+(n-k)n,$$

$$2A(n-k-B)+\frac{2k-n}{n-k}B^2+2(n-k)B-(n-k)n\geq 0,$$

$$2A(n-k-B)-(n-k-B)\left(n+\frac{2k-n}{n-k}B\right)\geq 0,$$

$$(n-k-B)\left(2A-n-\frac{2k-n}{n-k}B\right)\geq 0,$$

$$(n-k-B)\left[2(A+B-n)+\frac{n(n-k-B)}{n-k}\right]\geq 0.$$

The equality holds for

$$a_1=\cdots=a_k=\frac{p-n+k}{k},\qquad a_{k+1}=\cdots=a_n=1.$$

Second Solution. The desired inequalities can be proved by applying Karamata's inequality to the convex function $f(u)=u^2$. In the case (a), the decreasingly ordered sequence $(p-k+1,1,\ldots,1,0,\ldots,0)$ majorizes the decreasingly ordered sequence $(a_1,a_2,\ldots,a_n)$; that is

$$(p-k+1,1,\ldots,1,0,\ldots,0)\succ(a_1,a_2,\ldots,a_k,a_{k+1},\ldots,a_n).$$

Also, in the cases (b) and (c), we have

$$(a_1,a_2,\ldots,a_k,a_{k+1},\ldots,a_n)\succ\left(1,1,\ldots,1,\frac{p-k}{n-k},\ldots,\frac{p-k}{n-k}\right)$$

and

$$(a_1,a_2,\ldots,a_k,a_{k+1},\ldots,a_n)\succ\left(\frac{p-n+k}{k},\frac{p-n+k}{k},\ldots,\frac{p-n+k}{k},1,\ldots,1\right),$$

respectively

□

P 1.256. *Let $a_1,a_2,\ldots,a_n$ $(n\geq 3)$ be nonnegative real numbers such that*

$$a_1\geq\cdots\geq a_k\geq 1\geq a_{k+1}\geq\cdots\geq a_n,\qquad 1\leq k\leq n-1,$$

and

$$a_1+a_2+\cdots+a_n=n,\qquad a_1^2+a_2^2+\cdots+a_n^2=q,$$

where q is a fixed number. Prove that the product $r=a_1a_2\cdots a_n$ is maximum when

$$a_2=\cdots=a_k=1,\qquad a_{k+1}=\cdots=a_n.$$

(Vasile C., 2015)

Solution. We show first that there exists a unique n-tuple $(a_1, a_2, \ldots, a_n)$ such that

$$a_1 \ge a_2 = \cdots = a_k = 1 \ge a_{k+1} = \cdots = a_n.$$

By the Cauchy-Schwarz inequality

$$n(a_1^2 + a_2^2 + \cdots + a_n^2) \ge (a_1 + a_2 + \cdots + a_n)^2,$$

we get $q \ge n$. Since $q = n$ involves $a_1 = a_2 = \cdots = a_n = 1$, consider further that $q > n$. For

$$a_1 := x, \quad a_2 = \cdots = a_k = 1, \quad a_{k+1} = \cdots = a_n := y,$$

we get

$$x = 1 + \sqrt{\frac{(n-k)(q-n)}{n-k+1}}, \quad y = 1 - \sqrt{\frac{q-n}{(n-k)(n-k+1)}}.$$

Since $x \ge 1$ and $y \le 1$, we only need to show that $y \ge 0$. This is equivalent to

$$q \le (n-k+1)^2 + k - 1,$$

which is the inequality (a) in P 1.255.

Consider that r is maximum at $(b_1, b_2, \ldots, b_n)$, where

$$b_1 \ge \cdots \ge b_k \ge 1 \ge b_{k+1} \ge \cdots \ge b_n.$$

We will show now, by the contradiction method, that

$$b_2 = \cdots = b_k = 1, \quad b_{k+1} = \cdots = b_n.$$

To show that $b_{k+1} = \cdots = b_n$ for $1 \le k \le n-2$, assume that

$$b_{k+1} \ne b_n.$$

For

$$a_2 = b_2, \ \ldots, \ a_k = b_k, \qquad a_{k+2} = b_{k+2}, \ \ldots, \ a_{n-1} = b_{n-1},$$

we have $a_1 + a_{k+1} + a_n = constant$ and $a_1^2 + a_{k+1}^2 + a_n^2 = constant$, where

$$a_1 \ge 1 \ge a_{k+1} \ge a_n.$$

According to P 1.168, the product $a_1 a_{k+1} a_n$ is maximum for $a_{k+1} = a_n$, which contradicts the assumption that $b_{k+1} \ne b_n$. From this contradiction, it follows that $b_{k+1} = \cdots = b_n$.

To show that $b_2 = \cdots = b_k = 1$ for $2 \le k \le n-1$, assume that

$$b_2 \ne 1.$$

For

$$a_3 = b_3, \ \ldots, \ a_{n-1} = b_{n-1},$$

we have $a_1 + a_2 + a_n = constant$ and $a_1^2 + a_2^2 + a_n^2 = constant$, where

$$a_1 \geq a_2 \geq 1 \geq a_n.$$

According to P 1.171, the product $a_1 a_2 a_n$ is maximum for $a_2 = 1$ or $a_n = 1$. The first case contradicts the assumption that $b_2 \neq 1$, while the second case involves $b_n = 1$, hence $b_1 = b_2 = \cdots = b_n = 1$ (because $b_1 \geq b_2 \geq \cdots \geq b_n$ and $b_1 + b_2 + \cdots + b_n = n$), which also contradicts the assumption that $b_2 \neq 1$; as a consequence, we have $b_2 = 1$, which involves $b_2 = \cdots = b_k = 1$.

□

P 1.257. *If $a_1, a_2, \ldots, a_n$ are nonnegtive real numbers such that*

$$a_1 \leq 1 \leq a_2 \leq \cdots \leq a_n, \qquad a_1 + a_2 + \cdots + a_n = n,$$

then

$$(a_1 a_2 \cdots a_n)^{\frac{2}{n}}(a_1^2 + a_2^2 + \cdots + a_n^2) \leq n.$$

(Vasile C., 2015)

Solution. For $n = 2$, we need to show that $a_1 + a_2 = 2$ implies

$$a_1 a_2(a_1^2 + a_2^2) \leq 2.$$

Indeed, we have

$$16 - 8a_1 a_2(a_1^2 + a_2^2) = (a_1 + a_2)^4 - 8a_1 a_2(a_1^2 + a_2^2) = (a_1 - a_2)^4 \geq 0.$$

For $n \geq 3$, according to the preceding P 1.256, it suffices to consider the case $a_2 = \cdots = a_{n-1} = 1$. Thus, we only need to show that $a_1 + a_n = 2$ involves

$$(a_1 a_n)^{\frac{2}{n}}(a_1^2 + a_n^2 + n - 2) \leq n.$$

This is true if $f(x) \leq \ln n$ for $x \in (0, 2)$, where

$$f(x) = \frac{2}{n}[\ln x + \ln(2 - x)] + \ln(2x^2 - 4x + n + 2).$$

From the derivative

$$f'(x) = \frac{2}{n}\left(\frac{1}{x} - \frac{1}{2 - x}\right) + \frac{4(x - 1)}{2x^2 - 4x + n + 2} = \frac{4(n + 2)(1 - x)^3}{nx(2 - x)(2x2 - 4x + n + 2)},$$

it follows that $f(x)$ is increasing on $(0, 1]$ and decreasing on $[1, 2)$; therefore,

$$f(x) \leq f(1) = \ln n.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 1.258. *Let $a_1, a_2, \ldots, a_n$ ($n \ge 3$) be nonnegative real numbers such that*

$$a_1 \ge \cdots \ge a_k \ge 1 \ge a_{k+1} \ge \cdots \ge a_n, \qquad 1 \le k \le n-1,$$

and

$$a_1 + a_2 + \cdots + a_n = p, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = q,$$

where p and q are fixed numbers.

(a) For $p \le n$, the product $r = a_1a_2\cdots a_n$ is maximum when $a_2 = \cdots = a_k = 1$ and $a_{k+1} = \cdots = a_n$;

(b) For $p \ge n$ and $q \ge n-1+(p-n+1)^2$, the product $r = a_1a_2\cdots a_n$ is maximum when $a_2 = \cdots = a_k = 1$ and $a_{k+1} = \cdots = a_n$;

(c) For $p \ge n$ and $q < n-1+(p-n+1)^2$, the product $r = a_1a_2\cdots a_n$ is maximum when $a_2 = \cdots = a_k$ and $a_{k+1} = \cdots = a_n = 1$.

(Vasile Cîrtoaje and Linqaszayi, 2015)

Solution. (a) For $p = k$, we have

$$a_1 = \cdots = a_k = 1, \qquad a_{k+1} = \cdots = a_n = 0.$$

Consider further that $p > k$. We show first that there exists a unique n-tuple $(a_1, a_2, \ldots, a_n)$ such that

$$a_1 \ge a_2 = \cdots = a_k = 1 \ge a_{k+1} = \cdots = a_n.$$

According to P 1.255, we have

$$\frac{p^2 - 2pk + kn}{n-k} \le q \le (p-k+1)^2 + k - 1.$$

For $a_1 := x$, $a_2 = \cdots = a_k = 1$ and $a_{k+1} = \cdots = a_n := y$, from

$$a_1 + a_2 + \cdots + a_n = p, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = q,$$

we get

$$x + (n-k)y = p - k + 1, \quad x^2 + (n-k)y^2 = q - k + 1.$$

We need to show that this system has a unique solution (x, y) such that $x \ge 1 \ge y \ge 0$. From the system equations, we get $f(x) = 0$, where

$$f(x) = (n-k+1)x^2 - 2(p-k+1)x + (p-k+1)^2 - (n-k)(q-k+1).$$

We have

$$f(1) = (n-k)\left(\frac{p^2 - 2pk + kn}{n-k} - q\right) \le 0$$

and

$$f(p-k+1) = (n-k)[(p-k+1)^2 + k - 1 - q] \ge 0.$$

Therefore, the equation $f(x)=0$ has a single root $x\in[1,p-k+1]$. From $1\le x\le p-k+1$, we get

$$1\le p-k+1-(n-k)y\le p-k+1,$$

hence

$$1\ge\frac{p-k}{n-k}\ge y\ge 0.$$

Consider now that r is maximum at $(b_1,b_2,\ldots,b_n)$, where

$$b_1\ge\cdots\ge b_k\ge 1\ge b_{k+1}\ge\cdots\ge b_n.$$

Applying the contradiction method as in P 1.256, we get

$$b_2=\cdots=b_k=1,\quad b_{k+1}=\cdots=b_n.$$

(b) We show first that there exists a unique n-tuple $(a_1,a_2,\ldots,a_n)$ such that

$$a_1\ge a_2=\cdots=a_k=1\ge a_{k+1}=\cdots=a_n.$$

By hypothesis and P 1.255-(a), we have

$$n-1+(p-n+1)^2\le q\le(p-k+1)^2+k-1.$$

As in the case (a), we need to show that the system

$$x+(n-k)y=p-k+1,\quad x^2+(n-k)y^2=q-k+1,$$

has a unique solution (x,y) such that $x\ge 1\ge y\ge 0$. From the system equations, we get $g(y)=0$, where

$$g(y)=(n-k)(n-k+1)y^2-2(n-k)(p-k+1)y+(p-k+1)^2+k-1-q.$$

We have

$$g(0)=(p-k+1)^2+k-1-q\ge 0$$

and

$$g(1)=(p-n+1)^2+n-1-q\le 0.$$

Therefore, the equation $g(y)=0$ has a single root $y\in[0,1]$. From $y\le 1$, we get

$$y=\frac{p-k+1-x}{n-k}\le 1,$$

hence

$$x\ge p-n+1\ge 1.$$

Consider now that r is maximum at $(b_1,b_2,\ldots,b_n)$, where

$$b_1\ge\cdots\ge b_k\ge 1\ge b_{k+1}\ge\cdots\ge b_n.$$

Applying the contradiction method as in P 1.256, we get

$$b_{k+1} = \cdots = b_n.$$

We still need to show that

$$b_2 = \cdots = b_k = 1$$

for $k \ge 2$. Assume for the sake of contradiction, that

$$b_2 \ne 1.$$

For

$$a_3 = b_3,\ \cdots,\ a_{n-1} = b_{n-1},$$

we have $a_1 + a_2 + a_n = constant$ and $a_1^2 + a_2^2 + a_n^2 = constant$, where $a_1 \ge a_2 \ge 1 \ge a_n$. According to P 1.171, the product $a_1 a_2 a_{k+1}$ is maximum for $a_2 = 1$ or $a_n = 1$. The first case contradicts the assumption that $b_2 \ne 1$. The second case leads to $b_n = 1$, hence $b_{k+1} = \cdots = b_n = 1$. From the hypothesis $q \ge n-1+(p-n+1)^2$ and

$$q = b_1^2 + \cdots + b_k^2 + n - k, \qquad p = b_1 + \cdots + b_k + n - k,$$

we get

$$b_1^2 + \cdots + b_k^2 - k + 1 \ge (b_1 + \cdots + b_k - k + 1)^2,$$

which is equivalent to

$$(b_1 - 1)^2 + \cdots + (b_k - 1)^2 \ge [(b_1 - 1) + \cdots + (b_k - 1)]^2.$$

This is true only if

$$b_2 - 1 = \cdots = b_k - 1 = 0,$$

that is,

$$b_2 = \cdots = b_k.$$

This result contradicts also the assumption that $b_2 \ne 1$; as a consequence, we have $b_2 = 1$, which involves $b_2 = \cdots = b_k = 1$.

(c) By hypothesis and P 1.255-(c), we have

$$\frac{p^2 - 2(n-k)p + n(n-k)}{k} \le q \le n - 1 + (p - n + 1)^2.$$

For $k = 1$, these inequalities become

$$n - 1 + (p - n + 1)^2 \le q < n - 1 + (p - n + 1)^2,$$

which is not possible. Consider further that

$$k \ge 2.$$

We show first that there exists a unique n-tuple $(a_1, a_2, \ldots, a_n)$ such that

$$a_1 \ge a_2 = \cdots = a_k \ge a_{k+1} = \cdots = a_n = 1.$$

For $a_1 := x$, $a_2 = \cdots = a_k = y$ and $a_{k+1} = \cdots = a_n = 1$, from

$$a_1 + a_2 + \cdots + a_n = p, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = q,$$

we get

$$x + (k-1)y = p - n + k, \quad x^2 + (k-1)y^2 = q - n + k.$$

We need to show that this system has a unique solution (x, y) such that $x \geq y \geq 1$. From the system equations, we get $h(y) = 0$, where

$$h(y) = (k-1)ky^2 - 2(k-1)(p-n+k)y + (p-n+k)^2 + n - k - q.$$

We have

$$h(1) = (p-n+1)^2 + n - 1 - q > 0$$

and

$$h\left(\frac{p-n+k}{k}\right) = \frac{p^2 - 2(n-k)p + n(n-k)}{k} - q \leq 0.$$

Therefore, the equation $hy) = 0$ has a single root

$$y \in \left(1, \frac{p-n+k}{k}\right].$$

From

$$y = \frac{p-n+k-x}{k-1} \leq \frac{p-n+k}{k},$$

we get

$$x \geq \frac{p-n+k}{k} \geq y.$$

Consider now that r is maximum at $(b_1, b_2, \ldots, b_n)$, where

$$b_1 \geq \cdots \geq b_k \geq 1 \geq b_{k+1} \geq \cdots \geq b_n.$$

We need to show that

$$b_2 = \cdots = b_k, \qquad b_{k+1} = \cdots = b_n = 1.$$

To show that $b_2 = \cdots = b_k$ for $k \geq 3$, assume for the sake of contradiction that

$$b_2 \neq b_k.$$

For

$$a_3 = b_3, \cdots, a_{k-1} = b_{k-1}, \qquad a_{k+1} = b_{k+1}, \cdots, a_n = b_n,$$

we have $a_1 + a_2 + a_k = constant$ and $a_1^2 + a_2^2 + a_k^2 = constant$, where

$$a_1 \geq a_2 \geq a_k \geq 1.$$

According to P 1.169 the product $a_1 a_2 a_k$ is maximum for $a_2 = a_k$, which contradicts the assumption that $b_2 \neq b_k$.

To show that $b_{k+1} = \cdots = b_n$ for $k \le n-2$, assume for the sake of contradiction that

$$b_{k+1} \ne b_n.$$

For

$$a_2 = b_2, \cdots, a_k = b_k, \qquad a_{k+2} = b_{k+2}, \cdots, a_{n-1} = b_{n-1},$$

we have $a_1 + a_{k+1} + a_n = constant$ and $a_1^2 + a_{k+1}^2 + a_n^2 = constant$, where

$$a_1 \ge 1 \ge a_{k+1} \ge a_n.$$

According to P 1.168, the product $a_1 a_{k+1} a_n$ is maximum for $a_{k+1} = a_n$, which contradicts the assumption that $b_{k+1} \ne b_n$. Therefore, we have

$$b_2 = \cdots = b_k := x, \qquad b_{k+1} = \cdots = b_n := y.$$

To end the proof, we still need to show that $y = 1$. Assume, for the sake of contradiction that

$$y \ne 1.$$

For

$$a_3 = b_3, \cdots, a_{n-1} = b_{n-1},$$

we have $a_1 + a_2 + a_n = constant$ and $a_1^2 + a_2^2 + a_n^2 = constant$, where

$$a_1 \ge a_2 \ge 1 \ge a_n.$$

According to P 1.171, the product $a_1 a_2 a_n$ is maximum for $a_n = 1$ or $a_2 = 1$, hence for $y = 1$ or $x = 1$. The first case contradicts the assumption that $y \ne 1$. The second case leads to

$$b_2 = \cdots = b_k = 1, \qquad b_{k+1} = \cdots = b_n := y < 1.$$

From the hypothesis $q \le n - 1 + (p - n + 1)^2$ and

$$q = b_1^2 + k - 1 + (n-k)y^2, \qquad p = b_1 + k - 1 + (n-k)y,$$

we get

$$b_1^2 + (n-k)(y^2 - 1) \le [b_1 + (n-k)(y-1)]^2,$$

which is equivalent to

$$(1-y)[(n-k-1)(1-y) - 2(b_1 - 1)] \ge 0.$$

Under the assumption that $y < 1$, this inequality implies

$$(n-k-1)(1-y) \ge 2(b_1 - 1).$$

On the other hand, the condition $p \ge n$ is equivalent to

$$b_1 - 1 \ge (n-k)(1-y).$$

Thus, we have

$$(n-k-1)(1-y) \geq 2(b_1-1) \geq 2(n-k)(1-y),$$

which involves

$$-(n-k+1)(1-y) \geq 0.$$

This result contradicts also the assumption $y \neq 1$.
Remark 1. For $p = n$, from P 1.258 we get P 1.256.

Remark 2. From P 1.258, we get the following simplified statement.

- *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be nonnegative real numbers such that*

$$a_1 \geq \cdots \geq a_k \geq 1 \geq a_{k+1} \geq \cdots \geq a_n, \qquad 1 \leq k \leq n-1,$$

and

$$a_1 + a_2 + \cdots + a_n = p, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = q,$$

where p and q are fixed numbers. Then, the product $r = a_1 a_2 \cdots a_n$ is maximum when

$$a_2 = \cdots = a_k = 1, \qquad a_{k+1} = \cdots = a_n$$

or

$$a_2 = \cdots = a_k, \qquad a_{k+1} = \cdots = a_n = 1.$$

□

P 1.259. *If $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ are nonnegative real numbers such that*

$$a_1 \leq a_2 \leq 1 \leq a_3 \leq \cdots \leq a_n, \qquad a_1 + a_2 + \cdots + a_n = n-1,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 + 10a_1 a_2 \cdots a_n \leq n+1.$$

(Vasile C., 2015)

Solution. According to P 1.258-(a), it suffices to prove the inequality for

$$a_1 = a_2, \quad a_3 = \cdots = a_{n-1} = 1.$$

Thus, we need to show that

$$2a + b = 2, \qquad 0 \leq a \leq 1/2, \quad b \geq 1,$$

implies

$$2a^2 + (n-3) + b^2 + 10a^2 b \leq n+1,$$

which is equivalent to

$$2a^2 + b^2 + 10a^2 b \leq 4,$$

$$2a^2+(2-2a)^2+10a^2(2-2a)\le 4,$$

$$2a(1-2a)(4-5a)\ge 0.$$

The equality holds for

$$a_1=a_2=0,\quad a_3=\cdots=a_{n-1}=1,\quad a_n=2,$$

and for

$$a_1=a_2=1/2,\quad a_3=\cdots=a_n=1.$$

□

P 1.260. *If a,b,c,d,e are nonnegative real numbers such that*

$$a\le b\le 1\le c\le d\le e,\qquad a+b+c+d+e=8,$$

then

$$a^2+b^2+c^2+d^2+e^2+3abcde\le 38.$$

(Vasile C., 2015)

Solution. According to Remark 2 from P 1.258, it suffices to prove the inequality for

$$a=b,\quad c=d=1,$$

and for

$$a=b=1,\quad c=d.$$

Case 1: $a=b,\ \ c=d=1$. We need to show that

$$2a+e=6,\qquad 0\le a\le 1,\ \ e\ge 4,$$

implies

$$2a^2+2+e^2+3a^2e\le 38,$$

which is equivalent to

$$2a^2+e^2+3a^2e\le 36,$$

$$a^2+2(3-a)^2+3a^2(3-a)\le 18,$$

$$3a(a-2)^2\ge 0.$$

The equality holds for

$$a=b=0,\quad c=d=1,\quad e=6.$$

Case 2: $a=b=1,\ \ c=d$. We need to show that

$$2c+e=6,\qquad 1\le c\le 2\le e\le 4,$$

implies

$$2 + 2c^2 + e^2 + 3c^2 e \leq 38,$$

which is equivalent to

$$2c^2 + e^2 + 3c^2 e \leq 36,$$

$$c^2 + 2(3 - c)^2 + 3c^2(3 - c) \leq 18,$$

$$3c(c - 2)^2 \geq 0.$$

The equality holds for

$$a = b = 1, \quad c = d = e = 2.$$

□

Chapter 2

Noncyclic Inequalities

2.1 Applications

2.1. If a, b are positive real numbers, then

$$\frac{1}{4a^2+b^2}+\frac{3}{b^2+4ab} \ge \frac{16}{5(a+b)^2}.$$

2.2. If a, b are positive real numbers, then

$$3a\sqrt{3a}+3b\sqrt{6a+3b} \ge 5(a+b)\sqrt{a+b}.$$

2.3. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$(ab+c)(ac+b) \le 4.$$

2.4. If a, b, c are nonnegative real numbers, then

$$a^3+b^3+c^3-3abc \ge \frac{1}{4}(b+c-2a)^3.$$

2.5. Let a, b, c be nonnegative real numbers such that $a \ge b \ge c$. Prove that

(a) $$a^3+b^3+c^3-3abc \ge 2(2b-a-c)^3;$$

(b) $$a^3+b^3+c^3-3abc \ge (a-2b+c)^3.$$

2.6. Let a, b, c be nonnegative real numbers such that $a \ge b \ge c$. Prove that

(a) $$a^3 + b^3 + c^3 - 3abc \ge 3(a^2 - b^2)(b - c);$$

(b) $$a^3 + b^3 + c^3 - 3abc \ge \frac{9}{2}(a - b)(b^2 - c^2).$$

2.7. If a, b, c are nonnegative real numbers such that

$$c = \min\{a, b, c\}, \qquad a^2 + b^2 + c^2 = 3,$$

then

(a) $$5b + 2c \le 9;$$

(b) $$5(b + c) \le 9 + 3a.$$

2.8. Let a, b, c be nonnegative real numbers such that $a = \max\{a, b, c\}$. Prove that

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \ge 2(b^4 + c^4 + 4b^2c^2)(b - c)^2.$$

2.9. Let a, b, c be nonnegative real numbers such that $a = \max\{a, b, c\}$. Prove that

$$a^2 + b^2 + c^2 \ge \frac{9abc}{a + b + c} + \frac{5}{3}(b - c)^2.$$

2.10. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{(a + b)^2} + \frac{1}{(a + c)^2} + \frac{16}{(b + c)^2} \ge \frac{6}{ab + bc + ca}.$$

2.11. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{(a + b)^2} + \frac{1}{(a + c)^2} + \frac{2}{(b + c)^2} \ge \frac{5}{2(ab + bc + ca)}.$$

2.12. If a, b, c are nonnegative real numbers, no two of which are zero, then

$$\frac{1}{(a + b)^2} + \frac{1}{(a + c)^2} + \frac{25}{(b + c)^2} \ge \frac{8}{ab + bc + ca}.$$

2.13. If a, b, c are positive real numbers, then

$$(a+b)^3(a+c)^3 \geq 4a^2bc(2a+b+c)^2.$$

2.14. If a, b, c are positive real numbers such that $abc = 1$, then

(a) $$\frac{a}{b} + \frac{b}{c} + \frac{1}{a} \geq a + b + 1;$$

(b) $$\frac{a}{b} + \frac{b}{c} + \frac{1}{a} \geq \sqrt{3(a^2 + b^2 + 1)}.$$

2.15. If a, b, c are positive real numbers such that $abc \geq 1$, then

$$a^{\frac{a}{b}} b^{\frac{b}{c}} c^c \geq 1.$$

2.16. If a, b, c are positive real numbers such that $ab + bc + ca = 3$, then

$$ab^2c^3 < 4.$$

2.17. If a, b, c are positive real numbers such that $ab + bc + ca = \frac{5}{3}$, then

$$ab^2c^2 \leq \frac{1}{3}.$$

2.18. Let a, b, c be positive real numbers such that

$$a \leq b \leq c, \qquad ab + bc + ca = 3.$$

Prove that

(a) $$ab^2c \leq \frac{9}{8};$$

(b) $$ab^4c \leq 2;$$

(c) $$a^2b^3c \leq 2.$$

2.19. Let a,b,c be positive real numbers such that

$$a \le b \le c, \qquad a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Prove that

$$b \ge \frac{1}{a+c-1}.$$

2.20. Let a,b,c be positive real numbers such that

$$a \le b \le c, \qquad a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Prove that

$$ab^2c^3 \ge 1.$$

2.21. Let a,b,c be positive real numbers such that

$$a \le b \le c, \qquad a+b+c = abc+2.$$

Prove that

$$(1-b)(1-ab^3c) \ge 0.$$

2.22. Let a,b,c be real numbers, no two of which are zero. Prove that

(a) $$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{2(b^2+c^2)};$$

(b) $$\frac{(a+b)^2}{a^2+b^2}+\frac{(a+c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{2(b^2+c^2)}.$$

2.23. Let a,b,c be real numbers, no two of which are zero. If $bc \ge 0$, then

(a) $$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{(b+c)^2};$$

(b) $$\frac{(a+b)^2}{a^2+b^2}+\frac{(a+c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{(b+c)^2}.$$

2.24. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{|a-b|^3}{a^3+b^3}+\frac{|a-c|^3}{a^3+c^3}\geq\frac{|b-c|^3}{(b+c)^3}.$$

2.25. Let a, b, c be positive real numbers, $b \neq c$. Prove that

$$\frac{ab}{(a+b)^2}+\frac{ac}{(a+c)^2}\leq\frac{(b+c)^2}{4(b-c)^2}.$$

2.26. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{3bc+a^2}{b^2+c^2}\geq\frac{3ab-c^2}{a^2+b^2}+\frac{3ac-b^2}{a^2+c^2}.$$

2.27. Let a, b, c be nonnegative real numbers such that $a+b>0$. Prove that

$$abc\geq(b+c-a)(c+a-b)(a+b-c)+\frac{ab(a-b)^2}{a+b}.$$

2.28. Let a, b, c be nonnegative real numbers such that $a\geq b\geq c$. Prove that

(a) $$abc\geq(b+c-a)(c+a-b)(a+b-c)+\frac{2ab(a-b)^2}{a+b};$$

(b) $$abc\geq(b+c-a)(c+a-b)(a+b-c)+\frac{27b(a-b)^4}{4a^2}.$$

2.29. Let a, b, c be nonnegative real numbers such that $a+b>0$. Prove that

$$\sum a^2(a-b)(a-c)\geq a^2b^2\left(\frac{a-b}{a+b}\right)^2.$$

2.30. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$ab^2+bc^2+2ca^2\leq 8.$$

2.31. Let a, b, c be nonnegative real numbers such that $a+b+c=3$. Prove that

$$ab^2+bc^2+\frac{3}{2}abc\leq 4.$$

2.32. Let a, b, c be nonnegative real numbers such that $a+b+c=5$. Prove that

$$ab^2+bc^2+2abc \le 20.$$

2.33. If a, b, c are nonnegative real numbers, then

$$a^3+b^3+c^3-a^2b-b^2c-c^2a \ge \frac{8}{9}(a-b)(b-c)^2.$$

2.34. Let a, b, c be nonnegative real numbers such that $a \ge b \ge c$. Prove that

(a) $$\sum a^2(a-b)(a-c) \ge 4a^2b^2\left(\frac{a-b}{a+b}\right)^2;$$

(b) $$\sum a^2(a-b)(a-c) \ge \frac{27b(a-b)^4}{4a}.$$

2.35. If a, b, c are real numbers such that

$$a \ge b \ge 1 \ge c, \qquad a^2+b^2+c^2=3,$$

then

(a) $$1-abc \le 2(b-c)^2;$$

(b) $$1-abc \ge 2(a-b)^2;$$

(c) $$1-abc \ge \frac{1}{2}(a-c)^2;$$

(d) $$1-abc \le \frac{3}{4}(a-c)^2.$$

2.36. If a, b, c are real numbers such that

$$a \ge 1 \ge b \ge c, \qquad a^2+b^2+c^2=3,$$

then

$$1-abc \le \frac{2}{3}(a-c)^2.$$

2.37. If $a \ge 1 \ge b \ge c \ge 0$ and $a^2+b^2+c^2=3$, then

$$1-abc \le \frac{1}{\sqrt{2}}(a-c).$$

2.38. If $a \ge 1 \ge b \ge c \ge 0$ and $a^2 + b^2 + c^2 = 3$, then

$$1 - abc \le (1 + \sqrt{2})(a - b).$$

2.39. If $a \ge 1 \ge b \ge c \ge 0$ and $a^2 + b^2 + c^2 = 3$, then

$$1 - abc \le (3 + 2\sqrt{2})(a - b)^2.$$

2.40. If a, b, c are positive real numbers, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{(a-c)^2}{ab + bc + ca}.$$

2.41. If a, b, c are positive real numbers, then

(a) $$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{4(a-c)^2}{(a+b+c)^2};$$

(b) $$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{5(a-c)^2}{(a+b+c)^2}.$$

2.42. If $a \ge b \ge c > 0$, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{3(b-c)^2}{ab + bc + ca}.$$

2.43. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

(a) if $a \ge b \ge 1 \ge c$, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{2(a-b)^2}{ab};$$

(b) if $a \ge 1 \ge b \ge c$, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{2(b-c)^2}{bc}.$$

2.44. Let a, b, c be positive real numbers such that

$$a \geq 1 \geq b \geq c, \qquad abc = 1.$$

prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3 + \frac{9(b-c)^2}{ab+bc+ca}.$$

2.45. Let a, b, c be positive real numbers such that

$$a \geq 1 \geq b \geq c, \qquad a+b+c = 3.$$

prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3 + \frac{4(b-c)^2}{b^2+c^2}.$$

2.46. Let a, b, c be positive real numbers such that

$$a \geq b \geq 1 \geq c, \qquad a+b+c = 3.$$

Prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3 + \frac{3(a-b)^2}{ab}.$$

2.47. If a, b, c are positive real numbers, then

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3 + \frac{2(a-c)^2}{(a+c)^2}.$$

2.48. If a, b, c are positive real numbers, then

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a+b+c + \frac{4(a-c)^2}{a+b+c}.$$

2.49. If $a \geq b \geq c > 0$, then

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a+b+c + \frac{6(b-c)^2}{a+b+c}.$$

2.50. If $a \ge b \ge c > 0$, then

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} > 5(a-b).$$

2.51. Let a, b, c be positive real numbers such that

$$a \ge b \ge 1 \ge c, \qquad a+b+c=3.$$

Prove that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge 3 + \frac{11(a-c)^2}{4(a+c)}.$$

2.52. If a, b, c are positive real numbers, then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{27(b-c)^2}{16(a+b+c)^2}.$$

2.53. Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{9(b-c)^2}{4(a+b+c)^2}.$$

2.54. Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{(b-c)^2}{2(b+c)^2}.$$

2.55. Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{(b-c)^2}{4bc}.$$

2.56. Let a, b, c be positive real numbers such that

$$a \le 1 \le b \le c, \qquad a+b+c=3,$$

then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{3}{2} + \frac{3(b-c)^2}{4bc}.$$

2.57. Let a, b, c be nonnegative real numbers such that

$$a \geq 1 \geq b \geq c, \qquad a + b + c = 3,$$

then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} + \frac{(b-c)^2}{(b+c)^2}.$$

2.58. Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that

(a) $$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{2(b-c)^2}{3(b^2+c^2)} \leq 1;$$

(b) $$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(b-c)^2}{b^2+bc+c^2} \leq 1;$$

(c) $$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(a-b)^2}{2(a^2+b^2)} \leq 1.$$

2.59. Let a, b, c be positive real numbers such that

$$a \leq 1 \leq b \leq c, \qquad a + b + c = 3,$$

then

$$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(b-c)^2}{bc} \leq 1.$$

2.60. Let a, b, c be nonnegative real numbers such that $a = \max\{a, b, c\}$ and $b+c > 0$. Prove that

(a) $$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(b-c)^2}{2(ab+bc+ca)} \leq 1;$$

(b) $$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{2(b-c)^2}{(a+b+c)^2} \leq 1.$$

2.61. Let a, b, c be positive real numbers. Prove that

(a) if $a \geq b \geq c$, then

$$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(a-c)^2}{a^2-ac+c^2} \geq 1;$$

(b) if $a \geq 1 \geq b \geq c$ and $abc = 1$, then

$$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(b-c)^2}{b^2-bc+c^2} \leq 1.$$

2.62. Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that

(a) $$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{4(b-c)^2}{3(b+c)^2};$$

(b) $$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{(a-b)^2}{(a+b)^2}.$$

2.63. If a, b, c are positive real numbers, then

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{9(a-c)^2}{4(a+b+c)^2}.$$

2.64. Let a, b, c be nonnegative real numbers, no two of which are zero. If $a = \min\{a, b, c\}$, then

$$\frac{1}{\sqrt{a^2-ab+b^2}} + \frac{1}{\sqrt{b^2-bc+c^2}} + \frac{1}{\sqrt{c^2-ca+a^2}} \ge \frac{6}{b+c}.$$

2.65. If $a \ge 1 \ge b \ge c \ge 0$ such that

$$ab+bc+ca = abc+2,$$

then

$$ac \le 4-2\sqrt{2}.$$

2.66. If a, b, c are nonnegative real numbers such that

$$ab+bc+ca = 3, \qquad a \le 1 \le b \le c,$$

then

(a) $$a+b+c \le 4;$$

(b) $$2a+b+c \le 4.$$

2.67. Let a, b, c be nonnegative real numbers such that $a \le b \le c$. Prove that

(a) if $a+b+c = 3$, then

$$a^4(b^4+c^4) \le 2;$$

(b) if $a+b+c = 2$, then

$$c^4(a^4+b^4) \le 1.$$

2.68. If a, b, c are nonnegative real numbers such that $ab + bc + ca = 3$, then

(a) $$a^2 + b^2 + c^2 - a - b - c \geq \frac{5}{8}(a-c)^2;$$

(b) $$a^2 + b^2 + c^2 - a - b - c \geq \frac{5}{2}\min\{(a-b)^2, (b-c)^2, (c-a)^2\}.$$

2.69. If a, b, c are nonnegative real numbers such that $ab + bc + ca = 3$, then

$$\frac{a^3 + b^3 + c^3}{a+b+c} \geq 1 + \frac{5}{9}(a-c)^2.$$

2.70. If a, b, c are nonnegative real numbers such that

$$a \geq b \geq c, \quad ab + bc + ca = 3,$$

then

(a) $$\frac{a^3 + b^3 + c^3}{a+b+c} \geq 1 + \frac{7}{9}(a-b)^2;$$

(b) $$\frac{a^3 + b^3 + c^3}{a+b+c} \geq 1 + \frac{2}{3}(b-c)^2.$$

(c) $$\frac{a^3 + b^3 + c^3}{a+b+c} \geq 1 + \frac{7}{3}\min\{(a-b)^2, (b-c)^2\}.$$

2.71. If a, b, c are nonnegative real numbers such that $ab + bc + ca = 3$, then

$$a^4 + b^4 + c^4 - a^2 - b^2 - c^2 \geq \frac{11}{4}(a-c)^2.$$

2.72. If a, b, c are nonnegative real numbers such that

$$a \geq b \geq c, \quad ab + bc + ca = 3,$$

then

(a) $$a^4 + b^4 + c^4 - a^2 - b^2 - c^2 \geq \frac{11}{3}(a-b)^2;$$

(b) $$a^4 + b^4 + c^4 - a^2 - b^2 - c^2 \geq \frac{10}{3}(b-c)^2.$$

2.73. Let a,b,c be nonnegative real numbers such that

$$a \le b \le c, \quad a+b+c=3.$$

Find the greatest real number k such that

$$\sqrt{(56b^2+25)(56c^2+25)}+k(b-c)^2 \le 14(b+c)^2+25.$$

2.74. If $a \ge b \ge c > 0$ such that $abc=1$, then

$$3(a+b+c) \le 8+\frac{a}{c}.$$

2.75. If $a \ge b \ge c > 0$, then

$$(a+b-c)(a^2b-b^2c+c^2a) \ge (ab-bc+ca)^2.$$

2.76. If $a \ge b \ge c \ge 0$, then

$$\frac{(a-c)^2}{2(a+c)} \le a+b+c-3\sqrt[3]{abc} \le \frac{2(a-c)^2}{a+5c}.$$

2.77. If $a \ge b \ge c \ge d \ge 0$, then

$$\frac{(a-d)^2}{a+3d} \le a+b+c+d-4\sqrt[4]{abcd} \le \frac{3(a-d)^2}{a+5d}.$$

2.78. If $a \ge b \ge c > 0$, then

(a) $$a+b+c-3\sqrt[3]{abc} \ge \frac{3(a-b)^2}{5a+4b};$$

(b) $$a+b+c-3\sqrt[3]{abc} \ge \frac{64(a-b)^2}{7(11a+24b)}.$$

2.79. If $a \ge b \ge c > 0$, then

(a) $$a+b+c-3\sqrt[3]{abc} \ge \frac{3(b-c)^2}{4b+5c};$$

(b) $$a+b+c-3\sqrt[3]{abc} \ge \frac{25(b-c)^2}{7(3b+11c)}.$$

2.80. If $a \ge b \ge c > 0$, then

$$a+b+c-3\sqrt[3]{abc} \ge \frac{3(a-c)^2}{4(a+b+c)}.$$

2.81. If $a \ge b \ge c > 0$, then

(a) $$a^6+b^6+c^6-3a^2b^2c^2 \ge 12a^2c^2(b-c)^2;$$

(b) $$a^6+b^6+c^6-3a^2b^2c^2 \ge 10a^3c(b-c)^2.$$

2.82. If $a \ge b \ge c > 0$, then

$$\frac{ab+bc}{a^2+b^2+c^2} \le \frac{1+\sqrt{3}}{4}.$$

2.83. If $a \ge b \ge c \ge d > 0$, then

$$\frac{ab+bc+cd}{a^2+b^2+c^2+d^2} \le \frac{2+\sqrt{7}}{6}.$$

2.84. If

$$a \ge 1 \ge b \ge c \ge d \ge 0, \qquad a+b+c+d=4,$$

then

$$ab+bc+cd \le 3.$$

2.85. Let k and a,b,c be positive real numbers, and let

$$E=(ka+b+c)\left(\frac{k}{a}+\frac{1}{b}+\frac{1}{c}\right), \qquad F=(ka^2+b^2+c^2)\left(\frac{k}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

(a) If $k \ge 1$, then

$$\sqrt{\frac{F-(k-2)^2}{2k}}+2 \ge \frac{E-(k-2)^2}{2k};$$

(b) If $0 < k \le 1$, then

$$\sqrt{\frac{F-k^2}{k+1}}+2 \ge \frac{E-k^2}{k+1}.$$

2.86. If a, b, c are positive real numbers, then

$$\frac{a}{2b+6c}+\frac{b}{7c+a}+\frac{25c}{9a+8b}>1.$$

2.87. If a, b, c are positive real numbers such that

$$\frac{1}{a}\ge\frac{1}{b}+\frac{1}{c},$$

then

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\ge\frac{55}{12(a+b+c)}.$$

2.88. If a, b, c are positive real numbers such that

$$\frac{1}{a}\ge\frac{1}{b}+\frac{1}{c},$$

then

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}\ge\frac{189}{40(a^2+b^2+c^2)}.$$

2.89. Find the best real numbers k, m, n such that

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})\sqrt{a+b+c}\ge ka+mb+nc$$

for all $a\ge b\ge c\ge 0$.

2.90. Let $a, b\in(0,1]$, $a\le b$.

(a) If $a\le\frac{1}{e}$, then

$$2a^a\ge a^b+b^a;$$

(b) If $b\ge\frac{1}{e}$, then

$$2b^b\ge a^b+b^a.$$

2.91. If $0\le a\le b$ and $b\ge\frac{1}{2}$, then

$$2b^{2b}\ge a^{2b}+b^{2a}.$$

2.92. If $a \ge b \ge 0$, then

(a) $$a^{b-a} \le 1 + \frac{a-b}{\sqrt{a}};$$

(b) $$a^{a-b} \ge 1 - \frac{3(a-b)}{4\sqrt{a}}.$$

2.93. If a, b, c are positive real numbers such that

$$a \ge b \ge c, \qquad ab^2c^3 = 1,$$

then

$$a + 2b + 3c \ge \frac{1}{a} + \frac{2}{b} + \frac{3}{c}.$$

2.94. If a, b, c are positive real numbers such that

$$a + b + c = 3, \qquad a \le b \le c,$$

then

$$\frac{1}{a} + \frac{2}{b} \ge a^2 + b^2 + c^2.$$

2.95. If a, b, c are positive real numbers such that

$$a + b + c = 3, \qquad a \le b \le c,$$

then

$$\frac{2}{a} + \frac{3}{b} + \frac{1}{c} \ge 2(a^2 + b^2 + c^2).$$

2.96. If a, b, c are positive real numbers such that

$$a + b + c = 3, \qquad a \le b \le c,$$

then

$$\frac{31}{a} + \frac{25}{b} + \frac{25}{c} \ge 27(a^2 + b^2 + c^2).$$

2.97. If a, b, c are the lengths of the sides of a triangle, then

$$a^3(b + c) + bc(b^2 + c^2) \ge a(b^3 + c^3).$$

2.98. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{(a+b)^2}{2ab+c^2}+\frac{(a+c)^2}{2ac+b^2}\ge\frac{(b+c)^2}{2bc+a^2}.$$

2.99. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{a+b}{ab+c^2}+\frac{a+c}{ac+b^2}\ge\frac{b+c}{bc+a^2}.$$

2.100. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{b(a+c)}{ac+b^2}+\frac{c(a+b)}{ab+c^2}\ge\frac{a(b+c)}{bc+a^2}.$$

2.101. If a, b, c, d are positive real numbers such that

$$a\ge b\ge c\ge d,\qquad ab^2c^3d^6=1,$$

then

$$a+2b+3c+6d\ge\frac{1}{a}+\frac{2}{b}+\frac{3}{c}+\frac{6}{d}.$$

2.102. If a, b, c, d are positive real numbers such that

$$a\ge b\ge c\ge d,\qquad abc^2d^4\ge 1,$$

then

$$a+b+2c+4d\ge\frac{1}{a}+\frac{1}{b}+\frac{2}{c}+\frac{4}{d}.$$

2.103. If a, b, c, d are positive real numbers such that

$$abcd\ge 1,\qquad a\ge b\ge c\ge d,\qquad ad\ge bc,$$

then

$$a+b+c+d\ge\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

2.104. If a, b, c, d, e, f are positive real numbers such that

$$abcdef \geq 1, \qquad a \geq b \geq c \geq d \geq e \geq f, \qquad af \geq be \geq cd,$$

then

$$a + b + c + d + e + f \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \frac{1}{f}.$$

2.105. Let a, b, c, d be nonnegative real numbers such that

$$a^2 - ab + b^2 = c^2 - cd + d^2.$$

Prove that

$$(a + b)(c + d) \geq 2(ab + cd).$$

2.106. Let a, b, c, d be nonnegative real numbers such that

$$a^2 - ab + b^2 = c^2 - cd + d^2.$$

Prove that

$$\frac{1}{a^2 + ab + b^2} + \frac{1}{c^2 + cd + d^2} \leq \frac{8}{3(a + b)(c + d)}.$$

2.107. Let a, b, c, d be nonnegative real numbers such that

$$a^2 - ab + b^2 = c^2 - cd + d^2.$$

Prove that

$$\frac{1}{a^2 + ab + b^2} + \frac{1}{c^2 + cd + d^2} \leq \frac{8}{3(a + b)(c + d)}.$$

2.108. Let a, b, c, d be nonnegative real numbers such that

$$a^2 - ab + b^2 = c^2 - cd + d^2.$$

Prove that

$$\frac{1}{(ac + bd)^4} + \frac{1}{(ad + bc)^4} \leq \frac{2}{(ab + cd)^4}.$$

2.109. Let a, b, c, d be nonnegative real numbers such that $a \geq b \geq c \geq d$ and

$$a + b + c + d = 13, \qquad a^2 + b^2 + c^2 + d^2 = 43.$$

Prove that

$$ab \geq cd + 3.$$

2.110. Let a, b, c, d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and

$$a+b+c+d=13, \qquad a^2+b^2+c^2+d^2=43.$$

Prove that

$$ab-cd \ge \frac{7}{2}$$

(a) for $a \le \dfrac{39}{10}$;

(b) for $d \le \dfrac{31}{11}$.

2.111. Let a, b, c, d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and

$$a+b+c+d=13, \qquad a^2+b^2+c^2+d^2=43.$$

Prove that

$$\frac{83}{4} \le ac+bd \le \frac{169}{8}.$$

2.112. If a, b, c, d are positive real numbers such that

$$a+b+c+d=4, \qquad a \le b \le 1 \le c \le d,$$

then

$$9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \ge 4+8(a^2+b^2+c^2+d^2).$$

2.113. If a, b, c, d are positive real numbers such that

$$a^2+b^2+c^2+d^2=4, \qquad a \le b \le c \le d,$$

then

$$\frac{1}{a}+a+b+c+d \ge 5.$$

2.114. If a, b, c, d are real numbers, then

$$6(a^2+b^2+c^2+d^2)+(a+b+c+d)^2 \ge 12(ab+bc+cd).$$

2.115. If a,b,c,d are positive real numbers, then

$$\frac{1}{a^2+ab}+\frac{1}{b^2+bc}+\frac{1}{c^2+cd}+\frac{1}{d^2+da}\ge\frac{4}{ac+bd}.$$

2.116. If a,b,c,d are positive real numbers, then

$$\frac{1}{a(1+b)}+\frac{1}{b(1+a)}+\frac{1}{c(1+d)}+\frac{1}{d(1+c)}\ge\frac{16}{1+8\sqrt{abcd}}.$$

2.117. If a,b,c,d are positive real numbers such that $a\ge b\ge c\ge d$ and

$$a+b+c+d=4,$$

then

$$ac+bd\le 2.$$

2.118. If a,b,c,d are positive real numbers such that $a\ge b\ge c\ge d$ and

$$a+b+c+d=4,$$

then

$$2\left(\frac{1}{b}+\frac{1}{d}\right)\ge a^2+b^2+c^2+d^2.$$

2.119. Let a,b,c,d be positive real numbers such that $a\ge b\ge c\ge d$ and

$$ab+bc+cd+da=3.$$

Prove that

$$a^3bcd<4.$$

2.120. Let a,b,c,d be positive real numbers such that $a\ge b\ge c\ge d$ and

$$ab+bc+cd+da=6.$$

Prove that

$$acd\le 2.$$

2.121. Let a, b, c, d be positive real numbers such that $a \geq b \geq c \geq d$ and

$$ab + bc + cd + da = 9.$$

Prove that

$$abd \leq 4.$$

2.122. Let a, b, c, d be positive real numbers such that $a \geq b \geq c \geq d$ and

$$a^2 + b^2 + c^2 + d^2 = 10.$$

Prove that

$$2b + 4d \leq 3c + 5.$$

2.123. Let a, b, c, d be positive real numbers such that $a \leq b \leq c \leq d$ and

$$abcd = 1.$$

Prove that

$$4 + \frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a} \geq 2(a+b)(c+d).$$

2.124. Let a, b, c, d be positive real numbers such that $a \geq b \geq c \geq d$ and

$$3(a^2 + b^2 + c^2 + d^2) = (a+b+c+d)^2.$$

Prove that

(a) $$\frac{a+d}{b+c} \leq 2;$$

(b) $$\frac{a+c}{b+d} \leq \frac{7+2\sqrt{6}}{5};$$

(c) $$\frac{a+c}{c+d} \leq \frac{3+\sqrt{5}}{2}.$$

2.125. Let a, b, c, d be nonnegative real numbers such that $a \geq b \geq c \geq d$ and

$$2(a^2 + b^2 + c^2 + d^2) = (a+b+c+d)^2.$$

Prove that

$$a \geq b + 3c + (2\sqrt{3} - 1)d.$$

2.126. If $a \ge b \ge c \ge d \ge 0$, then

(a) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{3}{2}\left(\sqrt{b}-2\sqrt{c}+\sqrt{d}\right)^2;$$

(b) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{2}{9}\left(3\sqrt{b}-2\sqrt{c}-\sqrt{d}\right)^2;$$

(c) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{4}{19}\left(3\sqrt{b}-\sqrt{c}-2\sqrt{d}\right)^2;$$

(d) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{3}{8}\left(\sqrt{b}-3\sqrt{c}+2\sqrt{d}\right)^2;$$

(e) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{1}{2}\left(2\sqrt{b}-3\sqrt{c}+\sqrt{d}\right)^2;$$

(f) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{1}{6}\left(2\sqrt{b}+\sqrt{c}-3\sqrt{d}\right)^2.$$

2.127. If $a \ge b \ge c \ge d \ge 0$, then

(a) $$a+b+c+d-4\sqrt[4]{abcd} \ge \left(\sqrt{a}-\sqrt{d}\right)^2;$$

(b) $$a+b+c+d-4\sqrt[4]{abcd} \ge 2\left(\sqrt{b}-\sqrt{c}\right)^2;$$

(c) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{4}{3}\left(\sqrt{b}-\sqrt{d}\right)^2;$$

(d) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{3}{2}\left(\sqrt{c}-\sqrt{d}\right)^2.$$

2.128. If $a \ge b \ge c \ge d \ge e \ge 0$, then

$$a+b+c+d+e-5\sqrt[5]{abcde} \ge 2\left(\sqrt{b}-\sqrt{d}\right)^2.$$

2.129. If a, b, c, d, e are real numbers, then

$$\frac{ab+bc+cd+de}{a^2+b^2+c^2+d^2+e^2} \le \frac{\sqrt{3}}{2}.$$

2.130. If a, b, c, d, e are positive real numbers, then

$$\frac{a^2b^2}{bd+ce}+\frac{b^2c^2}{cd+ae}+\frac{c^2a^2}{ad+be} \ge \frac{3abc}{d+e}.$$

2.131. If a, b, c, d, e, f are nonnegative real numbers such that

$$a \geq b \geq c \geq d \geq e \geq f,$$

then

$$(a+b+c+d+e+f)^2 \geq 8(ac+bd+ce+df).$$

2.132. If $a \geq b \geq c \geq d \geq e \geq f \geq 0$, then

$$a+b+c+d+e+f-6\sqrt[6]{abcdef} \geq 2\left(\sqrt{b}-\sqrt{e}\right)^2.$$

2.133. Let a, b, c and x, y, z be positive real numbers such that

$$x+y+z=a+b+c.$$

Prove that

$$ax^2+by^2+cz^2+xyz \geq 4abc.$$

2.134. Let a, b, c and x, y, z be positive real numbers such that

$$x+y+z=a+b+c.$$

Prove that

$$\frac{x(3x+a)}{bc}+\frac{y(3y+b)}{ca}+\frac{z(3z+c)}{ab} \geq 12.$$

2.135. Let a, b, c be given positive numbers. Find the minimum value $F(a,b,c)$ of

$$E(x,y,z)=\frac{ax}{y+z}+\frac{by}{z+x}+\frac{cz}{x+y},$$

where x, y, z are nonnegative real numbers, no two of which are zero.

2.136. Let a, b, c and x, y, z be real numbers.

(a) If $ab+bc+ca>0$, then

$$[(b+c)x+(c+a)y+(a+b)z]^2 \geq 4(ab+bc+ca)(xy+yz+zx);$$

(b) If $a, b, c \geq 0$, then

$$[(b+c)x+(c+a)y+(a+b)z]^2 \geq 4(a+b+c)(ayz+bzx+cxy).$$

2.137. Let a, b, c and x, y, z be positive real numbers such that

$$\frac{a}{yz} + \frac{b}{zx} + \frac{c}{xy} = 1.$$

Prove that

(a) $$x + y + z \ge \sqrt{4(a + b + c + \sqrt{ab} + \sqrt{bc} + \sqrt{ca}\ ) + 3\sqrt[3]{abc}};$$

(b) $$x + y + z > \sqrt{a + b} + \sqrt{b + c} + \sqrt{c + a}.$$

2.138. If a, b, c and x, y, z are nonnegative real numbers, then

$$\frac{2}{(a + b)(x + y)} + \frac{2}{(b + c)(y + z)} + \frac{2}{(c + a)(z + x)} \ge \frac{9}{(b + c)x + (c + a)y + (a + b)z}.$$

2.139. Let a, b, c be the lengths of the sides of a triangle. If x, y, z are real numbers, then

$$(ya^2 + zb^2 + xc^2)(za^2 + xb^2 + yc^2) \ge (xy + yz + zx)(a^2b^2 + b^2c^2 + c^2a^2).$$

2.140. If $a_1 \ge a_2 \ge \cdots \ge a_8 \ge 0$, then

$$a_1 + a_2 + \cdots + a_8 - 8\sqrt[8]{a_1a_2\cdots a_8} \ge 3\left(\sqrt{a_6} - \sqrt{a_7}\right)^2.$$

2.141. Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Prove that

$$a_1b_1 + \cdots + a_nb_n + \sqrt{(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)} \ge \frac{2}{n}(a_1 + \cdots + a_n)(b_1 + \cdots + b_n).$$

2.142. Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 \ge 2a_2$. Prove that

$$(5n - 1)(a_1^2 + a_2^2 + \cdots + a_n^2) \ge 5(a_1 + a_2 + \cdots + a_n)^2.$$

2.143. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 \ge 4a_2$, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \ge \left(n + \frac{1}{2}\right)^2.$$

2.144. If $a_1 \ge a_2 \ge \cdots \ge a_n > 0$ such that $a_1 + a_2 + \cdots + a_n = n$, then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - n \ge \frac{4(n-1)^2}{n^3}(a_1 - a_2)^2.$$

2.145. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are real numbers such that

$$a_1 \le a_2 \le \cdots \le a_n, \quad a_1 + a_2 + \cdots + a_n = 0,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 + na_1a_n \le 0.$$

2.146. Let $a_1, a_2, \ldots, a_n$ $(n \ge 4)$ be nonnegative real numbers such that

$$a_1 \ge a_2 \ge \cdots \ge a_n$$

and

$$(a_1 + a_2 + \cdots + a_n)^2 = 4(a_1^2 + a_2^2 + \cdots + a_n^2).$$

Prove that

$$1 \le \frac{a_1 + a_2}{a_3 + a_4 + \cdots + a_n} \le 1 + \sqrt{\frac{2n-8}{n-2}}.$$

2.147. If $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$, then

(a) $$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{1}{3}\left(\sqrt{a_1} + \sqrt{a_2} - 2\sqrt{a_n}\right)^2;$$

(b) $$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{1}{4}\left(2\sqrt{a_1} - \sqrt{a_{n-1}} - \sqrt{a_n}\right)^2.$$

2.148. If $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0,\ n \ge 3$, then

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{n-1}{2n}\left(\sqrt{a_{n-2}} + \sqrt{a_{n-1}} - 2\sqrt{a_n}\right)^2.$$

2.149. Let $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$. If

$$\frac{n}{2} \le k \le n-1,$$

then

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{2k(n-k)}{n}\left(\sqrt{a_k} - \sqrt{a_{k+1}}\right)^2.$$

2.150. Let $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. If

$$1 \leq k < j \leq n, \qquad k+j \geq n+1,$$

then

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \geq \frac{2k(n-j+1)}{n+k-j+1}\left(\sqrt{a_k}-\sqrt{a_j}\right)^2.$$

2.151. If $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0,\ n \geq 4$, then

(a) $$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \geq \frac{1}{2}\left(1-\frac{1}{n}\right)\left(\sqrt{a_{n-2}}-3\sqrt{a_{n-1}}+2\sqrt{a_n}\right)^2;$$

(b) $$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \geq \left(1-\frac{2}{n}\right)\left(2\sqrt{a_{n-2}}-3\sqrt{a_{n-1}}+\sqrt{a_n}\right)^2.$$

2.2 Solutions

P 2.1. *If a, b are positive real numbers, then*

$$\frac{1}{4a^2+b^2}+\frac{3}{b^2+4ab}\geq\frac{16}{5(a+b)^2}.$$

Solution. Using the Cauchy-Schwarz inequality gives

$$\frac{1}{4a^2+b^2}+\frac{3}{b^2+4ab}\geq\frac{(1+3)^2}{(4a^2+b^2)+3(b^2+4ab)}=\frac{4}{a^2+b^2+3ab}.$$

Thus, we only need to show that

$$\frac{1}{a^2+b^2+3ab}\geq\frac{4}{5(a+b)^2},$$

which reduces to $(a-b)^2\geq 0$. The equality holds for $a=b$.

□

P 2.2. *If a, b are positive real numbers, then*

$$3a\sqrt{3a}+3b\sqrt{6a+3b}\geq 5(a+b)\sqrt{a+b}.$$

Solution. Due to homogeneity, we may assume that $a+b=3$. Thus, we need to show that

$$a\sqrt{a}+(3-a)\sqrt{3+a}\geq 5$$

for $0<a<3$. Substituting

$$\sqrt{a}=x, \qquad 0<x<\sqrt{3},$$

the inequality becomes

$$(3-x^2)\sqrt{3+x^2}\geq 5-x^3.$$

For $\sqrt[3]{5}\leq x<\sqrt{3}$, the inequality is trivial. For $0<x<\sqrt[3]{5}$, squaring both sides of the inequality gives

$$(3-x^2)(9-x^4)\geq(5-x^3)^2,$$

$$3x^4-10x^3+9x^2-2\leq 0,$$

$$(x-1)^2(3x^2-4x-2)\leq 0.$$

Since $3x^2-4x-2\leq 0$ for $\dfrac{2-\sqrt{10}}{3}\leq x\leq\dfrac{2+\sqrt{10}}{3}$, we only need to prove that

$$\sqrt[3]{5}\leq\frac{2+\sqrt{10}}{3}.$$

Indeed, we have

$$\left(\frac{2+\sqrt{10}}{3}\right)^3 - 5 = \frac{22\sqrt{10}-67}{27} > 0.$$

The equality holds for $a = b/2$.

□

P 2.3. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

$$(ab + c)(ac + b) \le 4.$$

Solution. By the AM-GM inequality, we have

$$(ab+c)(ac+b) \le \left[\frac{(ab+c)+(ac+b)}{2}\right]^2 = \frac{(a+1)^2(b+c)^2}{4}.$$

Therefore, it suffices to show that

$$(a+1)(b+c) \le 4.$$

Indeed,

$$(a+1)(b+c) \le \left[\frac{(a+1)+(b+c)}{2}\right]^2 = 4.$$

The equality holds for $a = b = c = 1$, for $a = 1$, $b = 0$, $c = 2$, and for $a = 1$, $b = 2$, $c = 0$.

□

P 2.4. *If a, b, c are nonnegative real numbers, then*

$$a^3 + b^3 + c^3 - 3abc \ge \frac{1}{4}(b + c - 2a)^3.$$

Solution. Write the inequality as

$$2(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] \ge (b+c-2a)^3.$$

Consider the non-trivial case $b + c - 2a \ge 0$. Since $(b-c)^2 \ge 0$ and

$$a + b + c \ge b + c - a,$$

it suffices to show that

$$2(a-b)^2 + 2(c-a)^2 \ge (b+c-2a)^2.$$

Indeed, we have

$$2(a-b)^2 + 2(c-a)^2 - (b+c-2a)^2 = (b-c)^2 \ge 0.$$

The equality holds for $a = b = c$, and also for $a = 0$ and $b = c$.

□

P 2.5. *Let a, b, c be nonnegative real numbers such that $a \ge b \ge c$. Prove that*

(a) $$a^3 + b^3 + c^3 - 3abc \ge 2(2b - a - c)^3;$$

(b) $$a^3 + b^3 + c^3 - 3abc \ge (a - 2b + c)^3.$$

Solution. (a) Write the inequality as

$$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \ge 2(2b - a - c)^3.$$

For the non-trivial case $2b - a - c \ge 0$, since

$$a + b + c \ge 2(2b - a - c),$$

it suffices to show that

$$a^2 + b^2 + c^2 - ab - bc - ca \ge (2b - a - c)^2.$$

This is equivalent to the obvious inequality

$$3(a - b)(b - c) \ge 0.$$

The equality holds for $a = b = c$, and also for $a = b$ and $c = 0$.

(b) Write the inequality as

$$(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) \ge (a - 2b + c)^3.$$

For the non-trivial case $a - 2b + c \ge 0$, since

$$a + b + c \ge a - 2b + c,$$

it suffices to show that

$$a^2 + b^2 + c^2 - ab - bc - ca \ge (a - 2b + c)^2,$$

which is equivalent to

$$3(a - b)(b - c) \ge 0.$$

The equality holds for $a = b = c$, and also for $b = c = 0$.

□

P 2.6. *Let a, b, c be nonnegative real numbers such that $a \ge b \ge c$. Prove that*

(a) $$a^3 + b^3 + c^3 - 3abc \ge 3(a^2 - b^2)(b - c);$$

(b) $$a^3 + b^3 + c^3 - 3abc \ge \frac{9}{2}(a - b)(b^2 - c^2).$$

Solution. (a) Write the inequality as

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \ge 3(a+b)(a-b)(b-c)).$$

Since

$$a+b+c \ge a+b,$$

it suffices to show that

$$a^2+b^2+c^2-ab-bc-ca \ge 3(a-b)(b-c).$$

Indeed,

$$a^2+b^2+c^2-ab-bc-ca-3(a-b)(b-c) = (a-2b+c)^2 \ge 0.$$

The equality holds for $a=b=c$, and also for $a=2b$ and $c=0$.

(b) Write the inequality as

$$(a+b+c)(a^2+b^2+c^2-ab-bc-ca) \ge \frac{9}{2}(a-b)(b-c)(b+c).$$

Since

$$a+b+c \ge \frac{3}{2}(b+c),$$

it suffices to show that

$$a^2+b^2+c^2-ab-bc-ca \ge 3(a-b)(b-c).$$

This is equivalent to the obvious inequality

$$(a-2b+c)^2 \ge 0.$$

The equality holds for $a=b=c$.

□

P 2.7. *If a,b,c are nonnegative real numbers such that*

$$c = \min\{a,b,c\}, \qquad a^2+b^2+c^2 = 3,$$

then

(a) $$5b+2c \le 9;$$

(b) $$5(b+c) \le 9+3a.$$

Solution. (a) It suffices to show that

$$5b + 2c + (a - c) \leq 9;$$

that is,

$$9 \geq a + 5b + c.$$

This follows from the Cauchy-Schwarz inequality

$$(1 + 25 + 1)(a^2 + b^2 + c^2) \geq (a + 5b + c)^2.$$

The equality holds for $a = c = \frac{1}{3}$ and $b = \frac{5}{3}$.

(b) It suffices to show that

$$5(b + c) + 4(a - c) \leq 9 + 3a;$$

that is,

$$9 \geq a + 5b + c.$$

As we have shown at (a), this follows from the Cauchy-Schwarz inequality

$$(1 + 25 + 1)(a^2 + b^2 + c^2) \geq (a + 5b + c)^2.$$

The equality holds for $a = c = \frac{1}{3}$ and $b = \frac{5}{3}$.

□

P 2.8. *Let a, b, c be nonnegative real numbers such that $a = \max\{a, b, c\}$. Prove that*

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 2(b^4 + c^4 + 4b^2c^2)(b - c)^2.$$

Solution. Because the inequality is symmetric in b and c, we may assume that $b \geq c$; that is,

$$a \geq b \geq c.$$

We will show that

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 2b^6 + c^6 - 3b^4c^2 \geq 2(b^4 + c^4 + 4b^2c^2)(b - c)^2.$$

The left inequality is equivalent to the obvious inequality

$$(a^2 - b^2)(a^4 + a^2b^2 + b^4 - 3b^2c^2) \geq 0.$$

The right inequality is equivalent to

$$(b^2 - c^2)^2(2b^2 + c^2) \geq 2(b^4 + c^4 + 4b^2c^2)(b - c)^2,$$

$$(b - c)^2[(b + c)^2(2b^2 + c^2) - 2(b^4 + c^4 + 4b^2c^2)] \geq 0,$$

$$c(b - c)^3(4b^2 - bc + c^2) \geq 0.$$

The equality holds for $a = b = c$, for $a = b$ and $c = 0$, and for $a = c$ and $b = 0$.

□

P 2.9. *Let a, b, c be nonnegative real numbers such that $a = \max\{a, b, c\}$. Prove that*

$$a^2 + b^2 + c^2 \geq \frac{9abc}{a+b+c} + \frac{5}{3}(b-c)^2.$$

Solution. Because the inequality is symmetric in b and c, we may assume that $b \geq c$, hence

$$a \geq b \geq c.$$

Write the inequality as follows:

$$(a^2 + b^2 + c^2)(a+b+c) - 9abc \geq \frac{5}{3}(a+b+c)(b-c)^2;$$

$$a^3 + b^3 + c^3 - 3abc + \sum a(b-c)^2 \geq \frac{5}{3}(a+b+c)(b-c)^2;$$

$$(a+b+c)\sum(b-c)^2 + 2\sum a(b-c)^2 \geq \frac{10}{3}(a+b+c)(b-c)^2.$$

It suffices to show that

$$(a+b+c)[(a-c)^2 + (b-c)^2] + 2a(b-c)^2 + 2b(a-c)^2 \geq \frac{10}{3}(a+b+c)(b-c)^2.$$

This inequality is true if

$$(a+b+c)[(b-c)^2 + (b-c)^2] + 2a(b-c)^2 + 2b(b-c)^2 \geq \frac{10}{3}(a+b+c)(b-c)^2.$$

Thus, we only need to show that

$$2(a+b+c) + 2a + 2b \geq \frac{10}{3}(a+b+c),$$

which reduces to $a + b - 2c \geq 0$. The equality holds for $a = b = c$.

□

P 2.10. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{16}{(b+c)^2} \geq \frac{6}{ab+bc+ca}.$$

(Vasile C., 2014)

Solution *(by Nguyen Van Quy)*. Since the equality holds for $a = 0$ and $b = c$, we write the desired inequality in the form

$$\frac{16}{(b+c)^2} + \left(\frac{1}{a+b} + \frac{1}{a+c}\right)^2 \ge \frac{6}{ab+bc+ca} + \frac{2}{(a+b)(a+c)}$$

and apply then the AM-GM inequality

$$\frac{16}{(b+c)^2} + \left(\frac{1}{a+b} + \frac{1}{a+c}\right)^2 \ge \frac{8}{b+c}\left(\frac{1}{a+b} + \frac{1}{a+c}\right).$$

Therefore, it suffices to show that

$$\frac{8}{b+c}\left(\frac{1}{a+b} + \frac{1}{a+c}\right) \ge \frac{6}{ab+bc+ca} + \frac{2}{(a+b)(a+c)}.$$

Since $(a+b)(a+c) \ge ab+bc+ca$, it is enough to show that

$$\frac{8}{b+c}\left(\frac{1}{a+b} + \frac{1}{a+c}\right) \ge \frac{8}{ab+bc+ca},$$

which is equivalent to

$$(2a+b+c)(ab+bc+ca) \ge (a+b)(b+c)(c+a).$$

We have

$$\begin{aligned}(2a+b+c)(ab+bc+ca) &\ge (a+b+c)(ab+bc+ca)\\ &\ge (a+b)(b+c)(c+a).\end{aligned}$$

This completes the proof. The equality holds for $a = 0$ and $b = c$.

□

P 2.11. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{2}{(b+c)^2} \ge \frac{5}{2(ab+bc+ca)}.$$

Solution. This inequality follows from Iran 1996 inequality (see P 1.72 in Volume 2, for $k = 2$), namely

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{1}{(b+c)^2} \ge \frac{9}{4(ab+bc+ca)},$$

and the inequality in P 2.10, namely

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{16}{(b+c)^2} \ge \frac{6}{ab+bc+ca}.$$

Indeed, summing the first inequality multiplied by 14 and the second inequality, we get the desired inequality. The equality holds for $a = 0$ and $b = c$.

□

P 2.12. *If a, b, c are nonnegative real numbers, no two of which are zero, then*

$$\frac{1}{(a+b)^2}+\frac{1}{(a+c)^2}+\frac{25}{(b+c)^2}\ge\frac{8}{ab+bc+ca}.$$

(Vasile C., 2014)

Solution. Write the inequality as

$$\left(\frac{1}{a+b}+\frac{1}{a+c}\right)^2+\frac{25}{(b+c)^2}\ge\frac{8}{ab+bc+ca}+\frac{2}{(a+b)(a+c)}.$$

By the AM-GM inequality, we have

$$\left(\frac{1}{a+b}+\frac{1}{a+c}\right)^2+\frac{25}{(b+c)^2}\ge\frac{10}{b+c}\left(\frac{1}{a+b}+\frac{1}{a+c}\right).$$

Therefore, it suffices to show that

$$\frac{10}{b+c}\left(\frac{1}{a+b}+\frac{1}{a+c}\right)\ge\frac{8}{ab+bc+ca}+\frac{2}{(a+b)(a+c)}.$$

Since $(a+b)(a+c)\ge ab+bc+ca$, it is enough to show that

$$\frac{10}{b+c}\left(\frac{1}{a+b}+\frac{1}{a+c}\right)\ge\frac{10}{ab+bc+ca},$$

which is equivalent to

$$(2a+b+c)(ab+bc+ca)\ge(a+b)(b+c)(c+a).$$

Indeed,

$$\begin{aligned}(2a+b+c)(ab+bc+ca)&\ge(a+b+c)(ab+bc+ca)\\&\ge(a+b)(b+c)(c+a).\end{aligned}$$

This completes the proof. The equality holds for $a=0$ and $\frac{b}{c}+\frac{c}{b}=3$.

□

P 2.13. *If a, b, c are positive real numbers, then*

$$(a+b)^3(a+c)^3\ge 4a^2bc(2a+b+c)^2.$$

(XZLBQ, 2014)

Solution *(by Nguyen Van Quy)*. Write the inequality as

$$\frac{(a+b)^2(a+c)^2}{4a^2bc} \geq \frac{(2a+b+c)^2}{(a+b)(a+c)}.$$

Since

$$\begin{aligned}(a+b)^2(a+c)^2 &= [(a-b)^2+4ab][(a-c)^2+4ac] \\ &\geq 4ac(a-b)^2+4ab(a-c)^2+16a^2bc,\end{aligned}$$

it suffices to show that

$$\frac{(a-b)^2}{ab}+\frac{(a-c)^2}{ac}+4 \geq \frac{(2a+b+c)^2}{(a+b)(a+c)},$$

which is equivalent to

$$\frac{(a-b)^2}{ab}+\frac{(a-c)^2}{ac} \geq \frac{(b-c)^2}{(a+b)(a+c)}.$$

Indeed, by the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{ab}+\frac{(a-c)^2}{ac} \geq \frac{(a-b-a+c)^2}{ab+ac} \geq \frac{(b-c)^2}{(a+b)(a+c)}.$$

The equality holds for $a=b=c$.

□

P 2.14. *If a,b,c are positive real numbers such that $abc=1$, then*

(a)
$$\frac{a}{b}+\frac{b}{c}+\frac{1}{a} \geq a+b+1;$$

(b)
$$\frac{a}{b}+\frac{b}{c}+\frac{1}{a} \geq \sqrt{3(a^2+b^2+1)}.$$

(Vasile C., 2007)

Solution. (a) ***First Solution.*** Write the inequality as

$$\left(2\frac{a}{b}+\frac{b}{c}\right)+\left(\frac{b}{c}+\frac{1}{a}\right)+\left(\frac{1}{a}+a\right) \geq 3a+2b+2.$$

By the AM-GM inequality, we have

$$\left(2\frac{a}{b}+\frac{b}{c}\right)+\left(\frac{b}{c}+\frac{1}{a}\right)+\left(\frac{1}{a}+a\right) \geq 3\sqrt[3]{\frac{a^2}{bc}}+2\sqrt{\frac{b}{ca}}+2=3a+2b+2.$$

The equality holds for $a=b=c=1$.

Second Solution. Since $c = \dfrac{1}{ab}$, the inequality becomes as follows:

$$\frac{a}{b} + ab^2 + \frac{1}{a} \ge a + b + 1,$$

$$\frac{1}{b} + b^2 + \frac{1}{a^2} \ge 1 + \frac{b}{a} + \frac{1}{a},$$

$$\frac{1}{a^2} - (b+1)\frac{1}{a} + b^2 + \frac{1}{b} - 1 \ge 0,$$

$$\left(\frac{1}{a} - \frac{b+1}{2}\right)^2 + \frac{(b-1)^2(3b+4)}{4b} \ge 0.$$

(b) Write the inequality as

$$a\left(\frac{1}{b} + b^2\right) + \frac{1}{a} \ge \sqrt{3(a^2+b^2+1)}.$$

By squaring, this inequality becomes

$$a^2\left(b^4 + 2b - 3 + \frac{1}{b^2}\right) + \frac{1}{a^2} \ge b^2 + 3 - \frac{2}{b}.$$

Since

$$b^4 + 2b - 3 + \frac{1}{b^2} > 2b - 3 + \frac{1}{b^2} = \frac{(b-1)^2(2b+1)}{b^2} \ge 0,$$

by the AM-GM inequality, we have

$$a^2\left(b^4 + 2b - 3 + \frac{1}{b^2}\right) + \frac{1}{a^2} \ge 2\sqrt{b^4 + 2b - 3 + \frac{1}{b^2}}.$$

Thus, it suffices to prove that

$$2\sqrt{b^4 + 2b - 3 + \frac{1}{b^2}} \ge b^2 + 3 - \frac{2}{b}.$$

Squaring again, we get the inequality

$$b^5 - 2b^3 + 4b^2 - 7b + 4 \ge 0,$$

which is equivalent to the obvious inequality

$$b(b^2-1)^2 + 4(b-1)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.15. *If a, b, c are positive real numbers such that $abc \ge 1$, then*

$$a^{\frac{a}{b}} b^{\frac{b}{c}} c^c \ge 1.$$

(Vasile C., 2011)

Solution. Write the inequality as

$$\frac{a}{b}\ln a + \frac{b}{c}\ln b + c\ln c \ge 0.$$

Since $f(x) = x\ln x$ is a convex function on $(0, \infty)$, apply Jensen's inequality to get

$$pa\ln a + qb\ln b + rc\ln c \ge (p+q+r)\left(\frac{pa+qb+rc}{p+q+r}\right)\ln\left(\frac{pa+qb+rc}{p+q+r}\right)$$

$$= (pa+qb+rc)\ln\left(\frac{pa+qb+rc}{p+q+r}\right),$$

where $p, q, r > 0$. Choosing

$$p = \frac{1}{b}, \quad q = \frac{1}{c}, \quad r = 1,$$

we get

$$\frac{a}{b}\ln a + \frac{b}{c}\ln b + c\ln c \ge (\frac{a}{b} + \frac{b}{c} + c)\ln\left(\frac{\frac{a}{b} + \frac{b}{c} + c}{\frac{1}{b} + \frac{1}{c} + 1}\right).$$

Thus, it suffices to show that

$$\frac{a}{b} + \frac{b}{c} + c \ge \frac{1}{b} + \frac{1}{c} + 1.$$

Since $a \ge \frac{1}{bc}$, we need to show that

$$\frac{1}{b^2c} + \frac{b}{c} + c \ge \frac{1}{b} + \frac{1}{c} + 1.$$

This is equivalent to

$$\frac{1}{b^2} + b + c^2 \ge \frac{c}{b} + 1 + c,$$

$$c^2 - \left(1 + \frac{1}{b}\right)c + b - 1 + \frac{1}{b^2} \ge 0,$$

$$\left(c - \frac{b+1}{2b}\right)^2 + \frac{(b-1)^2(4b+3)}{4b^2} \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.16. *If a,b,c are positive real numbers such that $ab+bc+ca=3$, then*

$$ab^2c^3 < 4.$$

(Vasile C., 2012)

Solution. From $ab+bc+ca=3$, we get

$$c=\frac{3-ab}{a+b}<\frac{3}{a+b}.$$

Therefore,

$$\begin{aligned}(a+b)^3(4-ab^2c^3) &> 4(a+b)^3-27ab^2\\ &= 4a^3+12a^2b-15ab^2+4b^3\\ &= (a+4b)(2a-b)^2 \geq 0.\end{aligned}$$

□

P 2.17. *If a,b,c are positive real numbers such that $ab+bc+ca=\frac{5}{3}$, then*

$$ab^2c^2 \leq \frac{1}{3}.$$

(Vasile C., 2012)

Solution. By the AM-GM inequality, we have

$$ab+ca \geq 2a\sqrt{bc}.$$

Thus, from $ab+bc+ca=\frac{5}{3}$, we get

$$2a\sqrt{bc}+bc \leq \frac{5}{3}.$$

Therefore, it suffices to show that

$$\frac{(5-3bc)b^2c^2}{6\sqrt{bc}} \leq \frac{1}{3}.$$

Setting $\sqrt{bc}=t$, this inequality becomes

$$3t^5-5t^3+2 \geq 0.$$

Indeed, be the AM-GM inequality, we have

$$3t^5+2=t^5+t^5+t^5+1+1 \geq 5\sqrt[5]{t^5\cdot t^5\cdot t^5\cdot 1\cdot 1}=5t^3.$$

The equality holds for $a=\frac{1}{3}$ and $b=c=1$.

□

P 2.18. *Let a, b, c be positive real numbers such that*

$$a \le b \le c, \qquad ab + bc + ca = 3.$$

Prove that

(a) $$ab^2c \le \frac{9}{8};$$

(b) $$ab^4c \le 2;$$

(c) $$ab^3c^2 \le 2.$$

(Vasile C., 2012)

Solution. From $(b-a)(b-c) \le 0$, we get

$$b^2 + ac \le b(a+c),$$

$$b^2 + ac \le 3 - ac,$$

$$b^2 + 2ac \le 3.$$

(a) We have

$$9 - 8ab^2c \ge 9 - 4b^2(3-b^2) = (2b^2-3)^2 \ge 0.$$

The equality holds for $a = \frac{1}{2}\sqrt{\frac{3}{2}}$ and $b = c = \sqrt{\frac{3}{2}}$.

(b) We have

$$4 - 2ab^4c \ge 4 - b^4(3-b^2) = (b^2-2)^2(b^2+1) \ge 0.$$

The equality holds for $a = \frac{\sqrt{2}}{4}$ and $b = c = \sqrt{2}$.

(c) Write the desired inequality as follows:

$$2(ab+bc+ca)^3 \ge 27ab^3c^2,$$

$$2\left(a + c + \frac{ca}{b}\right)^3 \ge 27ac^2.$$

Since $ca/b \ge a$, it suffices to show that

$$2(2a+c)^3 \ge 27ac^2,$$

which is equivalent to the obvious inequality

$$(a+2c)(4a-c)^2 \ge 0.$$

The equality holds for $a = \frac{\sqrt{2}}{4}$ and $b = c = \sqrt{2}$.

□

P 2.19. *Let a,b,c be positive real numbers such that*

$$a \le b \le c, \qquad a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Prove that

$$b \ge \frac{1}{a+c-1}.$$

(Vasile C., 2007)

Solution. Let us show that

$$a \le 1, \qquad c \ge 1.$$

From $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ and

$$a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-6=\frac{(a-1)^2}{a}+\frac{(b-1)^2}{b}+\frac{(c-1)^2}{c} \ge 0,$$

we get

$$a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge 3.$$

Then,

$$\frac{1}{a} \ge \frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 1, \qquad c \ge \frac{a+b+c}{3} \ge 1.$$

Further, consider the following two cases.

Case 1: $abc \ge 1$. Write the desired inequality as

$$a+c-1-\frac{1}{b} \ge 0.$$

We have

$$a+c-1-\frac{1}{b}=(1-a)(c-1)+\frac{abc-1}{b} \ge 0.$$

Case 2: $abc \le 1$. Since

$$a+c-1-\frac{1}{b}=\frac{1}{a}+\frac{1}{c}-1-b,$$

the desired inequality is equivalent to

$$\frac{1}{a}+\frac{1}{c}-1-b \ge 0.$$

We have

$$\frac{1}{a}+\frac{1}{c}-1-b=\left(\frac{1}{a}-1\right)\left(1-\frac{1}{c}\right)+\frac{1-abc}{ac} \ge 0.$$

This completes the proof. The equality holds for $a=b=c=1$.

□

P 2.20. *Let a, b, c be positive real numbers such that*

$$a \le b \le c, \qquad a+b+c = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Prove that

$$ab^2c^3 \ge 1.$$

(Vasile C., 1998)

First Solution. Write the inequality in the homogeneous form

$$ab^2c^3 \ge \left[\frac{abc(a+b+c)}{ab+bc+ca}\right]^3,$$

which is equivalent to

$$(ab+bc+ca)^3 \ge a^2b(a+b+c)^3.$$

Since

$$(ab+bc+ca)^2 \ge 3abc(a+b+c),$$

it suffices to show that

$$3c(ab+bc+ca) \ge a(a+b+c)^2.$$

Indeed,

$$3c(ab+bc+ca) - a(a+b+c)^2 \ge (a+b+c)(ab+bc+ca) - a(a+b+c)^2$$

$$= (a+b+c)(bc-a^2) \ge 0.$$

The equality holds for $a=b=c=1$.

Second Solution. Let us show that

$$a \le 1, \qquad bc \ge 1.$$

Indeed, if $a > 1$, then $1 < a \le b \le c$ and

$$a+b+c-\frac{1}{a}-\frac{1}{b}-\frac{1}{c} = \frac{1-a^2}{a}+\frac{1-b^2}{b}+\frac{1-c^2}{c} < 0,$$

which is false. On the other hand, from $a \le 1$ and

$$a-\frac{1}{a} = (b+c)\left(\frac{1}{bc}-1\right),$$

we get $bc \ge 1$. Similarly, we can prove that

$$c \ge 1, \qquad ab \le 1.$$

Since $bc \ge 1$, it suffices to show that

$$abc^2 \ge 1.$$

Taking account of $ab \le 1$, we have

$$c - \frac{1}{c} = (a+b)\left(\frac{1}{ab} - 1\right) \ge 2\sqrt{ab}\left(\frac{1}{ab} - 1\right) = 2\left(\frac{1}{\sqrt{ab}} - \sqrt{ab}\right) \ge \frac{1}{\sqrt{ab}} - \sqrt{ab},$$

hence

$$\left(c - \frac{1}{\sqrt{ab}}\right)\left(1 + \frac{\sqrt{ab}}{c}\right) \ge 0.$$

The last inequality involves

$$abc^2 \ge 0.$$

□

P 2.21. *Let a, b, c be positive real numbers such that*

$$a \le b \le c, \qquad a + b + c = abc + 2.$$

Prove that

$$(1-b)(1-ab^3c) \ge 0.$$

(Vasile C., 1999)

Solution. Let us show that

$$a \le 1, \qquad c \ge 1.$$

To do this, we write the hypothesis $a + b + c = abc + 2$ in the equivalent form

$$(1-a)(1-c) + (1-b)(1-ac) = 0, \tag{*}$$

If $a > 1$, then $1 < a \le b \le c$, which contradicts (*). Similarly, if $c < 1$, then $a \le b \le c < 1$, which also contradicts (*). Therefore, we have $a \le 1$ and $c \ge 1$. According to (*), we get

$$(1-b)(1-ac) = (1-a)(c-1) \ge 0. \tag{**}$$

There are two cases to consider.

Case 1: $b \ge 1$. According to (**), we have $ac \ge 1$. Therefore,

$$ab^3c = ac \cdot b^3 \ge 1,$$

hence $(1-b)(1-ab^3c) \ge 0$.

Case 2: $b \le 1$. According to (**), we have $ac \le 1$. Therefore,

$$ab^3c = ac \cdot b^3 \le 1,$$

and hence

$$(1-b)(1-ab^3c) \ge 0.$$

This completes the proof. The equality holds for $a = b = 1 \le c$ or $a \le 1 = b = c$.

□

P 2.22. *Let a, b, c be real numbers, no two of which are zero. Prove that*

(a) $$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{2(b^2+c^2)};$$

(b) $$\frac{(a+b)^2}{a^2+b^2}+\frac{(a+c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{2(b^2+c^2)}.$$

Solution. (a) Consider two cases.
Case 1: $2a^2 \le b^2+c^2$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2)}{a^2+c^2} \ge \frac{[(b-a)+(a-c)]^2}{(a^2+b^2)+(a^2+c^2)} = \frac{(b-c)^2}{2a^2+b^2+c^2}.$$

Thus, it suffices to show that

$$\frac{1}{2a^2+b^2+c^2} \ge \frac{1}{2(b^2+c^2)},$$

which reduces to $b^2+c^2 \ge 2a^2$.
Case 2: $2a^2 \ge b^2+c^2$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{[c(b-a)+b(a-c)]^2}{c^2(a^2+b^2)+b^2(a^2+c^2)} = \frac{a^2(b-c)^2}{a^2(b^2+c^2)+2b^2c^2}.$$

Therefore, it suffices to prove that

$$\frac{a^2}{a^2(b^2+c^2)+2b^2c^2} \ge \frac{1}{2(b^2+c^2)},$$

which reduces to $a^2(b^2+c^2) \ge 2b^2c^2$. This is true since

$$2a^2(b^2+c^2)-4b^2c^2 \ge (b^2+c^2)^2-4b^2c^2 = (b^2-c^2)^2.$$

The equality holds for $a=b=c$.

(b) The inequality follows from the inequality in (a) by replacing a with $-a$. The equality holds for $-a=b=c$.

□

P 2.23. *Let a, b, c be real numbers, no two of which are zero. If $bc \ge 0$, then*

(a) $$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{(b+c)^2};$$

(b) $$\frac{(a+b)^2}{a^2+b^2}+\frac{(a+c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{(b+c)^2}.$$

(Vasile C., 2011)

Solution. (a) Consider two cases: $a^2 \le bc$ and $a^2 \ge bc$.
Case 1: $a^2 \le bc$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2)}{a^2+c^2} \ge \frac{[(b-a)+(a-c)]^2}{(a^2+b^2)+(a^2+c^2)} = \frac{(b-c)^2}{2a^2+b^2+c^2}.$$

Thus, it suffices to show that

$$\frac{1}{2a^2+b^2+c^2} \ge \frac{1}{(b+c)^2},$$

which is equivalent to $a^2 \le bc$.
Case 2: $a^2 \ge bc$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{[c(b-a)+b(a-c)]^2}{c^2(a^2+b^2)+b^2(a^2+c^2)} = \frac{a^2(b-c)^2}{a^2(b^2+c^2)+2b^2c^2}.$$

Therefore, it suffices to prove that

$$\frac{a^2}{a^2(b^2+c^2)+2b^2c^2} \ge \frac{1}{(b+c)^2},$$

which reduces to $bc(a^2-bc) \ge 0$. The equality holds for $a=b=c$, for $b=0$ and $a=c$, and for $c=0$ and $a=b$.

(b) The inequality follows from the inequality in (a) by replacing a with $-a$. The equality holds for $-a=b=c$, for $b=0$ and $a+c=0$, and for $c=0$ and $a+b=0$.

□

P 2.24. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{|a-b|^3}{a^3+b^3}+\frac{|a-c|^3}{a^3+c^3} \ge \frac{|b-c|^3}{(b+c)^3}.$$

(Vasile C., 2013)

Solution. Without loss of generality, assume that $b \ge c$. Thus, we have three cases to consider: $a \ge b \ge c$, $b \ge c \ge a$ and $b \ge a \ge c$.
Case 1: $a \ge b \ge c$. It suffices to show that

$$\frac{|a-c|^3}{(a+c)^3} \ge \frac{|b-c|)^3}{(b+c)^3},$$

which is equivalent to

$$\frac{a-c}{a+c} \ge \frac{b-c}{b+c}.$$

Indeed,

$$\frac{a-c}{a+c}-\frac{b-c}{b+c}=\frac{2c(a-b)}{(a+c)(b+c)}\geq 0.$$

Case 2: $b\geq c\geq a$. It suffices to show that

$$\frac{(b-a)^3}{a^3+b^3}\geq\frac{(b-c)^3}{(b+c)^3}.$$

Indeed,

$$\frac{(b-a)^3}{a^3+b^3}\geq\frac{(b-c)^3}{a^3+b^3}\geq\frac{(b-c)^3}{b^3+c^3}\geq\frac{b-c)^3}{(b+c)^3}.$$

Case 3: $b\geq a\geq c$. We need to prove that

$$\frac{(b-a)^3}{a^3+b^3}+\frac{(a-c)^3}{a^3+c^3}\geq\frac{(b-c)^3}{(b+c)^3}.$$

Using the substitution

$$x=\frac{b-a}{a+b},\quad y=\frac{a-c}{a+c},\quad 0\leq x<1,\quad 0\leq y\leq 1,$$

we have

$$b=\frac{1+x}{1-x}a,\quad c=\frac{1-y}{1+y}a,$$

$$(b-a)^3=\frac{8x^3}{(1-x)^3}a^3,\quad (a-c)^3=\frac{8y^3}{(1+y)^3}a^3,$$

$$a^3+b^3=\frac{2(1+3x^3)}{(1-x)^3},\quad a^3+c^3=\frac{2(1+3y^2)}{(1+y)^3},$$

$$\frac{b-c}{b+c}=\frac{x+y}{1+xy}.$$

Thus, the desired inequality becomes

$$\frac{4x^3}{1+3x^2}+\frac{4y^3}{1+3y^2}\geq\frac{(x+y)^3}{(1+xy)^3},$$

$$\frac{x^2+y^2-xy+3x^2y^2}{(1+3x^2)(1+3y^2)}\geq\frac{(x+y)^2}{4(1+xy)^3},$$

$$\frac{s-p+3p^2}{1+3s+9p^2}\geq\frac{s+2p}{4(1+p)^3},$$

where

$$s=x^2+y^2,\quad p=xy,\quad 0\leq p<1,\quad 2p\leq s\leq 1+p^2.$$

Therefore, we need to show that $f(s)\geq 0$, where

$$f(s)=4(1+p)^3(s-p+3p^2)-(s+2p)(3s+1+9p^2).$$

Since f is a concave function, it suffices to show that $f(2p) \geq 0$ and $f(1+p^2) \geq 0$. Indeed, we have

$$f(2p) = 4p^3(3p+1)(p+3) \geq 0,$$

$$f(1+p^2) = 16p^3(p+1)^2 \geq 0.$$

Thus, the proof is completed. The equality holds for $a = b = c$, for $b = 0$ and $a = c$, and for $c = 0$ and $a = b$.

□

P 2.25. *Let a, b, c be positive real numbers, $b \neq c$. Prove that*

$$\frac{ab}{(a+b)^2} + \frac{ac}{(a+c)^2} \leq \frac{(b+c)^2}{4(b-c)^2}.$$

(Vasile C., 2010)

Solution. Write the inequality in the form

$$\frac{(a-b)^2}{(a+b)^2} + \frac{(a-c)^2}{(a+c)^2} + \frac{(b+c)^2}{(b-c)^2} \geq 2.$$

Replacing a be $-a$, the inequality becomes

$$\frac{(a+b)^2}{(a-b)^2} + \frac{(a+c)^2}{(a-c)^2} + \frac{(b+c)^2}{(b-c)^2} \geq 2. \qquad (*)$$

Making the substitution

$$x = \frac{a+b}{a-b}, \quad y = \frac{b+c}{b-c}, \quad z = \frac{c+a}{c-a},$$

we can write the inequality as

$$x^2 + y^2 + z^2 \geq 2.$$

From

$$x+1 = \frac{2a}{a-b}, \quad y+1 = \frac{2b}{b-c}, \quad z+1 = \frac{2c}{c-a}$$

and

$$x-1 = \frac{2b}{a-b}, \quad y-1 = \frac{2c}{b-c}, \quad z-1 = \frac{2a}{c-a},$$

we get

$$(x+1)(y+1)(z+1) = (x-1)(y-1)(z-1),$$

$$xy + yz + zx + 1 = 0.$$

Therefore, we have

$$x^2 + y^2 + z^2 - 2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = (x+y+z)^2 \geq 0.$$

The inequality (*) is an equality for $x+y+z=0$; that is,

$$(a+b+c)(ab+bc+ca)-9abc=0.$$

Therefore, the original inequality is an equality for

$$(b+c-a)(bc-ab-ac)+9abc=0.$$

□

P 2.26. *Let a,b,c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{3bc+a^2}{b^2+c^2} \ge \frac{3ab-c^2}{a^2+b^2}+\frac{3ac-b^2}{a^2+c^2}.$$

(Vasile C., 2014)

Solution *(by Nguyen Van Quy)*. Write the inequality as

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{a^2+c^2}+\frac{c^2}{a^2+b^2}+\frac{3bc}{b^2+c^2} \ge \frac{3ab}{a^2+b^2}+\frac{3ac}{a^2+c^2}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{b^2}{a^2+c^2}+\frac{c^2}{a^2+b^2} \ge \frac{(b^2+c^2)^2}{b^2(a^2+c^2)+c^2(a^2+b^2)} = \frac{(b^2+c^2)^2}{a^2(b^2+c^2)+2b^2c^2}$$
$$\ge 2-\frac{a^2(b^2+c^2)+2b^2c^2}{(b^2+c^2)^2} = 2-\frac{a^2}{b^2+c^2}-\frac{2b^2c^2}{(b^2+c^2)^2},$$

hence

$$\frac{a^2}{b^2+c^2}+\frac{b^2}{a^2+c^2}+\frac{c^2}{a^2+b^2} \ge 2-\frac{2b^2c^2}{(b^2+c^2)^2}.$$

Therefore, it suffices to show that

$$2-\frac{2b^2c^2}{(b^2+c^2)^2}+\frac{3bc}{b^2+c^2} \ge \frac{3ab}{a^2+b^2}+\frac{3ac}{a^2+c^2}.$$

This inequality is equivalent to

$$\left[\frac{1}{2}-\frac{2b^2c^2}{(b^2+c^2)^2}\right]+\left(\frac{3}{2}-\frac{3ab}{a^2+b^2}\right)+\left(\frac{3}{2}-\frac{3ac}{a^2+c^2}\right) \ge \left(\frac{3}{2}-\frac{3bc}{b^2+c^2}\right),$$

$$\frac{(b^2-c^2)^2}{3(b^2+c^2)^2}+\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{b^2+c^2}.$$

Using the inequality in P 2.23-(a), namely

$$\frac{(a-b)^2}{a^2+b^2}+\frac{(a-c)^2}{a^2+c^2} \ge \frac{(b-c)^2}{(b+c)^2},$$

it is enough to prove that

$$\frac{(b+c)^2}{3(b^2+c^2)^2}+\frac{1}{(b+c)^2}\ge\frac{1}{b^2+c^2},$$

which is equivalent to

$$\frac{1}{(b+c)^2}\ge\frac{2(b^2-bc+c^2)}{3(b^2+c^2)^2}.$$

We have

$$\begin{aligned}3(b^2+c^2)^2-2(b+c)^2(b^2-bc+c^2)&=3(b^2+c^2)^2-2(b+c)(b^3+c^3)\\&=b^4+c^4+6b^2c^2-2bc(b^2+c^2)\\&\ge(b^2+c^2)^2-2bc(b^2+c^2)\\&=(b^2+c^2)(b-c)^2\ge0.\end{aligned}$$

The equality holds for $a=b=c$.

□

P 2.27. *Let a,b,c be nonnegative real numbers such that $a+b>0$. Prove that*

$$abc\ge(b+c-a)(c+a-b)(a+b-c)+\frac{ab(a-b)^2}{a+b}.$$

(Vasile C., 2011)

Solution. Since

$$(b+c-a)(c+a-b)(a+b-c)=\frac{2(a^2b^2+b^2c^2+c^2a^2)-a^4-b^4-c^4}{a+b+c},$$

we can rewrite the inequality as

$$a^4+b^4+c^4+abc(a+b+c)\ge2(a^2b^2+b^2c^2+c^2a^2)+\frac{ab(a+b+c)(a-b)^2}{a+b}.$$

By Schur's inequality of fourth degree, we have

$$a^4+b^4+c^4+abc(a+b+c)\ge\sum ab(a^2+b^2).$$

Therefore, it suffices to prove that

$$\sum ab(a^2+b^2)\ge2(a^2b^2+b^2c^2+c^2a^2)+\frac{ab(a+b+c)(a-b)^2}{a+b},$$

which is equivalent to

$$\sum ab(a-b)^2\ge\frac{ab(a+b+c)(a-b)^2}{a+b},$$

or

$$bc(b-c)^2+ca(c-a)^2 \ge \frac{abc(a-b)^2}{a+b}.$$

This inequality follows immediately from the Cauchy-Schwarz inequality

$$(a+b)[bc(b-c)^2+ca(c-a)^2] \ge [\sqrt{abc}(b-c)+\sqrt{abc}(c-a)]^2.$$

The equality holds for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 2.28. *Let a,b,c be nonnegative real numbers such that $a \ge b \ge c$. Prove that*

(a) $$abc \ge (b+c-a)(c+a-b)(a+b-c)+\frac{2ab(a-b)^2}{a+b};$$

(b) $$abc \ge (b+c-a)(c+a-b)(a+b-c)+\frac{27b(a-b)^4}{4a^2}.$$

(Vasile C., 2011)

Solution. (a) Write the inequality as

$$\sum a(a-b)(a-c) \ge \frac{2ab(a-b)^2}{a+b}.$$

Since

$$c(c-a)(c-b) \ge 0,$$

it suffices to show that

$$a(a-b)(a-c)+b(b-c)(c-a) \ge \frac{2ab(a-b)^2}{a+b}.$$

Since

$$a(a-b)(a-c)=a(a-b)[(a-b)+(b-c)]=a(a-b)^2+a(a-b)(b-c)$$

$$\ge \frac{2ab(a-b)^2}{a+b}+a(a-b)(b-c),$$

it suffices to show that

$$a(a-b)(b-c)+b(b-c)(b-a) \ge 0.$$

This inequality is equivalent to

$$(a-b)^2(b-c) \ge 0.$$

The equality holds for $a=b=c$, and for $a=b$ and $c=0$.

(b) Write the inequality as

$$\sum a(a-b)(a-c) \geq \frac{27b(a-b)^4}{4a^2}.$$

Since

$$c(c-a)(c-b) \geq 0,$$

it suffices to show that

$$a(a-b)(a-c)+b(b-c)(c-a) \geq \frac{27b(a-b)^4}{4a^2},$$

which is equivalent to

$$a(a-b)^2+a(a-b)(b-c)+b(b-c)(c-a) \geq \frac{27b(a-b)^4}{4a^2}.$$

Since

$$a(a-b)^2-\frac{27b(a-b)^4}{4a^2}=\frac{(a-b)^2(a-3b)^2}{4a^2},$$

it suffices to show that

$$a(a-b)(b-c)+b(b-c)(b-a) \geq 0.$$

This inequality is equivalent to

$$(a-b)^2(b-c) \geq 0.$$

The equality holds for $a=b=c$, and for $a/3=b=c$.

□

P 2.29. *Let a,b,c be nonnegative real numbers such that $a+b>0$. Prove that*

$$\sum a^2(a-b)(a-c) \geq a^2b^2\left(\frac{a-b}{a+b}\right)^2.$$

(Vasile C., 2011)

Solution. Without loss of generality, assume that $a \geq b$. There three cases to consider.

Case 1. $c \geq a \geq b$. Since

$$\begin{aligned} a^2(a-b)(a-c)+c^2(c-a)(c-b) &\geq a^2(a-b)(a-c)+c^2(c-a)(a-b) \\ &= (a-b)(c-a)^2(c+a) \geq 0, \end{aligned}$$

it suffices to show that

$$b^2(a-b)(c-b) \geq a^2b^2\left(\frac{a-b}{a+b}\right)^2.$$

Since $c-b \geq a-b$, this is true if

$$1 \geq \left(\frac{a}{a+b}\right)^2,$$

which is true.

Case 2. $a \geq b \geq c$. Since

$$c^2(c-a)(c-b) \geq 0$$

and

$$a^2(a-b)(a-c)+b^2(b-c)(b-a) = (a-b)^2[a^2+ab+b^2-c(a+b)]$$

$$\geq (a-b)^2[a^2+ab+b^2-b(a+b)] = a^2(a-b)^2,$$

it suffices to show that

$$1 \geq \left(\frac{b}{a+b}\right)^2,$$

which is true.

Case 3. $a \geq c \geq b$. Since

$$b^2(b-c)(b-a) \geq b^2(c-b)^2$$

and

$$a^2(a-b)(a-c)+c^2(c-a)(c-b) = (a-c)^2[a^2+ac+c^2-b(a+c)]$$

$$\geq (a-c)^2[a^2+ac+c^2-c(a+c)] = a^2(a-c)^2,$$

it suffices to show that

$$b^2(c-b)^2+a^2(a-c)^2 \geq a^2b^2\left(\frac{a-b}{a+b}\right)^2.$$

By the Cauchy-Schwarz inequality, we have

$$\left(\frac{1}{b^2}+\frac{1}{a^2}\right)\left[b^2(c-b)^2+a^2(a-c)^2\right] \geq [(c-b)+(a-c)]^2 = (a-b)^2.$$

Therefore, it suffices to prove that

$$\frac{a^2b^2(a-b)^2}{a^2+b^2} \geq a^2b^2\left(\frac{a-b}{a+b}\right)^2,$$

which is clearly true.

This completes the proof. The equality holds for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

• *Let a,b,c be nonnegative real numbers such that $a+b>0$. If k is a positive natural number, then*

$$\sum a^k(a-b)(a-c)\ge\left(\frac{ab}{a+b}\right)^k(a-b)^2.$$

□

P 2.30. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$ab^2+bc^2+2ca^2\le 8.$$

Solution. Since the equality holds for $a=2$, $b=0$, $c=1$, we apply the AM-GM inequality to get

$$\frac{ca^2}{4}=c\cdot\frac{a}{2}\cdot\frac{a}{2}\le\frac{1}{27}\left(c+\frac{a}{2}+\frac{a}{2}\right)^3=\frac{1}{27}(c+a)^3\le\frac{1}{27}(a+b+c)^3=1.$$

Therefore, it suffices to show that

$$ab^2+bc^2+ca^2\le 4,$$

which is the inequality in P 1.1.

□

P 2.31. *Let a,b,c be nonnegative real numbers such that $a+b+c=3$. Prove that*

$$ab^2+bc^2+\frac{3}{2}abc\le 4.$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2007)

Solution. Consider two cases.

Case 1: $c\ge 2b$. We have

$$ab^2+bc^2+\frac{3}{2}abc=b(a+c)^2-ab\left(a-b+\frac{c}{2}\right)\le b(a+c)^2$$

$$=4b\left(\frac{a+c}{2}\right)\left(\frac{a+c}{2}\right)\le 4\left(\frac{b+\frac{a+c}{2}+\frac{a+c}{2}}{3}\right)^3=4.$$

Case 2: $2b > c$. Write the desired inequality as $f(a) \ge 0$, where

$$f(a) = 4\left(\frac{a+b+c}{3}\right)^3 - ab^2 - bc^2 - \frac{3}{2}abc,$$

with the derivative

$$f'(a) = 4\left(\frac{a+b+c}{3}\right)^2 - b^2 - \frac{3}{2}bc.$$

The equation $f'(a) = 0$ has the positive root

$$a_1 = \frac{3}{2}\sqrt{\frac{b(2b+3c)}{2}} - b - c = \frac{(2b-c)(5b+8c)}{6\sqrt{2b(2b+c)} + 8(b+c)}.$$

Since $f'(a) < 0$ for $0 \le a < a_1$ and $f'(a) > 0$ for $a > a_1$, $f(a)$ is decreasing on $[0, a_1]$ and increasing on $[a_1, \infty)$; consequently, $f(a) \ge f(a_1)$. To complete the proof, it suffices to show that $f(a_1) \ge 0$. Indeed, since

$$4\left(\frac{a_1+b+c}{3}\right)^2 = b^2 + \frac{3}{2}bc,$$

we have

$$\begin{aligned}
f(a_1) &= 4\left(\frac{a_1+b+c}{3}\right)^3 - a_1\left(b^2 + \frac{3}{2}bc\right) - bc^2 \\
&= \frac{a_1+b+c}{3}\left(b^2 + \frac{3}{2}bc\right) - a_1\left(b^2 + \frac{3}{2}bc\right) - bc^2 \\
&= \frac{b+c-2a_1}{3}\left(b^2 + \frac{3}{2}bc\right) - bc^2 \\
&= \left(b + c - \sqrt{\frac{2b^2+3bc}{2}}\right)\left(b^2 + \frac{3}{2}bc\right) - bc^2 \\
&= \frac{b}{4}\left[4b^2 + 10bc + 2c^2 - (2b+3c)\sqrt{2b(2b+3c)}\right] \\
&= \frac{bc(2b-c)^2(b+2c)}{2[4b^2 + 10bc + 2c^2 + (2b+3c)\sqrt{2b(2b+3c)}]} \ge 0.
\end{aligned}$$

Thus, the proof is completed. The equality holds for $a = 0$, $b = 1$, $c = 2$, and for $a = 1$, $b = 2$, $c = 0$.

□

P 2.32. *Let a, b, c be nonnegative real numbers such that $a + b + c = 5$. Prove that*

$$ab^2 + bc^2 + 2abc \le 20.$$

(Vo Quoc Ba Can, 2011)

Solution. Write the inequality as

$$b(ab+c^2+2ac) \le 20.$$

We see that the equality holds for $a=1$ and $b=c=2$. From $(a-b/2)^2 \ge 0$, it follows that

$$ab \le a^2+\frac{b^2}{4}.$$

Therefore, for $b \le 4$, we have

$$\begin{aligned} b(ab+c^2+2ac)-20 &\le b\left(a^2+\frac{b^2}{4}+c^2+2ac\right)-20 = b\left[(a+c)^2+\frac{b^2}{4}\right]-20 \\ &= b\left[(5-b)^2+\frac{b^2}{4}\right]-20 = \frac{5}{4}(b-4)(b-2)^2 \le 0. \end{aligned}$$

Consider now that $b>4$. Since

$$a = 5-b-c \le 5-b,$$

We have

$$\begin{aligned} ab^2+bc^2+2abc-20 &= ab^2+b(5-a-b)^2+2ab(5-a-b)-20 \\ &= b^3+ab^2-10b^2-a^2b+25b-20 \\ &\le b^3+ab^2-10b^2+25b-20 \\ &\le b^3+(5-b)b^2-10b^2+25b-20 \\ &= -5(b-4)(b-1) < 0. \end{aligned}$$

□

P 2.33. *If a,b,c are nonnegative real numbers, then*

$$a^3+b^3+c^3-a^2b-b^2c-c^2a \ge \frac{8}{9}(a-b)(b-c)^2.$$

Solution. Since

$$3(a^3+b^3+c^3-a^2b-b^2c-c^2a) = \sum(2a^3-3a^2b+b^3) = \sum(2a+b)(a-b)^2,$$

we can write the inequality as

$$(2a+b)(a-b)^2+(2b+c)(b-c)^2+(2c+a)(c-a)^2 \ge \frac{8}{3}(a-b)(b-c)^2.$$

If $a \le b$, then

$$(2a+b)(a-b)^2+(2b+c)(b-c)^2+(2c+a)(c-a)^2 \ge 0 \ge \frac{8}{3}(a-b)(b-c)^2.$$

If $a \ge b$, then there are two cases to consider: $b \ge c$ and $b \le c$.
Case 1: $a \ge b \ge c$. It suffices to show that

$$(2c+a)(a-c)^2 \ge \frac{8}{3}(a-b)(b-c)^2.$$

By the AM-GM inequality, we have

$$\begin{aligned}(a-b)(b-c)^2 &= 4(a-b)\left(\frac{b-c}{2}\right)\left(\frac{b-c}{2}\right)\\ &\le 4\left[\frac{(a-b)+(b-c)/2+(b-c)/2}{3}\right]^3\\ &= \frac{4}{27}(a-c)^3.\end{aligned}$$

Therefore, it suffices to show that

$$(2c+a)(a-c)^2 \ge \frac{32}{81}(a-c)^3,$$

which is obvious.
Case 2: $a \ge b$, $c \ge b$. Making the substitution

$$a = b+p, \qquad c = b+q, \qquad p,q \ge 0,$$

the inequality becomes

$$(3b+2p)p^2 + (3b+q)q^2 + (3b+p+2q)(p-q)^2 \ge \frac{8}{3}pq^2,$$

$$3[p^2+q^2+(p-q)^2]b + 2p^3 + q^3 + (p+2q)(p-q)^2 \ge \frac{8}{3}pq^2.$$

It suffices to show that

$$2p^3 + q^3 + (p+2q)(p-q)^2 \ge \frac{8}{3}pq^2,$$

which is equivalent to

$$2p^3 + 2q^3 \ge \frac{34}{9}pq^2.$$

By the AM-GM inequality, we have

$$2p^3 + 2q^3 = 2p^3 + q^3 + q^3 \ge 3\sqrt[3]{2p^3q^6} \ge \frac{34}{9}pq^2,$$

because

$$3\sqrt[3]{2} > \frac{34}{9}.$$

The equality holds for $a = b = c$.

□

P 2.34. *Let a, b, c be nonnegative real numbers such that $a \geq b \geq c$. Prove that*

(a) $$\sum a^2(a-b)(a-c) \geq 4a^2b^2\left(\frac{a-b}{a+b}\right)^2;$$

(b) $$\sum a^2(a-b)(a-c) \geq \frac{27b(a-b)^4}{4a}.$$

(Vasile C., 2011)

Solution. (a) Since $c^2(c-a)(c-b) \geq 0$, it suffices to show that

$$a^2(a-b)(a-c) + b^2(b-c)(b-a) \geq 4a^2b^2\left(\frac{a-b}{a+b}\right)^2.$$

Since

$$a^2(a-b)(a-c) = a^2(a-b)[(a-b)+(b-c)]$$

$$= a^2(a-b)^2 + a^2(a-b)(b-c) \geq 4a^2b^2\left(\frac{a-b}{a+b}\right)^2 + a^2(a-b)(b-c),$$

it suffices to show that

$$a^2(a-b)(b-c) + b^2(b-c)(b-a) \geq 0.$$

This inequality is equivalent to

$$(a-b)^2(a+b)(b-c) \geq 0.$$

The equality holds for $a = b = c$, and for $a = b$ and $c = 0$.

(b) Since $c^2(c-a)(c-b) \geq 0$, it suffices to show that

$$a^2(a-b)(a-c) + b^2(b-c)(c-a) \geq \frac{27b(a-b)^4}{4a},$$

which is equivalent to

$$a^2(a-b)^2 + a(a-b)(b-c) + b^2(b-c)(c-a) \geq \frac{27b(a-b)^4}{4a}.$$

Since

$$a^2(a-b)^2 - \frac{27b(a-b)^4}{4a} = \frac{(a-b)^2(a-3b)^2(4a-3b)}{4a} \geq 0,$$

it suffices to show that

$$a^2(a-b)(b-c) + b^2(b-c)(b-a) \geq 0.$$

This inequality is equivalent to

$$(a-b)^2(a+b)(b-c) \geq 0.$$

The equality holds for $a = b = c$, and for $a/3 = b = c$.

□

P 2.35. *If a, b, c are real numbers such that*

$$a \ge b \ge 1 \ge c, \qquad a^2 + b^2 + c^2 = 3,$$

then

(a) $$1 - abc \le 2(b - c)^2;$$

(b) $$1 - abc \ge 2(a - b)^2;$$

(c) $$1 - abc \ge \frac{1}{2}(a - c)^2;$$

(d) $$1 - abc \le \frac{3}{4}(a - c)^2.$$

(Vasile Cîrtoaje, 2020)

Solution. (a) Write the inequality as follows:

$$1 - abc \le 2(3 - a^2 - 2bc),$$

$$5 - 2a^2 \ge (4 - a)bc.$$

From $(b^2 - 1)(c^2 - 1) \le 0$, we get

$$b^2c^2 \le b^2 + c^2 - 1 = 2 - a^2, \qquad bc \le \sqrt{2 - a^2}, \qquad a \le \sqrt{2}.$$

Thus, it is enough to show that

$$5 - 2a^2 \ge (4 - a)\sqrt{2 - a^2},$$

which, by squaring, becomes

$$5a^4 - 8a^3 - 6a^2 + 16a - 7 \ge 0,$$

$$(a - 1)^3(5a + 7) \ge 0.$$

The equality occurs for $a = b = c = 1$.

(b) From

$$3 = a^2 + b^2 + c^2 \ge 1 + 1 + c^2,$$

it follows that $c \in [-1, 1]$. Write the required inequality as follows:

$$1 - abc \ge 2(3 - c^2 - 2ab),$$

$$(4 - c)ab \ge 5 - 2c^2.$$

From $(a^2 - 1)(b^2 - 1) \ge 0$, we get

$$ab \ge \sqrt{a^2 + b^2 - 1} = \sqrt{2 - c^2}.$$

Thus, it is enough to show that

$$(4-c)\sqrt{2-c^2} \geq 5-2c^2,$$

which, by squaring, becomes

$$5c^4-8c^3-6c^2+16c-7 \leq 0,$$

$$(c-1)^3(5c+7) \leq 0.$$

(c) Write the inequality as follows:

$$2-2abc \geq 3-b^2-2ac,$$

$$b^2-1 \geq 2ac(b-1),$$

which is true if

$$b+1 \geq 2ac.$$

It is enough to show that

$$b+1 \geq a^2+c^2,$$

which is equivalent to

$$b+1 \geq 3-b^2,$$

$$(b-1)(b+2) \geq 0.$$

The equality occurs for $a=b=c=1$.

(d) Write the inequality as follows:

$$4-4abc \leq 3(3-b^2)-6ac,$$

$$2(3-2b)ac \leq 5-3b^2.$$

From

$$(a^2-b^2)(b^2-c^2) \geq 0,$$

it follows that

$$ac \leq \sqrt{b^2(a^2+c^2)-b^4} = b\sqrt{3-2b^2}, \quad 1 \leq b \leq \sqrt{\frac{3}{2}}.$$

Thus, it suffices to show that

$$2b(3-2b)\sqrt{3-2b^2} \leq 5-3b^2.$$

By squaring, the inequality becomes

$$32b^6-96b^5+33b^4+144b^3-138b^2+25 \geq 0,$$

$$(b-1)^2(32b^4-32b^3-63b^2+50b+25) \geq 0.$$

It is true because

$$32b^4 - 32b^3 - 63b^2 + 50b + 25 > 32b^4 - 32b^3 - 64b^2 + 48b + 24$$

$$= 8(3 - 2b^2)(1 + 2b - 2b^2) \geq 0.$$

The equality occurs for $a = b = c = 1$.

□

P 2.36. *If a, b, c are real numbers such that*

$$a \geq 1 \geq b \geq c, \qquad a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \leq \frac{2}{3}(a - c)^2.$$

First Solution. There are two cases to consider: $b \leq 0$ and $b \geq 0$.

Case 1: $b \leq 0$. Since $0 \geq b \geq c$, hence $c^2 \geq b^2$, we have

$$3abc + 2(a - c)^2 - 3 \geq 2(a - c)^2 - 3$$

$$> a^2 + 2c^2 - 3 \geq a^2 + b^2 + c^2 - 3 = 0.$$

Case 2: $b \geq 0$. Write the inequality as follows:

$$3abc + 2(3 - b^2 - 2ac) \geq 3,$$

$$3 - 2b^2 \geq (4 - 3b)ac.$$

From $(b^2 - c^2)(b^2 - a^2) \leq 0$, we get

$$a^2c^2 \leq b^2(a^2 + c^2) - b^4 = b^2(3 - b^2) - b^4, \qquad ac \leq b\sqrt{3 - 2b^2}.$$

Thus, it is enough to show that

$$3 - 2b^2 \geq b(4 - 3b)\sqrt{3 - 2b^2},$$

which, by squaring, becomes

$$6b^6 - 16b^5 + 3b^4 + 24b^3 - 20b^2 + 3 \geq 0,$$

$$(1 - b)^3(3 + 9b - 2b^2 - 6b^3) \geq 0.$$

$$(1 - b)^3[3 + b + 2b(1 - b) + 6b(1 - b^2)] \geq 0.$$

The equality occurs for $a = b = c = 1$.

Second Solution (*by Mudok*). We will prove the stronger inequality

$$1-abc \le \frac{2}{3}\left[(a-c)^2-(a-b)(b-c)\right],$$

which is equivalent to

$$1-abc \le \frac{2}{3}(3-ab-bc-ca),$$

$$3-3abc \le 9-p^2,$$

where

$$p=a+b+c.$$

From

$$(a-1)(b-1)(c-1) \ge 0,$$

we get

$$abc \ge \frac{p^2-2p-1}{2}.$$

Thus it suffices to show that

$$3-\frac{3(p^2-2p-1)}{2} \le 9-p^2,$$

which is equivalent to

$$(p-3)^2 \ge 0.$$

□

P 2.37. *If $a \ge 1 \ge b \ge c \ge 0$ and $a^2+b^2+c^2=3$, then*

$$1-abc \le \frac{1}{\sqrt{2}}(a-c).$$

(Vasile Cîrtoaje, 2020)

Solution. Denoting $x=ac$, we need to show that $f(x) \ge 0$, where

$$f(x)=bx+\sqrt{\frac{3-b^2-2x}{2}}-1.$$

For fixed b, we have $x \in [0,M]$, where

$$M=b\sqrt{3-2b^2}.$$

Indeed, $(b^2-a^2)(b^2-c^2) \le 0$ yields

$$a^2c^2 \le b^2(a^2+c^2)-b^4=3b^2-2b^4, \qquad x \le b\sqrt{3-2b^2}=M.$$

We have $x = 0$ for $c = 0$, and $x = M$ for $a = b = 1$ or $b = c$. Since

$$f''(x) = -\frac{1}{\sqrt{2}} \cdot \frac{1}{(3-b^2-2x)^{3/2}} \le 0,$$

f is a concave function, therefore it suffices to show that $f(0) \ge 0$ and $f(M) \ge 0$. We have

$$f(0) = \sqrt{\frac{3-b^2}{2}} - 1 \ge 0.$$

Since

$$\sqrt{3-b^2-2M} = \sqrt{3-b^2-2b\sqrt{3-2b^2}} = \sqrt{3-2b^2} - b,$$

we have

$$f(M) \ge bM + \frac{2}{3} \cdot \sqrt{3-b^2-2M} - 1 = \frac{2}{3} \cdot \sqrt{3-b^2-2M} - (1-bM)$$

$$= \frac{2}{3} \cdot \left(\sqrt{3-2b^2} - b\right) - (1 - b^2\sqrt{3-2b^2})$$

$$= \left(b^2 + \frac{2}{3}\right)\sqrt{3-2b^2} - \frac{2b}{3} - 1.$$

So, we need to show that

$$(3b^2+2)\sqrt{3-2b^2} \ge 2b+3.$$

By squaring, the inequality becomes

$$6b^6 - b^4 - 8b^2 + 4b - 1 \le 0,$$

$$(b-1)(6b^5 + 6b^4 + 5b^3 + 5b^2 - 3b + 1) \le 0.$$

It is true because

$$5b^2 - 3b + 1 = b^2 + b + (2b-1)^2 > 0.$$

The equality occurs for $a = b = c = 1$, and also for $a = \sqrt{2}$, $b = 1$, $c = 0$.

□

P 2.38. *If $a \ge 1 \ge b \ge c \ge 0$ and $a^2 + b^2 + c^2 = 3$, then*

$$1 - abc \le (1+\sqrt{2})(a-b).$$

(Vasile Cîrtoaje, 2020)

Solution. Denoting $x = ab$, we need to show that $f(x) \ge 0$, where

$$f(x) = cx + k\sqrt{3-c^2-2x} - 1, \qquad k = 1+\sqrt{2}.$$

For fixed c, we have $x \in [m, M]$, where

$$m = c\sqrt{3-2c^2}, \qquad M = \sqrt{2-c^2}.$$

Indeed, $(a^2-c^2)(b^2-c^2) \ge 0$ yields

$$a^2b^2 \ge c^2(a^2+b^2) - c^4 = 3c^2 - 2c^4, \qquad x \ge c\sqrt{3-2c^2} = m,$$

and $(a^2-1)(b^2-1) \le 0$ yields

$$ab \le \sqrt{a^2+b^2-1} = \sqrt{2-c^2} = M.$$

We have $x = m$ for $b = c$, and $x = M$ for $b = 1$. Since

$$f''(x) = \frac{-k}{(3-c^2-2x)^{3/2}} \le 0,$$

f is a concave function, therefore it suffices to show that $f(m) \ge 0$ and $f(M) \ge 0$. We have

$$\begin{aligned}
f(m) &= c^2\sqrt{3-2c^2} + k\sqrt{3-c^2-2c\sqrt{3-2c^2}} - 1 \\
&= c^2\sqrt{3-2c^2} + k\left(\sqrt{3-2c^2} - c\right) - 1 \\
&\ge c^2\sqrt{3-2c^2} + 2\left(\sqrt{3-2c^2} - c\right) - 1 \\
&= (c^2+2)\sqrt{3-2c^2} - 2c - 1 \ge (2c+1)\left(\sqrt{3-2c^2} - 1\right) \ge 0.
\end{aligned}$$

Also,

$$\begin{aligned}
f(M) &= c\sqrt{2-c^2} + k\sqrt{3-c^2-2\sqrt{2-c^2}} - 1 \\
&= c\sqrt{2-c^2} - 1 + k\left(\sqrt{2-c^2} - 1\right) \\
&= \frac{-(1-c^2)^2}{c\sqrt{2-c^2}+1} + \frac{k(1-c^2)}{\sqrt{2-c^2}+1}.
\end{aligned}$$

So, we need to show that

$$\frac{k}{\sqrt{2-c^2}+1} \ge \frac{1-c^2}{c\sqrt{2-c^2}+1}.$$

It is true because

$$\frac{k}{\sqrt{2-c^2}+1} \ge 1 \ge \frac{1-c^2}{1+c\sqrt{2-c^2}}.$$

The equality occurs for $a = b = c = 1$, and also for $a = \sqrt{2}$, $b = 1$, $c = 0$.

□

P 2.39. *If $a \ge 1 \ge b \ge c \ge 0$ and $a^2+b^2+c^2=3$, then*

$$1-abc \le (3+2\sqrt{2})(a-b)^2.$$

(Vasile Cîrtoaje, 2020)

Solution. Write the inequality as follows:

$$abc+k(3-c^2-2ab)\ge 1,$$

$$3k-1-kc^2 \ge (2k-c)ab, \qquad k=3+2\sqrt{2}.$$

From $(a^2-1)(b^2-1)\le 0$, we get

$$ab \le \sqrt{a^2+b^2-1}=\sqrt{2-c^2}.$$

Thus, it is enough to show that

$$3k-1-kc^2 \ge (2k-c)\sqrt{2-c^2}.$$

Write this inequality as follows:

$$k\left(3-c^2-2\sqrt{2-c^2}\right)\ge 1-c\sqrt{2-c^2},$$

$$\frac{k(1-c^2)^2}{3-c^2+2\sqrt{2-c^2}} \ge \frac{(1-c^2)^2}{1+c\sqrt{2-c^2}},$$

which is true if

$$k\left(1+c\sqrt{2-c^2}\right)\ge 3-c^2+2\sqrt{2-c^2},$$

$$k-3+c^2 \ge (2-kc)\sqrt{2-c^2}.$$

For the nontrivial case $2-kc\ge 0$, we have

$$k-3+c^2 \ge k-3 = 2\sqrt{2} \ge (2-kc)\sqrt{2-c^2}.$$

The equality occurs for $a=b=c=1$, and also for $a=\sqrt{2},\ b=1,\ c=0$.

□

P 2.40. *If a,b,c are positive real numbers, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{(a-c)^2}{ab+bc+ca}.$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2008)

First Solution. By expanding, the inequality can be written as

$$b^2+\frac{bc^2}{a}+\frac{ca^2}{b}+\frac{ab^2}{c}\ge 2ab+2bc.$$

We can get this inequality by summing the AM-GM inequalities

$$ab+\frac{bc^2}{a}\ge 2bc,$$

$$b^2+\frac{ca^2}{b}+\frac{ab^2}{c}\ge 3ab.$$

The equality holds for $a=b=c$.

Second Solution. From

$$\begin{aligned}(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right)&=\sum\frac{a^2}{b}+\sum\frac{bc}{a}-2\sum a\\&=\sum\left(\frac{a^2}{b}-2a+b\right)+\sum\left(\frac{bc}{a}-b\right)\\&=\sum\left(\frac{a^2}{b}-2a+b\right)+\frac{1}{2}\sum\left(\frac{ab}{c}+\frac{ac}{b}-2a\right)\\&=\sum\frac{(a-b)^2}{b}+\frac{1}{2}\sum\frac{a(b-c)^2}{bc},\end{aligned}$$

we get

$$(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right)\ge\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(c-a)^2}{a}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}\ge\frac{(a-c)^2}{b+c}.$$

Therefore,

$$(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right)\ge\frac{(a-c)^2}{b+c}+\frac{(c-a)^2}{a},$$

which is equivalent to

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\ge\frac{(a-c)^2}{a(b+c)}.$$

From this result, the desired inequality follows immediately.

□

P 2.41. *If a, b, c are positive real numbers, then*

(a) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{4(a-c)^2}{(a+b+c)^2};$$

(b) $$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{5(a-c)^2}{(a+b+c)^2}.$$

(Vo Quoc Ba Can and Vasile Cîrtoaje, 2009)

Solution. As we have shown at the second solution of the preceding problem P 2.40:

$$(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right)=\sum\frac{(a-b)^2}{b}+\frac{1}{2}\sum\frac{a(b-c)^2}{bc},$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3 \ge \frac{(a-c)^2}{a(b+c)}.$$

(a) According to the upper inequality, it suffices to show that

$$\frac{1}{a(b+c)} \ge \frac{4}{(a+b+c)^2}.$$

Indeed,

$$\frac{1}{a(b+c)}-\frac{4}{(a+b+c)^2}=\frac{(a-b-c)^2}{a(b+c)(a+b+c)^2} \ge 0.$$

The equality holds for $a=b=c$.

(b) According to the upper identity, write the inequality as

$$(a+b+c)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right) \ge \frac{5(a-c)^2}{a+b+c},$$

$$\sum\frac{(a-b)^2}{b}+\frac{1}{2}\sum\frac{a(b-c)^2}{bc} \ge \frac{5(a-c)^2}{a+b+c},$$

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{c(a-b)^2}{2ab}+\frac{a(b-c)^2}{2bc} \ge \left(\frac{5}{a+b+c}-\frac{1}{a}-\frac{b}{2ac}\right)(a-c)^2.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c} \ge \frac{[(a-b)+(b-c)]^2}{b+c},$$

$$\frac{c(a-b)^2}{2ab}+\frac{a(b-c)^2}{2bc} \ge \frac{[(a-b)+(b-c)]^2}{\frac{2ab}{c}+\frac{2bc}{a}}=\frac{ac(a-c)^2}{2b(a^2+c^2)}.$$

Thus, we only need to show that

$$\frac{1}{b+c}+\frac{ac}{2b(a^2+c^2)} \ge \frac{5}{a+b+c}-\frac{1}{a}-\frac{b}{2ac},$$

which is equivalent to

$$\left(\frac{1}{a}+\frac{1}{b+c}\right)+\frac{ac}{2b(a^2+c^2)}+\frac{b}{2ac}\ge\frac{5}{a+b+c}.$$

This inequality is true because, by the Cauchy-Schwarz inequality and the AM-GM inequality, we have

$$\frac{1}{a}+\frac{1}{b+c}\ge\frac{4}{a+(b+c)}$$

and

$$\frac{ac}{2b(a^2+c^2)}+\frac{b}{2ac}\ge\frac{1}{\sqrt{a^2+c^2}}>\frac{1}{a+c}>\frac{1}{a+b+c}.$$

The equality holds for $a=b=c$.

□

P 2.42. *If $a\ge b\ge c>0$, then*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 3+\frac{3(b-c)^2}{ab+bc+ca}.$$

First Solution. Since

$$\frac{a}{b}+\frac{c}{a}-1-\frac{c}{b}=\frac{(a-b)(a-c)}{ab}\ge 0,$$

it suffices to show that

$$\frac{b}{c}+\frac{c}{b}-2\ge\frac{3(b-c)^2}{ab+bc+ca}.$$

Indeed, we have

$$\frac{b}{c}+\frac{c}{b}-2-\frac{3(b-c)^2}{ab+bc+ca}=\frac{(b-c)^2(ab+ac-2bc)}{bc(ab+bc+ca)}.$$

The equality holds for $a=b=c$.

Second Solution. Since

$$ab+bc+ca\ge 3bc,$$

it suffices to show that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 3+\frac{(b-c)^2}{bc},$$

which is equivalent to

$$\frac{a}{b}+\frac{c}{a}\ge 1+\frac{c}{b},$$

$$\frac{(a-b)(a-c)}{ab}\ge 0.$$

□

P 2.43. *Let a, b, c be positive real numbers such that $abc = 1$. Prove that*

(a) if $a \ge b \ge 1 \ge c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{2(a-b)^2}{ab};$$

(b) if $a \ge 1 \ge b \ge c$, then

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3+\frac{2(b-c)^2}{bc}.$$

(Vasile C., 2010)

Solution. (a) Write the inequality as

$$f(c) \ge \frac{a}{b}+2\frac{b}{a}-1,$$

where

$$f(c)=\frac{b}{c}+\frac{c}{a}.$$

From

$$b^3 \ge 1 = abc,$$

we find

$$b^2 \ge ac.$$

We will show that

$$f(c) \ge f\left(\frac{b^2}{a}\right) \ge \frac{a}{b}+2\frac{b}{a}-1.$$

The left inequality is equivalent to

$$\frac{b}{c}+\frac{c}{a} \ge \frac{a}{b}+\frac{b^2}{a^2},$$

$$\frac{b^2-ac}{bc} \ge \frac{b^2-ac}{a^2} \ge 0,$$

$$(a^2-bc)(b^2-ac) \ge 0.$$

The right inequality reduces to

$$\left(\frac{b}{a}-1\right)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

(b) Write the inequality as

$$f(a) \geq \frac{b}{c} + 2\frac{c}{b} - 1,$$

where

$$f(a) = \frac{a}{b} + \frac{c}{a}.$$

From

$$b^3 \leq 1 = abc,$$

we find

$$b^2 \leq ac.$$

We will show that

$$f(a) \geq f\left(\frac{b^2}{c}\right) \geq \frac{b}{c} + 2\frac{c}{b} - 1$$

The left inequality is equivalent to

$$\frac{a}{b} + \frac{c}{a} \geq \frac{b}{c} + \frac{c^2}{b^2},$$

$$\frac{ac - b^2}{bc} \geq \frac{c(ac - b^2}{ab^2} \geq 0,$$

$$(ab - c^2)(ac - b^2) \geq 0.$$

The right inequality reduces to

$$\left(\frac{c}{b} - 1\right)^2 \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.44. *Let a, b, c be positive real numbers such that*

$$a \geq 1 \geq b \geq c, \qquad abc = 1.$$

prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3 + \frac{9(b-c)^2}{ab + bc + ca}.$$

(Vasile C., 2010)

Solution. From $b^3 \le 1 = abc$, we find $b^2 \le ac$. We will show that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge \frac{2b}{c} + \frac{c^2}{b^2} \ge 3 + \frac{9(b-c)^2}{ab+bc+ca}.$$

The left inequality is equivalent to

$$\frac{a}{b} + \frac{c}{a} \ge \frac{b}{c} + \frac{c^2}{b^2},$$

$$\frac{a}{b} - \frac{b}{c} + \left(\frac{c}{a} - \frac{c^2}{b^2}\right) \ge 0,$$

$$\frac{ac-b^2}{bc} + \frac{c(b^2-ac)}{ab^2} \ge 0,$$

$$\frac{(ac-b^2)(ab-c^2)}{ab^2c} \ge 0.$$

The right inequality is equivalent to

$$\frac{2b}{c} + \frac{c^2}{b^2} - 3 \ge \frac{9(b-c)^2}{ab+bc+ca}.$$

$$\frac{(b-c)^2(2b+c)}{b^2c} \ge \frac{9(b-c)^2}{ab+bc+ca}.$$

We need to show that

$$\frac{(2b+c)}{b^2c} \ge \frac{9}{a(b+c)+bc}.$$

This is true if

$$\frac{(2b+c)}{b^2c} \ge \frac{9}{b(b+c)+bc},$$

which is equivalent to

$$\frac{2(b-c)^2}{b^2c(b+2c)} \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.45. *Let a, b, c be positive real numbers such that*

$$a \ge 1 \ge b \ge c, \qquad a + b + c = 3.$$

prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{4(b-c)^2}{b^2+c^2}.$$

(Vasile C., 2010)

Solution. From
$$3b \le 3 = a + b + c,$$
we find
$$2b \le a + c, \qquad a \ge 2b - c.$$
We will show that
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge \frac{2b-c}{b} + \frac{b}{c} + \frac{c}{2b-c} \ge 3 + \frac{4(b-c)^2}{b^2+c^2}.$$
The left inequality is equivalent to
$$\frac{a}{b} + \frac{c}{a} \ge \frac{2b-c}{b} + \frac{c}{2b-c},$$
$$\frac{a+c-2b}{b} - \frac{c(a+c-2b)}{a(2b-c)} \ge 0,$$
$$\frac{(a+c-2b)[a(b-c)+b(a-c)]}{ab(2b-c)} \ge 0.$$
The right inequality is equivalent to
$$\frac{(b-c)^2(2b+c)}{bc(2b-c)} \ge \frac{4(b-c)^2}{b^2+c^2}.$$
We need to show that
$$\frac{(2b+c)}{bc(2b-c)} \ge \frac{4}{b^2+c^2},$$
which is equivalent to
$$2b^3 - 7b^2c + 6bc^2 + c^3 \ge 0,$$
$$2b(b-2c)^2 + (b-c)^2c \ge 0.$$
The equality holds for $a = b = c = 1$.

□

P 2.46. *Let a, b, c be positive real numbers such that*
$$a \ge b \ge 1 \ge c, \qquad a + b + c = 3.$$
Prove that
$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{3(a-b)^2}{ab}.$$
(Vasile C., 2008)

Solution. From

$$3b \ge 3 = a + b + c,$$

we get

$$2b \ge a + c, \qquad c \le 2b - a.$$

We will show that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge \frac{a}{b} + \frac{b}{2b-a} + \frac{2b-a}{a} \ge 3 + \frac{3(a-b)^2}{ab}.$$

The left inequality is equivalent to

$$\frac{b}{c} + \frac{c}{a} \ge \frac{b}{2b-a} + \frac{2b-a}{a},$$

$$(2b-a-c)[b(a-c)+c(a-b)] \ge 0.$$

The right inequality is equivalent to

$$\frac{a}{b} + \frac{b}{2b-a} + \frac{2b-a}{a} - 3 \ge \frac{3(a-b)^2}{ab},$$

$$\frac{(a-b)^2(4b-a)}{ab(2b-a)} \ge \frac{3(a-b)^2}{ab},$$

$$\frac{2(a-b)^3}{ab(2b-a)} \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 2.47. *If a, b, c are positive real numbers, then*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \ge 3 + \frac{2(a-c)^2}{(a+c)^2}.$$

Solution. Since

$$\frac{a}{b} + \frac{b}{c} \ge 2\sqrt{\frac{a}{c}},$$

it suffices to show that

$$\frac{c}{a} + 2\sqrt{\frac{a}{c}} \ge 3 + \frac{2(a-c)^2}{(a+c)^2}.$$

Using the substitution $x = \sqrt{\frac{a}{c}}$, this inequality becomes as follows:

$$\frac{1}{x^2} + 2x \ge 3 + \frac{2(x^2-1)^2}{(x^2+1)^2},$$

$$\frac{(x-1)^2(2x+1)}{x^2} \ge \frac{2(x^2-1)^2}{(x^2+1)^2}.$$

We need to show that

$$\frac{2x+1}{x^2} \ge \frac{2(x+1)^2}{(x^2+1)^2},$$

which is equivalent to

$$2x^5-3x^4+2x+1 \ge 0.$$

For $0 < x \le 1$, we have

$$2x^5-3x^4+2x+1 > -3x^4+2x+1 \ge -3x+2x+1 \ge 0.$$

Also, for $x \ge 1$, we have

$$2x^5-3x^4+2x+1 > 2x^5-3x^4+2x-1 = (x-1)^2(2x^3+x^2-1) \ge 0.$$

The equality holds for $a=b=c$.

□

P 2.48. *If a,b,c are positive real numbers, then*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \ge a+b+c+\frac{4(a-c)^2}{a+b+c}.$$

(Balkan MO, 2005, 2008)

Solution. Write the inequality as follows:

$$\left(\frac{a^2}{b}+b-2a\right)+\left(\frac{b^2}{c}+c-2b\right)+\left(\frac{c^2}{a}+a-2c\right) \ge \frac{4(a-c)^2}{a+b+c},$$

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(a-c)^2}{a} \ge \frac{4(a-c)^2}{a+b+c}.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(a-c)^2}{a} \ge \frac{[(a-b)+(b-c)+(a-c)]^2}{b+c+a} = \frac{4(a-c)^2}{a+b+c}.$$

The equality holds for $a=b=c$, and also for $a=b+c$ and $\frac{b}{c}=\frac{1+\sqrt{5}}{2}$.

□

P 2.49. *If $a \ge b \ge c > 0$, then*

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a} \ge a+b+c+\frac{6(b-c)^2}{a+b+c}.$$

(Vasile C., 2014)

Solution. Write the inequality as follows:

$$\left(\frac{a^2}{b}+b-2a\right)+\left(\frac{b^2}{c}+c-2b\right)+\left(\frac{c^2}{a}+a-2c\right)\geq\frac{6(b-c)^2}{a+b+c},$$

$$\frac{(a-b)^2}{b}+\frac{(b-c)^2}{c}+\frac{(a-c)^2}{a}\geq\frac{6(b-c)^2}{a+b+c},$$

$$\frac{(a-b)^2}{b}+\frac{(a-c)^2}{a}+\frac{(a+b-5c)(b-c)^2}{c(a+b+c)}\geq 0.$$

Since

$$(a-c)^2=[(a-b)+(b-c)]^2=(a-b)^2+2(a-b)(b-c)+(b-c)^2,$$

we have

$$\frac{(a-b)^2}{b}+\frac{(a-c)^2}{a}\geq\frac{(a-c)^2}{a}\geq\frac{2(a-b)(b-c)+(b-c)^2}{a}.$$

Therefore, it suffices to show that

$$\frac{2(a-b)(b-c)+(b-c)^2}{a}+\frac{(a+b-5c)(b-c)^2}{c(a+b+c)}\geq 0,$$

which can be written as

$$\frac{2(a-b)(b-c)}{a}+\frac{(a-c)^2+ab+bc-2ca}{ac(a+b+c)}(b-c)^2\geq 0.$$

Since

$$(a-c)^2+ab+bc-2ca=(a-c)^2+a(b-c)-c(a-b)\geq -c(a-b),$$

it is enough to prove that

$$\frac{2(a-b)(b-c)}{a}-\frac{a-b}{a(a+b+c)}(b-c)^2\geq 0.$$

Indeed,

$$\frac{2(a-b)(b-c)}{a}-\frac{a-b}{a(a+b+c)}(b-c)^2=\frac{(a-b)(b-c)}{a}\left(2-\frac{b-c}{a+b+c}\right)\geq 0.$$

The equality holds for $a=b=c$.

□

P 2.50. *If $a \ge b \ge c > 0$, then*

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} > 5(a-b).$$

(Vasile C., 2014)

Solution. Consider two cases: $a \le 2b$ and $a \ge 2b$.

Case 1: $a \le 2b$. It suffices to show that

$$\frac{a^2}{b} + \frac{b^2}{b} \ge 5(a-b),$$

which is equivalent to the obvious inequality

$$(2b-a)(3b-a) \ge 0.$$

Case 2: $a \ge 2b$. Since

$$\frac{b^2}{c} + \frac{c^2}{a} - b - \frac{b^2}{a} = (b-c)\left(\frac{b}{c} - \frac{b+c}{a}\right)$$

$$\ge (b-c)\left(\frac{b}{c} - \frac{b+c}{2b}\right) = \frac{(b-c)^2(2b+c)}{2bc} \ge 0,$$

it suffices to show that

$$\frac{a^2}{b} + b + \frac{b^2}{a} \ge 5(a-b),$$

which is equivalent to

$$x(x-2)(3-x) < 1,$$

where $x = a/b \ge 2$. For the non-trivial case $2 \le x \le 3$, we have

$$x(x-2)(3-x) \ge x\left[\frac{(x-2)+(3-x)}{2}\right]^2 = \frac{x}{4} < 1.$$

□

P 2.51. *Let a, b, c be positive real numbers such that*

$$a \ge b \ge 1 \ge c, \qquad a+b+c = 3.$$

Prove that

$$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \ge 3 + \frac{11(a-c)^2}{4(a+c)}.$$

(Vasile C., 2010)

Solution. We have

$$a+b+c=3\le b, \qquad 2b\ge a+c.$$

Thus, we need to prove the homogeneous inequality

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\ge a+b+c+\frac{11(a-c)^2}{4(a+c)}$$

for

$$a\ge b\ge \frac{a+c}{2}.$$

Denote

$$f(a,b,c)=\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}-a-b-c.$$

We will show that

$$f(a,b,c)\ge f\left(a,\frac{a+c}{2},c\right)\ge \frac{11(a-c)^2}{4(a+c)}.$$

Write the left inequality as follows:

$$\left(\frac{a^2}{b}-\frac{2a^2}{a+c}\right)+\left[\frac{b^2}{c}-\frac{(a+c)^2}{4c}\right]-\left(b-\frac{a+c}{2}\right)\ge 0,$$

$$(2b-a-c)\left[-\frac{a^2}{b(a+c)}+\frac{2b+a+c}{4c}-\frac{1}{2}\right]\ge 0.$$

Since $2b-a-c\ge 0$, we only need to show that

$$\frac{2b+a+c}{4c}\ge \frac{a^2}{b(a+c)}+\frac{1}{2}.$$

It suffices to prove this inequality for $b=\dfrac{a+c}{2}$. Making this, the inequality becomes

$$\frac{a(a-c)^2}{2c(a+c)^2}\ge 0.$$

To prove the right inequality, we find

$$f\left(a,\frac{a+c}{2},c\right)=\frac{(a-c)^2(a^2+7ac+4c^2)}{4ac(a+c)},$$

hence

$$f\left(a,\frac{a+c}{2},c\right)-\frac{11(a-c)^2}{4(a+c)}=\frac{(a-c)^2(a-2c)^2}{4ac(a+c)}\ge 0.$$

The equality holds for $a=b=c=1$, and also for $\dfrac{a}{4}=\dfrac{b}{3}=\dfrac{c}{2}$ (that is, for $a=\dfrac{4}{3}$, $b=1$, $c=\dfrac{2}{3}$).

□

P 2.52. *If a, b, c are positive real numbers, then*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}+\frac{27(b-c)^2}{16(a+b+c)^2}.$$

(Vasile C., 2014)

Solution. Write the inequality as follows:

$$\sum\left(\frac{a}{b+c}+1\right)\ge\frac{9}{2}+\frac{27(b-c)^2}{16(a+b+c)^2},$$

$$\left[\sum(b+c)\right]\left(\sum\frac{1}{b+c}\right)\ge 9+\frac{27(b-c)^2}{2\left[\sum(b+c)\right]^2}.$$

Replacing $b+c$, $c+a$, $a+b$ by a, b, c, respectively, we need to show that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge 9+\frac{27(b-c)^2}{2(a+b+c)^2},$$

where a, b, c are the side-lengths of a non-degenerate triangle. Write this inequality in the form

$$\frac{a+b+c}{a}+(a+b+c)\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{54bc}{(a+b+c)^2}\ge 9+\frac{27(b+c)^2}{2(a+b+c)^2}.$$

Applying the AM-GM inequality gives

$$(a+b+c)\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{54bc}{(a+b+c)^2}\ge 6\sqrt{\frac{6(b+c)}{a+b+c}}.$$

Therefore, it suffices to show that

$$\frac{a+b+c}{a}+6\sqrt{\frac{6(b+c)}{a+b+c}}\ge 9+\frac{27(b+c)^2}{2(a+b+c)^2},$$

which can be rewritten as

$$\frac{1}{1-\dfrac{b+c}{a+b+c}}+6\sqrt{\frac{6(b+c)}{a+b+c}}\ge 9+\frac{27(b+c)^2}{2(a+b+c)^2}.$$

Using the substitution

$$\frac{b+c}{a+b+c}=\frac{2}{3}t^2,\quad t^2>\frac{3}{4},$$

this inequality becomes

$$\frac{1}{3-2t^2}+4t\ge 3+2t^4,$$

$$2t^6 - 3t^4 - 4t^3 + 3t^2 + 6t - 4 \geq 0,$$
$$(t-1)^2(2t^4 + 4t^3 + 3t^2 - 2t - 4) \geq 0,$$
$$(t-1)^2\left[(4t^2-3)(t^2+2t+2) + t^2 + 2t - 2\right] \geq 0.$$

Clearly, the last inequality is true for $t^2 > 3/4$. The original inequality is an equality for $a = b = c$.

□

P 2.53. *Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} + \frac{9(b-c)^2}{4(a+b+c)^2}.$$

(Vasile C., 2014)

Solution. Write the inequality as

$$\sum\left(\frac{a}{b+c} + 1\right) \geq \frac{9}{2} + \frac{9(b-c)^2}{4(a+b+c)^2},$$
$$\left[\sum(b+c)\right]\left(\sum\frac{1}{b+c}\right) \geq 9 + \frac{18(b-c)^2}{[(b+c)+(c+a)+(a+b)]^2}.$$

Replacing $b+c$, $c+a$, $a+b$ by a, b, c, respectively, we need to show that

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9 + \frac{18(b-c)^2}{(a+b+c)^2},$$

where a, b, c are the side-lengths of a non-degenerate triangle, $a = \max\{a, b, c\}$. Since

$$(a+b+c)^2 \geq \frac{9}{4}(b+c)^2 \geq 9bc,$$

it suffices to show that

$$(a+b+c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9 + \frac{2(b-c)^2}{bc}.$$

Write the inequality as follows:

$$\frac{(a-b)^2}{ab} + \frac{(a-c)^2}{ac} + \frac{(b-c)^2}{bc} \geq \frac{2(b-c)^2}{bc},$$
$$c(a-b)^2 + b(a-c)^2 \geq a(b-c)^2,$$
$$(b+c)a^2 - (b+c)^2a + bc(b+c) \geq 0,$$
$$(b+c)(a-b)(a-c) \geq 0.$$

Clearly, the last inequality is true. The original inequality is an equality for $a = b = c$.

□

P 2.54. *Let a, b, c be nonnegative real numbers, no two of which are zero. Prove that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{3}{2}+\frac{(b-c)^2}{2(b+c)^2}.$$

(Vasile C., 2014)

First Solution. Write the inequality as follows:

$$\frac{2bc}{(b+c)^2}+\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge 2,$$

$$\frac{a(b+c)+2bc}{(b+c)^2}+\frac{b}{c+a}+\frac{c}{a+b} \ge 2,$$

By the Cauchy-Schwarz inequality, we have

$$\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{(b+c)^2}{b(c+a)+c(a+b)} = \frac{(b+c)^2}{a(b+c)+2bc}.$$

Therefore, it suffices to prove that

$$\frac{a(b+c)+2bc}{(b+c)^2}+\frac{(b+c)^2}{a(b+c)+2bc} \ge 2,$$

which is obvious. The original inequality is an equality for $a=b=c$, for $a=b$ and $c=0$, and for $a=c$ and $b=0$.

Second Solution. Write the inequality as follows:

$$\sum\left(\frac{a}{b+c}+1\right) \ge \frac{9}{2}+\frac{(b-c)^2}{2(b+c)^2},$$

$$\left[\sum(b+c)\right]\left(\sum\frac{1}{b+c}\right) \ge 9+\frac{(b-c)^2}{(b+c)^2}.$$

Replacing $b+c$, $c+a$, $a+b$ by a, b, c, respectively, we need to show that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 9+\frac{(b-c)^2}{a^2},$$

where a, b, c are the lengths of the sides of a triangle. Write this inequality as

$$\frac{(a-b)^2}{ab}+\frac{(a-c)^2}{ac}+\frac{(b-c)^2}{bc} \ge \frac{(b-c)^2}{a^2},$$

$$a[c(a-b)^2+b(a-c)^2] \ge (bc-a^2)(b-c)^2.$$

Without loss of generality, assume that $b \ge c$. Since $a \ge b-c$, it suffices to show that

$$c(a-b)^2+b(a-c)^2 \ge (bc-a^2)(b-c).$$

Indeed, we have

$$c(a-b)^2+b(a-c)^2-(bc-a^2)(b-c) = 2b(a-c)^2 \ge 0.$$

□

P 2.55. *Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that*

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} + \frac{(b-c)^2}{4bc}.$$

(Vasile C., 2014)

First Solution *(by Nguyen Van Quy).* Notice that for $a = \min\{a, b, c\}$, we have

$$4bc = (2b)(2c) \geq (a+b)(a+c) \geq 2a(b+c),$$

hence

$$\frac{a}{b+c} \geq \frac{2a^2}{(a+b)(a+c)}, \quad \frac{(b-c)^2}{4bc} \leq \frac{(b-c)^2}{(a+b)(a+c)}.$$

So, it suffices to show that

$$\frac{2a^2}{(a+b)(a+c)} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2} + \frac{(b-c)^2}{(a+b)(a+c)},$$

which is equivalent to the obvious inequality

$$(a-b)(a-c) \geq 0.$$

The proof is completed. The original inequality is an equality for $a = b = c$.

Second Solution. Let

$$E(a, b, c) = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}.$$

Without loss of generality, assume that $b \leq c$, hence $a \leq b \leq c$. We will show that

$$E(a, b, c) \geq E(b, b, c) \geq \frac{3}{2} + \frac{(b-c)^2}{4bc}.$$

We have

$$\begin{aligned}
E(a, b, c) - E(b, b, c) &= \frac{a-b}{b+c} + \frac{b(b-a)}{(a+c)(b+c)} + \frac{c(b-a)}{2b(a+b)} \\
&= (b-a)\left[\frac{(b-a)-c}{(a+c)(b+c)} + \frac{c}{2b(a+b)}\right] \\
&= \frac{(b-a)[2b(b^2-a^2) + c(c-b)(a+2b+c)]}{2b(a+b)(a+c)(b+c)} \geq 0
\end{aligned}$$

and

$$\begin{aligned}
E(b, b, c) - \frac{3}{2} - \frac{(b-c)^2}{4bc} &= \left(\frac{2b}{b+c} + \frac{c}{2b} - \frac{3}{2}\right) - \frac{(b-c)^2}{4bc} \\
&= \frac{(b-c)^2}{2b(b+c)} - \frac{(b-c)^2}{4bc} \\
&= \frac{(c-b)^3}{4bc(b+c)} \geq 0.
\end{aligned}$$

□

P 2.56. *Let a,b,c be positive real numbers such that*

$$a \le 1 \le b \le c, \qquad a+b+c=3,$$

then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{3}{2}+\frac{3(b-c)^2}{4bc}.$$

(Vasile C., 2014)

Solution. From

$$3b \ge 3 = a+b+c,$$

we get

$$a \le 2b-c, \qquad 2b > c.$$

Let

$$E(a,b,c)=\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

We will show that

$$E(a,b,c) \ge E(2b-c,b,c) \ge \frac{3}{2}+\frac{3(b-c)^2}{4bc}.$$

We have

$$E(a,b,c)-E(2b-c,b,c)=(2b-a-c)F,$$

where

$$F=\frac{-1}{b+c}+\frac{1}{2(c+a)}+\frac{c}{(a+b)(3b-c)}.$$

Since $2b-a-c \ge 0$, we need to show that $F \ge 0$. This is true because

$$\begin{aligned}
F &= \frac{1}{2}\left(-\frac{1}{b+c}+\frac{1}{c+a}\right)-\frac{1}{2(b+c)}+\frac{c}{(a+b)(3b-c)}\\
&\ge -\frac{1}{2(b+c)}+\frac{c}{(a+b)(3b-c)} \ge -\frac{1}{2(a+b)}+\frac{c}{(a+b)(3b-c)}\\
&= \frac{3(c-b)}{2(a+b)(3b-c)} \ge 0.
\end{aligned}$$

In what concerns the right inequality, we have

$$\begin{aligned}
E(2b-c,b,c)-\frac{3}{2}-\frac{3(b-c)^2}{4bc} &= 3(b-c)^2\left[\frac{1}{(b+c)(3b-c)}-\frac{1}{4bc}\right]\\
&= \frac{-3(b-c)^3(3b+c)}{4bc(b+c)(3b-c)} \ge 0.
\end{aligned}$$

The proof is completed. The original inequality is an equality for $a=b=c=1$.

□

P 2.57. *Let a, b, c be nonnegative real numbers such that*

$$a \ge 1 \ge b \ge c, \qquad a+b+c=3,$$

then

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \ge \frac{3}{2}+\frac{(b-c)^2}{(b+c)^2}.$$

(Vasile C., 2014)

Solution. From

$$3b \le 3 = a+b+c,$$

we get

$$a \ge 2b-c.$$

Let

$$E(a,b,c)=\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

We will show that

$$E(a,b,c) \ge E(2b-c,b,c) \ge \frac{3}{2}+\frac{(b-c)^2}{(b+c)^2}.$$

We have

$$E(a,b,c)-E(2b-c,b,c)=(a-2b+c)F,$$

where

$$F=\frac{1}{b+c}-\frac{1}{2(c+a)}-\frac{c}{(a+b)(3b-c)}.$$

Since $a-2b+c \ge 0$, we need to show that $F \ge 0$. This is true because

$$\begin{aligned}
F &= \frac{1}{2}\left(\frac{1}{b+c}-\frac{1}{c+a}\right)+\frac{1}{2(b+c)}-\frac{c}{(a+b)(3b-c)} \\
&\ge \frac{1}{2(b+c)}-\frac{c}{(a+b)(3b-c)} \ge \frac{1}{2(a+b)}-\frac{c}{(a+b)(3b-c)} \\
&= \frac{3(b-c)}{2(a+b)(3b-c)} \ge 0.
\end{aligned}$$

The right inequality is also true because

$$\begin{aligned}
E(2b-c,b,c)-\frac{3}{2}-\frac{(b-c)^2}{(b+c)^2} &= \frac{(b-c)^2}{b+c}\left[\frac{3}{3b-c}-\frac{1}{b+c}\right] \\
&= \frac{4c(b-c)^2}{(b+c)^2(3b-c)} \ge 0.
\end{aligned}$$

The proof is completed. The original inequality is an equality for $a=b=c=1$, and also for $a=2$, $b=1$, $c=0$.

□

P 2.58. *Let a,b,c be positive real numbers such that $a=\min\{a,b,c\}$. Prove that*

(a) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{2(b-c)^2}{3(b^2+c^2)}\le 1;$$

(b) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{b^2+bc+c^2}\le 1;$$

(c) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-b)^2}{2(a^2+b^2)}\le 1.$$

(Vasile C., 2014)

Solution. (a) *First Solution.* Since

$$3(b^2+c^2)\ge 2(a^2+b^2+c^2),$$

it suffices to show that

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{a^2+b^2+c^2}\le 1.$$

This inequality is equivalent to

$$(a-b)(a-c)\ge 0,$$

which is clearly true. The equality holds for $a=b=c$.

Second Solution. Write the inequality as follows:

$$\frac{4(b-c)^2}{3(b^2+c^2)}\le\frac{(b-c)^2+(a-b)^2+(a-c)^2}{a^2+b^2+c^2},$$

$$3(b^2+c^2)[(a-b)^2+(a-c)^2]\ge(b-c)^2(4a^2+b^2+c^2),$$

$$3(b^2+c^2)[(b-c)^2+2(a-b)(a-c)]\ge(b-c)^2(4a^2+b^2+c^2),$$

$$6(b^2+c^2)(a-b)(a-c)+2(b-c)^2(b^2+c^2-2a^2)\ge 0.$$

The last inequality is true because $(a-b)(a-c)\ge 0$ and $b^2+c^2-2a^2\ge 0$.

(b) Without loss of generality, assume that

$$a\le b\le c.$$

Write the inequality as

$$\frac{ab+bc+ca}{a^2+b^2+c^2}\le\frac{3bc}{b^2+bc+c^2};$$

that is,

$$E(a,b,c)\ge 0,$$

where

$$E(a,b,c) = 3bca^2 - (b+c)(b^2+c^2+bc)a + bc(2b^2+2c^2-bc).$$

We will show that

$$E(a,b,c) \ge E(b,b,c) \ge 0.$$

We have

$$\begin{aligned} E(a,b,c) - E(b,b,c) &= 3bc(a^2-b^2) - (b+c)(b^2+c^2+bc)(a-b) \\ &= (b-a)[(b+c)(b^2+c^2+bc) - 3bc(a+b)] \\ &\ge (b-a)[(b+c)(b^2+c^2+bc) - 3bc(c+b)] \\ &= (b-a)(b+c)(b-c)^2 \ge 0. \end{aligned}$$

Also,

$$E(b,b,c) = b(c-b)^3 \ge 0.$$

The equality holds for $a = b = c$, and also for $a = b = 0$ or $a = c = 0$.

(c) Write the inequality as follows:

$$\frac{ab+(a+b)c}{a^2+b^2+c^2} \le \frac{(a+b)^2}{2(a^2+b^2)},$$

$$(a+b)^2c^2 - 2(a+b)(a^2+b^2)c + (a^2+b^2)^2 \ge 0,$$

$$[(a+b)c - (a^2+b^2)]^2 \ge 0.$$

The equality holds for $c = \dfrac{a^2+b^2}{a+b}$.

□

P 2.59. *Let a,b,c be positive real numbers such that*

$$a \le 1 \le b \le c, \qquad a+b+c = 3,$$

then

$$\frac{ab+bc+ca}{a^2+b^2+c^2} + \frac{(b-c)^2}{bc} \le 1.$$

(Vasile C., 2014)

Solution. From

$$3b \ge 3 = a+b+c,$$

we get

$$a \le 2b - c.$$

Write the inequality as follows:

$$\frac{2(b-c)^2}{bc} \le \frac{(b-c)^2+(a-b)^2+(a-c)^2}{a^2+b^2+c^2},$$

$$(b-a)^2+(c-a)^2 \ge \left(\frac{2a^2+2b^2+2c^2}{bc}-1\right)(c-b)^2,$$

$$(c-b)^2+2(b-a)(c-a) \ge \left(\frac{2a^2+2b^2+2c^2}{bc}-1\right)(c-b)^2,$$

$$(b-a)(c-a) \ge \left(\frac{a^2+b^2+c^2}{bc}-1\right)(c-b)^2.$$

Since

$$b-a \ge b-(2b-c) = c-b \ge 0, \qquad c-a \ge c-(2b-c) = 2(c-b) \ge 0,$$

it suffices to show that

$$2 \ge \frac{a^2+b^2+c^2}{bc}-1,$$

which is equivalent to

$$3bc \ge a^2+b^2+c^2.$$

This is true if

$$3bc \ge (2b-c)^2+b^2+c^2,$$

which reduces to

$$7bc \ge 5b^2+2c^2,$$

$$(c-b)(5b-2c) \ge 0.$$

Thus, we only need to show that $5b-2c \ge 0$. Indeed, we have

$$5b-2c > 2(2b-c) \ge 2a > 0.$$

The equality holds for $a=b=c=1$.

□

P 2.60. *Let a,b,c be nonnegative real numbers such that $a=\max\{a,b,c\}$ and $b+c>0$. Prove that*

(a) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{2(ab+bc+ca)} \le 1;$$

(b) $$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{2(b-c)^2}{(a+b+c)^2} \le 1.$$

(Vasile C., 2014)

Solution. Without loss of generality, assume that $a \ge b \ge c$.

(a) Write the inequality as follows:

$$\frac{(b-c)^2}{ab+bc+ca} \le \frac{(b-c)^2+(a-b)^2+(a-c)^2}{a^2+b^2+c^2},$$

$$(ab+bc+ca)[(a-b)^2+(a-c)^2] \ge (b-c)^2(a^2+b^2+c^2-ab-bc-ca).$$

Since

$$ab+bc+ca \ge ab \ge b^2 \ge (b-c)^2,$$

it suffices to show that

$$(a-b)^2+(a-c)^2 \ge a^2+b^2+c^2-ab-bc-ca.$$

Indeed,

$$(a-b)^2+(a-c)^2-(a^2+b^2+c^2-ab-bc-ca) = (a-b)(a-c) \ge 0.$$

The equality holds for $a=b=c$, for $a=b$ and $c=0$, and for $a=c$ and $b=0$.

(b) Write the inequality as follows:

$$\frac{4(b-c)^2}{(a+b+c)^2} \le \frac{(b-c)^2+(a-b)^2+(a-c)^2}{a^2+b^2+c^2},$$

$$(a+b+c)^2[(a-b)^2+(a-c)^2] \ge (b-c)^2[3(a^2+b^2+c^2)-2(ab+bc+ca)],$$

$$(a+b+c)^2[(b-c)^2+2(a-b)(a-c)] \ge (b-c)^2[3(a^2+b^2+c^2)-2(ab+bc+ca)],$$

$$(a+b+c)^2(a-b)(a-c) \ge (b-c)^2[a^2+b^2+c^2-2(ab+bc+ca)].$$

Since

$$a^2+b^2+c^2-2(ab+bc+ca) = (a-b)^2-c(2a+2b-c) \le (a-b)^2,$$

it suffices to show that

$$(a+b+c)^2(a-c) \ge (b-c)^2(a-b).$$

This inequality is true because

$$(a+b+c)^2 \ge (b-c)^2$$

and

$$a-c \ge a-b.$$

The equality holds for $a=b=c$, for $a=b$ and $c=0$, and for $a=c$ and $b=0$.

□

P 2.61. *Let a, b, c be positive real numbers. Prove that*

(a) if $a \ge b \ge c$, then

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(a-c)^2}{a^2-ac+c^2}\ge 1;$$

(b) if $a \ge 1 \ge b \ge c$ and $abc = 1$, then

$$\frac{ab+bc+ca}{a^2+b^2+c^2}+\frac{(b-c)^2}{b^2-bc+c^2}\le 1.$$

(Vasile C., 2014)

Solution. (a) Write the inequality as follows:

$$\frac{ab+bc+ca}{a^2+b^2+c^2}\ge\frac{ac}{a^2-ac+c^2},$$

$$acb^2-(a+c)(a^2-ac+c^2)b+a^2c^2\le 0,$$

$$acb^2-(a^3+c^3)b+a^2c^2\le 0,$$

$$(ab-c^2)(bc-a^2)\le 0.$$

Because $ab-c^2\ge 0$ and $bc-a^2\le 0$, the conclusion follows. The equality holds for $a=b=c$.

(b) From

$$b^3\le 1=abc,$$

it follows that

$$b^2\le ac.$$

Write the inequality as follows:

$$\frac{ab+bc+ca}{a^2+b^2+c^2}\le\frac{bc}{b^2-bc+c^2},$$

$$bca^2-(b+c)(b^2-bc+c^2)a+b^2c^2\ge 0,$$

$$bca^2-(b^3+c^3)a+b^2c^2\ge 0,$$

$$(ab-c^2)(ac-b^2)\ge 0.$$

The inequality is true because $ab-c^2\ge 0$ and $ac-b^2\ge 0$. The equality holds for $a=b=c=1$.

□

P 2.62. *Let a, b, c be positive real numbers such that $a = \min\{a, b, c\}$. Prove that*

(a) $$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{4(b-c)^2}{3(b+c)^2};$$

(b) $$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{(a-b)^2}{(a+b)^2}.$$

(Vasile C., 2014)

Solution. (a) *First Solution.* Since

$$3(b+c)^2 \ge 12bc \ge 4(ab+bc+ca),$$

it suffices to prove that

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{(b-c)^2}{ab+bc+ca},$$

which is equivalent to the obvious inequality

$$(a-b)(a-c) \ge 0.$$

The equality holds for $a = b = c$.

Second Solution. Since $(b+c)^2 \ge 4bc$, it suffices to prove that

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{(b-c)^2}{3bc}.$$

Write this inequality as follows:

$$\frac{(a-b)^2+(a-c)^2+(b-c)^2}{ab+bc+ca} \ge \frac{2(b-c)^2}{3bc},$$

$$3bc[(a-b)^2+(a-c)^2] \ge (b-c)^2(2ab+2ac-bc),$$

$$3bc[(b-c)^2+2(a-b)(a-c)] \ge (b-c)^2(2ab+2ac-bc),$$

$$6bc(a-b)(a-c)+2(b-c)^2(2bc-ab-ac) \ge 0.$$

The last inequality is true because $(a-b)(a-c) \ge 0$ and

$$2bc-ab-ac = b(c-a)+c(b-a) \ge 0.$$

(b) Write the inequality as follows:

$$\frac{a^2+b^2+c^2}{ab+(a+b)c} \ge \frac{2(a^2+b^2)}{(a+b)^2},$$

$$(a+b)^2c^2-2(a+b)(a^2+b^2)c+(a^2+b^2)^2 \ge 0,$$

$$[(a+b)c-(a^2+b^2)]^2 \ge 0.$$

The equality holds for $c = \dfrac{a^2+b^2}{a+b}$.

□

P 2.63. *If a, b, c are positive real numbers, then*

$$\frac{a^2+b^2+c^2}{ab+bc+ca} \ge 1 + \frac{9(a-c)^2}{4(a+b+c)^2}.$$

(Vasile C., 2014)

Solution. Write the inequality as follows:

$$\frac{(b-c)^2+(a-b)^2+(a-c)^2}{ab+bc+ca} \ge \frac{9(a-c)^2}{2(a+b+c)^2},$$

$$2(a+b+c)^2[(b-c)^2+(a-b)^2] \ge (a-c)^2[5(ab+bc+ca)-2(a^2+b^2+c^2)],$$

$$2(a+b+c)^2[(a-c)^2-2(a-b)(b-c)] \ge (a-c)^2[5(ab+bc+ca)-2(a^2+b^2+c^2)],$$

$$(a-c)^2[4(a^2+b^2+c^2)-(ab+bc+ca)] \ge 4(a+b+c)^2(a-b)(a-c).$$

Consider further the nontrivial case $(a-b)(a-c) \ge 0$. Since

$$(a-c)^2 = [(a-b)+(b-c)]^2 \ge 4(a-b)(b-c),$$

it suffices to show that

$$4(a^2+b^2+c^2)-(ab+bc+ca) \ge (a+b+c)^2.$$

Indeed,

$$4(a^2+b^2+c^2)-(ab+bc+ca)-(a+b+c)^2 = 3(a^2+b^2+c^2-ab-bc-ca) \ge 0.$$

The equality holds for $a = b = c$.

□

P 2.64. *Let a, b, c be nonnegative real numbers, no two of which are zero. If $a = \min\{a, b, c\}$, then*

$$\frac{1}{\sqrt{a^2-ab+b^2}} + \frac{1}{\sqrt{b^2-bc+c^2}} + \frac{1}{\sqrt{c^2-ca+a^2}} \ge \frac{6}{b+c}.$$

Solution. Since

$$\frac{1}{\sqrt{a^2-ab+b^2}} + \frac{1}{\sqrt{b^2-bc+c^2}} + \frac{1}{\sqrt{c^2-ca+a^2}} \ge \frac{1}{b} + \frac{1}{\sqrt{b^2-bc+c^2}} + \frac{1}{c},$$

it suffices to show that

$$\frac{1}{b} + \frac{1}{\sqrt{b^2-bc+c^2}} + \frac{1}{c} \ge \frac{6}{b+c}.$$

Write this inequality as

$$\frac{b}{c}+\frac{c}{b}+\sqrt{\frac{b^2+c^2+2bc}{b^2+c^2-bc}}\geq 4,$$

which is equivalent to

$$\sqrt{\frac{x+2}{x-1}}\geq 4-x,$$

where $x=\dfrac{b}{c}+\dfrac{c}{b}$, $x\geq 2$. Consider the non-trivial case $2\leq x\leq 4$. The inequality is true if

$$\frac{x+2}{x-1}\geq (4-x)^2,$$

which is equivalent to

$$(x-2)(x^2-7x+9)\leq 0.$$

This inequality is true because

$$x^2-7x+9<x^2-7x+10=(x-2)(x-5)\leq 0.$$

The equality holds for $a=b=c$, and also $a=0$ and $b=c$.

□

P 2.65. *If* $a\geq 1\geq b\geq c\geq 0$ *such that*

$$ab+bc+ca=abc+2,$$

then

$$ac\leq 4-2\sqrt{2}.$$

(Vasile C., 2012)

Solution. By hypothesis, we have

$$a=\frac{2-bc}{b+c-bc}.$$

Since

$$ac\leq \frac{1}{2}a(b+c)=\frac{(2-bc)(b+c)}{2(b+c-bc)}=\frac{2-bc}{2-\frac{2bc}{b+c}}\leq \frac{2-bc}{2-\sqrt{bc}},$$

it suffices to show that

$$\frac{2-bc}{2-\sqrt{bc}}\leq 4-2\sqrt{2},$$

which is equivalent to

$$(\sqrt{bc}-2+\sqrt{2})^2\geq 0.$$

The equality holds for $a=2$ and $b=c=2-\sqrt{2}$.

□

P 2.66. *If a, b, c are nonnegative real numbers such that*

$$ab + bc + ca = 3, \qquad a \le 1 \le b \le c,$$

then

(a) $$a + b + c \le 4;$$

(b) $$2a + b + c \le 4.$$

Solution. From

$$(1 - b)(1 - c) \ge 0,$$

we get

$$bc \ge b + c - 1.$$

Therefore, we have

$$3 = a(b + c) + bc \ge a(b + c) + b + c - 1 = (a + 1)(b + c) - 1,$$

$$b + c \le \frac{4}{a + 1},$$

hence

$$a + b + c - 4 \le a + \frac{4}{a + 1} - 4 = \frac{a(a - 3)}{a + 1} \le 0,$$

$$2a + b + c - 4 \le 2a + \frac{4}{a + 1} - 4 = \frac{2a(a - 1)}{a + 1} \le 0.$$

The equality holds for $a = 0$, $b = 1$ and $c = 3$. In addition, the inequality (b) is also an equality for $a = b = c = 1$.

□

P 2.67. *Let a, b, c be nonnegative real numbers such that $a \le b \le c$. Prove that*

(a) if $a + b + c = 3$, then

$$a^4(b^4 + c^4) \le 2;$$

(b) if $a + b + c = 2$, then

$$c^4(a^4 + b^4) \le 1.$$

(Vasile C., 2012)

Solution. (a) Let x, y be nonnegative real numbers. We claim that

$$x^4 - y^4 \geq 4y^3(x-y).$$

Indeed, this inequality follows from

$$\begin{aligned} x^4 - y^4 - 4y^3(x-y) &= (x-y)(x^3 + x^2y + xy^2 - 3y^3) \\ &= (x-y)[(x^3-y^3) + y(x^2-y^2) + y^2(x-y)]. \end{aligned}$$

Using this inequality, we can show that

$$b^4 + c^4 \leq a^4 + (b+c-a)^4.$$

Indeed, we have

$$\begin{aligned} a^4 + (b+c-a)^4 - b^4 - c^4 &= (a^4 - b^4) + (b+c-a)^4 - c^4 \\ &\geq 4b^3(a-b) + 4c^3(b+c-a-c) \\ &= 4(a-b)(b^3-c^3) \geq 0. \end{aligned}$$

Thus, it suffices to show that

$$a^4[a^4 + (b+c-a)^4] \leq 2,$$

which is equivalent to $f(a) \leq 2$, where

$$f(a) = a^8 + a^4(3-2a)^4, \quad 0 \leq a \leq 1.$$

If $f'(a) \geq 0$ for $0 \leq a \leq 1$, then $f(a)$ is increasing, hence $f(a) \leq f(1) = 2$. From the derivative

$$f'(a) = 4a^3[2a^4 - (4a-3)(3-2a)^3],$$

we need to show that

$$2a^4 \geq (4a-3)(3-2a)^3.$$

This inequality is true for the trivial case $0 \leq a \leq 3/4$. Consider further that $3/4 < a \leq 1$. We need to show that $h(a) \geq 0$, where

$$h(a) = \ln 2 + 4\ln a - \ln(4a-3) - 3\ln(3-2a), \quad 3/4 < a \leq 1.$$

From

$$h'(a) = \frac{4}{a} - \frac{4}{4a-3} + \frac{6}{3-2a} = \frac{6(7a-6)}{a(4a-3)(3-2a)},$$

it follows that $h(a)$ is decreasing on $(3/4, 6/7]$ and increasing on $[6/7, 1]$. Thus,

$$h(a) \geq h\left(\frac{6}{7}\right) = \ln 2 + 4\ln\frac{6}{7} - \ln\frac{3}{7} - 3\ln\frac{9}{7} = \ln\frac{32}{27} > 0.$$

The equality holds for $a = b = c = 1$.

(b) Since $a^4+b^4 \le (a+b)^4$, it suffices to show that

$$c^4(a+b)^4 \le 1,$$

which is true if

$$c(a+b) \le 1.$$

Indeed, we have

$$1-c(a+b) = 1-c(2-c) = (c-1)^2 \ge 0.$$

The equality holds for $a=0$ and $b=c=1$.

□

P 2.68. *If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then*

(a) $$a^2+b^2+c^2-a-b-c \ge \frac{5}{8}(a-c)^2;$$

(b) $$a^2+b^2+c^2-a-b-c \ge \frac{5}{2}\min\{(a-b)^2,(b-c)^2,(c-a)^2\}.$$

(Vasile C., 2014)

Solution. Denote

$$E = a^2+b^2+c^2-a-b-c, \qquad S = a^2+b^2+c^2-ab-bc-ca.$$

From

$$(a+b+c)^2 \ge 3(ab+bc+ca),$$

it follows that

$$a+b+c \ge 3.$$

We have

$$a+b+c-\sqrt{3(ab+bc+ca)} = \frac{S}{a+b+c+\sqrt{3(ab+bc+ca)}},$$

$$a+b+c-3 = \frac{S}{a+b+c+3},$$

$$(a+b+c)^2-3(a+b+c) = \frac{(a+b+c)S}{a+b+c+3},$$

$$-3(a+b+c) = -(a+b+c)^2 + \frac{S}{1+\dfrac{3}{a+b+c}},$$

$$-3(a+b+c) \ge -(a+b+c)^2 + \frac{S}{2},$$

therefore

$$E \geq a^2 + b^2 + c^2 - \frac{1}{3}(a+b+c)^2 + \frac{S}{6},$$

which is equivalent to

$$E \geq \frac{5S}{6}.$$

(a) It suffices to show that

$$\frac{5S}{6} \geq \frac{5}{8}(a-c)^2,$$

which is equivalent to

$$\frac{S}{3} \geq \frac{(a-c)^2}{4},$$

$$\frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{3} \geq \frac{(a-c)^2}{2},$$

$$2(a-b)^2 + 2(b-c)^2 \geq (a-c)^2,$$

$$2(a-b)^2 + 2(b-c)^2 \geq [(a-b)+(b-c)]^2,$$

$$(a-b)^2 + (b-c)^2 \geq 2(a-b)(b-c),$$

$$(a-2b+c)^2 \geq 0.$$

The equality holds for $a = b = c = 1$.

(b) Due to symmetry, without loss of generality, assume that

$$a \geq b \geq c.$$

It suffices to show that

$$\frac{5S}{6} \geq \frac{5}{2}\min\{(a-b)^2, (b-c)^2\},$$

which is equivalent to

$$S \geq 3\min\{(a-b)^2, (b-c)^2\},$$

$$(a-b)^2 + (b-c)^2 + (a-c)^2 \geq 6\min\{(a-b)^2, (b-c)^2\},$$

$$(a-b)^2 + (b-c)^2 + [(a-b)+(b-c)]^2 \geq 6\min\{(a-b)^2, (b-c)^2\},$$

$$(a-b)^2 + (b-c)^2 + (a-b)(b-c) \geq 3\min\{(a-b)^2, (b-c)^2\}.$$

The last inequality is clearly true. The equality holds for $a = b = c = 1$.

□

P 2.69. *If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then*

$$\frac{a^3+b^3+c^3}{a+b+c} \ge 1+\frac{5}{9}(a-c)^2.$$

(Vasile C., 2014)

Solution. It suffices to consider the case

$$a \ge b \ge c.$$

Write the inequality as

$$E \ge \frac{5}{9}(a-c)^2,$$

where

$$E = \frac{a^3+b^3+c^3}{a+b+c} - 1.$$

We have

$$\begin{aligned} E &= \frac{a^3+b^3+c^3}{a+b+c} - \frac{ab+bc+ca}{3} \\ &= \frac{3(a^3+b^3+c^3)-(a+b+c)(ab+bc+ca)}{3(a+b+c)} \\ &= \frac{A+B}{3(a+b+c)}, \end{aligned}$$

where

$$A = \sum[a^3+b^3-ab(a+b)], \qquad B = \sum a^3 - 3abc.$$

Since

$$A = \sum(a+b)(a-b)^2,$$

$$B = \frac{1}{2}(a+b+c)\sum(a-b)^2,$$

we get

$$E = \frac{\sum(3a+3b+c)(a-b)^2}{6(a+b+c)}.$$

Thus, we need to show that

$$\sum(3a+3b+c)(a-b)^2 \ge \frac{10}{3}(a+b+c)(a-c)^2,$$

which is equivalent to

$$3(3a+3b+c)(a-b)^2 + 3(a+3b+3c)(b-c)^2 \ge (a+7b+c)(a-c)^2.$$

Using the substitution

$$a = c+x, \quad b = c+y, \quad x \ge y \ge 0,$$

the inequality becomes

$$3(7c+3x+3y)(x-y)^2+3(7c+x+3y)y^2 \ge (9c+x+7y)x^2,$$

which is equivalent to

$$6c(2x^2-7xy+7y^2)+2(x+y)(2x-3y)^2 \ge 0.$$

This inequality is true since

$$2x^2-7xy+7y^2=(2x^2+7y^2)-7xy \ge (2\sqrt{14}-7)xy \ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=3/\sqrt{2}$, $b=\sqrt{2}$, $c=0$.

□

P 2.70. *If a,b,c are nonnegative real numbers such that*

$$a \ge b \ge c, \quad ab+bc+ca=3,$$

then

(a) $$\frac{a^3+b^3+c^3}{a+b+c} \ge 1+\frac{7}{9}(a-b)^2;$$

(b) $$\frac{a^3+b^3+c^3}{a+b+c} \ge 1+\frac{2}{3}(b-c)^2.$$

(c) $$\frac{a^3+b^3+c^3}{a+b+c} \ge 1+\frac{7}{3}\min\{(a-b)^2,(b-c)^2\}.$$

(Vasile C., 2014)

Solution. As we have shown in the proof of the preceding problem P 2.69,

$$\frac{a^3+b^3+c^3}{a+b+c}-1=\frac{\sum(3a+3b+c)(a-b)^2}{6(a+b+c)}.$$

(a) Write the inequality as

$$\sum(3a+3b+c)(a-b)^2 \ge \frac{14}{3}(a+b+c)(a-b)^2,$$

$$3(a+3b+3c)(b-c)^2+3(3a+b+3c)(a-c)^2 \ge (5a+5b+11c)(a-b)^2.$$

It suffices to show that

$$3(3a+b+3c)(a-c)^2 \ge (5a+5b+11c)(a-b)^2.$$

This is true since
$$(a-c)^2 \ge (a-b)^2$$
and
$$3(3a+b+3c)-(5a+5b+11c) = 2(2a-b-c) \ge 0.$$
The equality holds for $a=b=c=1$.

(b) Write the desired inequality as
$$\sum (3a+3b+c)(a-b)^2 \ge 4(a+b+c)(b-c)^2,$$
$$(3a+3b+c)(a-b)^2+(3a+b+3c)(a-c)^2 \ge (3a+b+c)(b-c)^2.$$
It suffices to show that
$$(3a+b+3c)(a-c)^2 \ge (3a+b+c)(b-c)^2.$$
This is true since
$$(a-c)^2 \ge (a-b)^2$$
and
$$(3a+b+3c)-(3a+b+c) = 2c \ge 0.$$
The equality holds for $a=b=c=1$, and also for $a=b=\sqrt{3}$, $c=0$.

(c) Denote
$$m = \min\{(a-b)^2, (b-c)^2\},$$
then write the desired inequality as
$$\frac{\sum (3a+3b+c)(a-b)^2}{6(a+b+c)} \ge \frac{7}{3}m,$$
$$\sum (3a+3b+c)(a-b)^2 \ge 14(a+b+c)m,$$
$$(3a+3b+c)(a-b)^2+(a+3b+3c)(b-c)^2+(3a+b+3c)[(a-b)+(b-c)]^2 \ge 14(a+b+c)m,$$
$$(3a+2b+2c)(a-b)^2+(2a+2b+3c)(b-c)^2+(3a+b+3c)(a-b)(b-c) \ge 7(a+b+c)m.$$

Case 1: $a-2b+c \ge 0$. The inequality is true if
$$(3a+2b+2c)+(2a+2b+3c)+(3a+b+3c) \ge 7(a+b+c),$$
which is equivalent to $a-2b+c \ge 0$.

Case 2: $a-2b+c \le 0$. Since $a-b \le b-c$, we need to show that
$$(3a+2b+2c)(a-b)^2+(2a+2b+3c)(b-c)^2+(3a+b+3c)(a-b)(b-c) \ge 7(a+b+c)(a-b)^2,$$
which is equivalent to
$$(2a+2b+3c)(b-c)^2+(3a+b+3c)(a-b)(b-c) \ge (4a+5b+5c)(a-b)^2.$$

Since $b-c \ge a-b \ge 0$, it suffices to show that

$$(2a+2b+3c)(a-b)(b-c)+(3a+b+3c)(a-b)(b-c) \ge (4a+5b+5c)(a-b)^2.$$

This is true if

$$(2a+2b+3c)(b-c)+(3a+b+3c)(b-c) \ge (4a+5b+5c)(a-b),$$

which is equivalent to

$$(5a+3b+6c)(b-c) \ge (4a+5b+5c)(a-b),$$

$$2(2b-a-c)(2b+2a+3c) \ge 0.$$

The equality holds for $2b = a+c$ and $a^2+4ac+c^2 = 6$.

□

P 2.71. *If a,b,c are nonnegative real numbers such that $ab+bc+ca=3$, then*

$$a^4+b^4+c^4-a^2-b^2-c^2 \ge \frac{11}{4}(a-c)^2.$$

(Vasile C., 2014)

Solution. It suffices to consider the case $a \ge b \ge c$. Denote

$$S = a^2+b^2+c^2, \quad q = ab+bc+ca.$$

Summing the identities

$$a^4+b^4+c^4-\frac{1}{3}S^2 = \frac{(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2}{3}$$

and

$$\frac{1}{3}S^2-\frac{1}{3}Sq = S\cdot\frac{(a-b)^2+(b-c)^2+(c-a)^2}{6},$$

we get

$$a^4+b^4+c^4-a^2-b^2-c^2 = \frac{(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2}{3} + S\cdot\frac{(a-b)^2+(b-c)^2+(c-a)^2}{6}.$$

Therefore, we can write the desired inequality in the homogeneous form

$$\frac{(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2}{3}+S\cdot\frac{(a-b)^2+(b-c)^2+(c-a)^2}{6} \ge \frac{11}{12}q(a-c)^2.$$

Since

$$(a-b)^2+(b-c)^2 \ge \frac{1}{2}[(a-b)+(b-c)]^2 = \frac{1}{2}(a-c)^2,$$

it suffices to prove that

$$\frac{(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2}{3}+\frac{S(a-c)^2}{4} \ge \frac{11}{12}q(a-c)^2,$$

which is equivalent to

$$4(a+b)^2(a-b)^2+4(b+c)^2(b-c)^2+E(a-c)^2 \ge 0,$$

where

$$E = 4(a+c)^2+3S-11q.$$

Using the substitution

$$b = c+x, \quad a = c+x+y, \quad x,y \ge 0,$$

the inequality becomes

$$4(2c+2x+y)^2y^2+4(2c+x)^2x^2+E(x+y)^2 \ge 0,$$

where

$$E = -8c^2-16xc-x^2+7y^2+3xy.$$

Write this inequality as

$$Ac^2+D \ge 2Bc,$$

where

$$A = 8(x-y)^2, \;\; B = 8y(x-y)(2x+y), \quad D = 3x^4+11y^4+28x^2y^2+33xy^3+x^3y.$$

Since $Ac^2+D \ge 2c\sqrt{AD}$, it suffices to show that $AD \ge B^2$. Indeed,

$$\begin{aligned} AD-B^2 &= 8(x-y)^2[3x^4+11y^4+28x^2y^2+33xy^3+x^3y-8y^2(2x+y)^2] \\ &= 8(x-y)^2[x^4+y^4+2(x^2-y^2)^2+xy(x^2+y^2)] \ge 0. \end{aligned}$$

This completes the proof. The equality holds for $a=b=c=1$.

□

P 2.72. *If a,b,c are nonnegative real numbers such that*

$$a \ge b \ge c, \quad ab+bc+ca = 3,$$

then

(a) $$a^4+b^4+c^4-a^2-b^2-c^2 \ge \frac{11}{3}(a-b)^2;$$

(b) $$a^4+b^4+c^4-a^2-b^2-c^2 \ge \frac{10}{3}(b-c)^2.$$

(Vasile C., 2014)

Solution. Denote

$$S = a^2 + b^2 + c^2, \quad q = ab + bc + ca.$$

As we have shown in the proof of the preceding problem,

$$a^4 + b^4 + c^4 - a^2 - b^2 - c^2 = \frac{(a^2-b^2)^2 + (b^2-c^2)^2 + (c^2-a^2)^2}{3} + S \cdot \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{6}.$$

(a) Write the desired inequality in the homogeneous form

$$(a^2-b^2)^2 + (b^2-c^2)^2 + (c^2-a^2)^2 + S \cdot \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2} \ge \frac{11}{3} q(a-b)^2.$$

Since

$$(a^2-b^2)^2 + (b^2-c^2)^2 + (c^2-a^2)^2 \ge (a^2-b^2)^2 + (a^2-c^2)^2 \ge 2(a^2-b^2)^2$$

and

$$(a-b)^2 + (b-c)^2 + (c-a)^2 \ge (a-b)^2 + (a-c)^2 \ge 2(a-b)^2,$$

it suffices to prove that

$$2(a+b)^2 + a^2 + b^2 + c^2 \ge \frac{11}{3}(ab + bc + ca);$$

that is,

$$9(a^2+b^2) + ab + 3c^2 \ge 11c(a+b).$$

Since

$$9(a^2+b^2) + ab - \frac{19}{4}(a+b)^2 = \frac{17}{4}(a-b)^2 \ge 0,$$

we have

$$9(a^2+b^2) + ab + 3c^2 - 11c(a+b) \ge \frac{19}{4}(a+b)^2 + 3c^2 - 11c(a+b) = \frac{(a+b-2c)(19a+19b-6c)}{4} \ge 0.$$

The equality holds for $a = b = c = 1$.

(b) Write the desired inequality in the homogeneous form

$$(a^2-b^2)^2 + (b^2-c^2)^2 + (c^2-a^2)^2 + S \cdot \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{2} \ge \frac{10}{3} q(b-c)^2.$$

Since

$$(a^2-b^2)^2 + (b^2-c^2)^2 + (c^2-a^2)^2 \ge (b^2-c^2)^2 + (a^2-c^2)^2 \ge (b+c)^2(b-c)^2 + (a+c)^2(b-c)^2$$

and

$$(a-b)^2+(b-c)^2+(c-a)^2 \ge (b-c)^2+(a-c)^2 \ge 2(b-c)^2,$$

it suffices to prove that

$$(b+c)^2+(a+c)^2+a^2+b^2+c^2 \ge \frac{10}{3}(ab+bc+ca);$$

that is,

$$6(a^2+b^2)-10ab+9c^2 \ge 4c(a+b).$$

Since

$$6(a^2+b^2)-10ab-\frac{1}{2}(a+b)^2 = \frac{11}{2}(a-b)^2 \ge 0,$$

we have

$$\begin{aligned}6(a^2+b^2)-10ab+9c^2-4c(a+b) &\ge \frac{1}{2}(a+b)^2+9c^2-4c(a+b)\\ &\ge 2\sqrt{\frac{9}{2}}\,c(a+b)-4c(a+b)\\ &= (3\sqrt{2}-4)c(a+b) \ge 0.\end{aligned}$$

The equality holds for $a=b=c=1$.

Remark. Similarly, we can prove the following refinement of the inequality in (b):

$$a^4+b^4+c^4-a^2-b^2-c^2 \ge \frac{1+\sqrt{33}}{2}\,(b-c)^2,$$

with equality for $a=b=c=1$, and also for $a=b=\frac{3+\sqrt{33}}{4}\,c$.

□

P 2.73. *Let a,b,c be nonnegative real numbers such that*

$$a \le b \le c, \quad a+b+c=3.$$

Find the greatest real number k such that

$$\sqrt{(56b^2+25)(56c^2+25)}+k(b-c)^2 \le 14(b+c)^2+25.$$

(Vasile C., 2014)

Solution. For $a=b=0$ and $c=3$, the inequality becomes

$$115+9k \le 126+25, \quad k \le 4.$$

To show that 4 is the greatest possible value of k, we need to prove the inequality

$$\sqrt{(56b^2+25)(56c^2+25)}+4(b-c)^2 \le 14(b+c)^2+25,$$

which is equivalent to

$$\sqrt{(56b^2+25)(56c^2+25)} \le 10(b^2+c^2)+36bc+25.$$

By squaring, the inequality becomes as follows:

$$(10b^2+10c^2+36bc)^2-56^2b^2c^2 \ge 50[28(b^2+c^2)-(10b^2+10c^2+36bc)],$$

$$20(b-c)^2(5b^2+5c^2+46bc) \ge 900(b-c)^2,$$
$$20(b-c)^2(5b^2+5c^2+46bc-45) \ge 0.$$

Therefore, we need to show that

$$5(b+c)^2+36bc-45 \ge 0.$$

From $(a-b)(a-c) \ge 0$, we get

$$bc \ge a(b+c)-a^2 = a(3-a)-a^2 = 3a-2a^2.$$

Thus,

$$5(b+c)^2+36bc-45 \ge 5(3-a)^2+36(3a-2a^2)-45 = a(78-67a) \ge 0.$$

The proof is completed. If $k=4$, then the equality holds for $a=b=c=1$ and also for $a=b=0$ and $c=3$.

□

P 2.74. *If $a \ge b \ge c > 0$ such that $abc=1$, then*

$$3(a+b+c) \le 8+\frac{a}{c}.$$

(Vasile C., 2009)

Solution. Write the inequality in the homogeneous form

$$\frac{3(a+b+c)}{\sqrt[3]{abc}} \le 8+\frac{a}{c},$$

which is equivalent to

$$\frac{3(x^3+y^3+z^3)}{xyz} \le 8+\frac{x^3}{z^3}, \quad x \ge y \ge z > 0.$$

We show that
$$\frac{x^3+y^3+z^3}{xyz} \le \frac{x^3+2z^3}{xz^2} \le \frac{1}{3}\left(8+\frac{x^3}{z^3}\right).$$
Write the left inequality as
$$(y-z)[x^3+z^3-yz(y+z)] \ge 0.$$
This is true since
$$x^3+z^3-yz(y+z) \ge y^3+z^3-yz(y+z) = (y+z)(y-z)^2 \ge 0.$$
Write the right inequality as
$$(x-z)(x^3-2x^2z-2xz^2+6z^3) \ge 0.$$
This is also true since
$$x^3-2x^2z-2xz^2+6z^3 = (x-z)^3+z(x^2-5xz+7z^2) \ge 0.$$
The equality holds for $a=b=c=1$.

□

P 2.75. *If $a \ge b \ge c > 0$, then*
$$(a+b-c)(a^2b-b^2c+c^2a) \ge (ab-bc+ca)^2.$$

Solution. Making the substitution
$$a=(p+1)c, \quad b=(q+1)c, \quad p \ge q \ge 0,$$
we get
$$a+b-c=(p+q+1)c,$$
$$a^2b-b^2c+c^2a=(p^2q+p^2+2pq-q^2+3p-q+1)c^3,$$
$$ab-bc+ca=(pq+2p+1)c^2.$$
Thus, the inequality becomes
$$(p+q+1)(p^2q+p^2+2pq-q^2+3p-q+1) \ge (pq+2p+1)^2,$$
which is equivalent to the obvious inequality
$$p^3(q+1)+q^2(p-q)+2q(p-q) \ge 0.$$
The equality holds for $a=b=c$.

□

P 2.76. *If $a \ge b \ge c \ge 0$, then*

$$\frac{(a-c)^2}{2(a+c)} \le a+b+c-3\sqrt[3]{abc} \le \frac{2(a-c)^2}{a+5c}.$$

(Vasile C., 2007)

Solution. (a) To prove the inequality

$$a+b+c-3\sqrt[3]{abc} \ge \frac{(a-c)^2}{2(a+c)},$$

we will show that

$$a+b+c-3\sqrt[3]{abc} \ge a+c-2\sqrt{ac} \ge \frac{(a-c)^2}{2(a+c)}. \qquad (*)$$

The left inequality is equivalent to

$$b+2\sqrt{ac} \ge 3\sqrt[3]{abc},$$

which is a consequence of the AM-GM inequality. The right inequality in (*) can be written as follows:

$$a^2+c^2+6ac \ge 4(a+c)\sqrt{ac},$$

$$\left(\sqrt{a}-\sqrt{b}\right)^4 \ge 0.$$

The equality holds for $a=b=c$.

(b) To prove the inequality

$$a+b+c-3\sqrt[3]{abc} \le \frac{2(a-c)^2}{a+5c},$$

we will show that

$$a+b+c-3\sqrt[3]{abc} \le 2a+c-3\sqrt[3]{a^2c} \le \frac{2(a-c)^2}{a+5c}. \qquad (**)$$

Write the left inequality as

$$a-b-3\sqrt[3]{ac}\left(\sqrt[3]{a}-\sqrt[3]{b}\right) \ge 0,$$

$$\left(\sqrt[3]{a}-\sqrt[3]{b}\right)\left(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}-3\sqrt[3]{ac}\right) \ge 0.$$

This is true since

$$\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2} \ge 3\sqrt[3]{ab} \ge 3\sqrt[3]{ac}.$$

The right inequality in (**) is an equality for $c=0$. For $c>0$, due to homogeneity, we may assume that $c=1$. In addition, making the substitution $a=x^3$, $x \ge 1$, the right inequality in (**) becomes in succession

$$(x^3+5)(2x^3-3x^2+1) \le 2(x^3-1)^2,$$

$$(x-1)^2(x^3+2x^2-2x-1) \ge 0,$$
$$(x-1)^3(x^2+3x+1) \ge 0.$$

The equality holds for $a=b=c$, and also for $a=b$ and $c=0$.

□

P 2.77. *If $a \ge b \ge c \ge d \ge 0$, then*

$$\frac{(a-d)^2}{a+3d} \le a+b+c+d-4\sqrt[4]{abcd} \le \frac{3(a-d)^2}{a+5d}.$$

(Vasile C., 2009)

Solution. (a) To prove the inequality

$$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{(a-d)^2}{a+3d},$$

we will show that

$$a+b+c+d-4\sqrt[4]{abcd} \ge a+d-2\sqrt{ad} \le \frac{(a-d)^2}{a+3d}. \qquad (*)$$

The left inequality is equivalent to

$$b+c+2\sqrt{ad} \ge 4\sqrt[4]{abcd},$$

which is a consequence of the AM-GM inequality. The right inequality in (*) can be written as follows:

$$(a-d)^2 \ge (a+3d)\left(\sqrt{a}-\sqrt{d}\right)^2,$$

$$2\sqrt{d}\left(\sqrt{a}-\sqrt{d}\right)^3 \ge 0.$$

The equality holds for $a=b=c=d$, and also for $b=c=d=0$.

(b) To prove the inequality

$$a+b+c+d-4\sqrt[4]{abcd} \le \frac{3(a-d)^2}{a+5d},$$

we will show that

$$a+b+c+d-4\sqrt[4]{abcd} \le 2a+c+d-4\sqrt[4]{a^2cd} \le \frac{3(a-d)^2}{a+5d}. \qquad (**)$$

Write the left inequality as

$$a-b-4\sqrt[4]{acd}\left(\sqrt[4]{a}-\sqrt[4]{b}\right) \ge 0,$$

$$\left(\sqrt[4]{a}-\sqrt[4]{b}\right)\left(\sqrt[4]{a^3}+\sqrt[4]{a^2b}+\sqrt[4]{ab^2}+\sqrt[4]{b^3}-4\sqrt[4]{acd}\right)\geq 0.$$

The last inequality follows from the AM-GM inequality:

$$\begin{aligned}\sqrt[4]{a^3}+\sqrt[4]{a^2b}+\sqrt[4]{ab^2}+\sqrt[4]{b^3}-4\sqrt[4]{acd} &\geq \sqrt[4]{a^3}+\sqrt[4]{a^2b}+\sqrt[4]{b^3}-3\sqrt[4]{ab^2}\\ &\geq \sqrt[4]{a^3}+\sqrt[4]{b^3}+\sqrt[4]{b^3}-3\sqrt[4]{ab^2}\geq 0.\end{aligned}$$

Write the right inequality in (**) as

$$F(c)\geq 0,$$

where

$$F(c)=3(a-d)^2-(a+5d)\left(2a+c+d-4\sqrt[4]{a^2cd}\right).$$

Since F is a concave function and $d\leq c\leq a$, it suffices to show that $F(d)\geq 0$ and $F(a)\geq 0$. We have

$$F(d)=3(a-d)^2-2(a+5d)\left(\sqrt{a}-\sqrt{d}\right)^2=\left(\sqrt{a}-\sqrt{d}\right)^3\left(\sqrt{a}+7\sqrt{d}\right)\geq 0$$

and

$$F(a)=3(a-d)^2-(a+5d)\left(3a+d-4\sqrt[4]{a^3d}\right).$$

Setting $a=1$ (due to homogeneity) and substituting $d=x^4$, $0\leq x\leq 1$, the inequality $F(a)\geq 0$ becomes

$$3(1-x^4)^2-(1+5x^4)(3+x^4-4x)\geq 0.$$

Since $3+x^4-4x=(1-x)^2(3+2x+x^2)$, we need to show that

$$3(1+x+x^2+x^3)^2-(1+5x^4)(3+2x+x^2)\geq 0,$$

which is equivalent to

$$x(2+4x+6x^2-3x^3-2x^4-x^5)\geq 0.$$

This inequality is true since

$$2+4x+6x^2-3x^3-2x^4-x^5>6x^2-3x^3-2x^4-x^5\geq 0.$$

The equality holds for $a=b=c=d$, and also for $a=b=c$ and $d=0$.

Remark. The following generalization holds.

- *If* $a_1\geq a_2\geq\cdots\geq a_n\geq 0$, *then*

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n}\leq\frac{(n-1)(a_1-a_n)^2}{a_1+k_na_n},$$

where

$$k_n=\begin{cases}7-\dfrac{8}{n+1}, & n\ odd\\ 7-\dfrac{8}{n}, & n\ even\end{cases}.$$

□

P 2.78. *If $a \ge b \ge c > 0$, then*

(a) $$a+b+c-3\sqrt[3]{abc} \ge \frac{3(a-b)^2}{5a+4b};$$

(b) $$a+b+c-3\sqrt[3]{abc} \ge \frac{64(a-b)^2}{7(11a+24b)}.$$

(Vasile C., 2009)

Solution. We use the inequality

$$a+b+c-3\sqrt[3]{abc} \ge a+2b-3\sqrt[3]{ab^2},$$

which is equivalent to

$$3\sqrt[3]{ab}\left(\sqrt[3]{b}-\sqrt[3]{c}\right) \ge b-c,$$

$$\left(\sqrt[3]{b}-\sqrt[3]{c}\right)\left(3\sqrt[3]{ab}-\sqrt[3]{b^2}-\sqrt[3]{bc}-\sqrt[3]{c^2}\right) \ge 0.$$

Since $a \ge b \ge c$, the inequality is obvious.

(a) It suffices to show that

$$a+2b-3\sqrt[3]{ab^2} \ge \frac{3(a-b)^2}{5a+4b}.$$

Setting $b=1$ (due to homogeneity) and $a=x^3$, $x \ge 1$, this inequality becomes as follows:

$$(5x^3+4)(x^3-3x+2) \ge 3(x^3-1)^2,$$

$$(x-1)^2(2x^4+4x^3-9x^2-2x+5) \ge 0,$$

$$(x-1)^4(2x^2+8x+5) \ge 0.$$

The equality holds for $a=b=c$.

(b) It suffices to show that

$$a+2b-3\sqrt[3]{ab^2} \ge \frac{64(a-b)^2}{7(11a+24b)}.$$

Setting $b=1$ and $a=x^3$, $x \ge 1$, this inequality becomes in succession:

$$7(11x^3+24)(x^3-3x+2) \ge 64(x^3-1)^2,$$

$$(x-1)^2(13x^4+26x^3-192x^2+40x+272) \ge 0,$$

$$(x-1)^2(x-2)^2(13x^3+78x+68) \ge 0.$$

The equality holds for $a=b=c$, and for $\frac{a}{8}=b=c$.

□

P 2.79. *If $a \ge b \ge c > 0$, then*

(a)
$$a+b+c-3\sqrt[3]{abc} \ge \frac{3(b-c)^2}{4b+5c};$$

(b)
$$a+b+c-3\sqrt[3]{abc} \ge \frac{25(b-c)^2}{7(3b+11c)}.$$

(Vasile C., 2009)

Solution. We use the inequality

$$a+b+c-3\sqrt[3]{abc} \ge 2b+c-3\sqrt[3]{b^2c},$$

which is equivalent to

$$a-b \ge 3\sqrt[3]{bc}\left(\sqrt[3]{a}-\sqrt[3]{b}\right),$$

$$\left(\sqrt[3]{a}-\sqrt[3]{b}\right)\left(\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}-3\sqrt[3]{bc}\right) \ge 0.$$

Since $a \ge b \ge c$, the inequality is obvious.

(a) It suffices to show that

$$2b+c-3\sqrt[3]{b^2c} \ge \frac{3(b-c)^2}{4b+5c}.$$

Setting $c=1$ and $b=x^3$, $x \ge 1$, this inequality becomes as follows:

$$(4x^3+5)(2x^3-3x^2+1) \ge 3(x^3-1)^2,$$

$$(x-1)^2(5x^4-2x^3-9x^2+4x+2) \ge 0,$$

$$(x-1)^4(5x^2+8x+2) \ge 0.$$

The equality holds for $a=b=c$.

(b) It suffices to show that

$$2b+c-3\sqrt[3]{b^2c} \ge \frac{25(b-c)^2}{7(3b+11c)}.$$

Setting $c=1$ and $b=x^3$, $x \ge 1$, this inequality becomes in succession:

$$7(3x^3+11)(2x^3-3x^2+1) \ge 25(x^3-1)^2,$$

$$(x-1)^2(17x^4-29x^3-75x^2+104x+52) \ge 0,$$

$$(x-1)^2(x-2)^2(17x^3+39x+13) \ge 0.$$

The equality holds for $a=b=c$, and for $a=b=8c$.

Remark. The following generalization holds.

- *If* $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$*, then*

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{3i(n-j+1)(a_i-a_j)^2}{2(2n+i-2j+2)a_i+2(n+2i-j+1)a_j}$$

for all $i<j$.

□

P 2.80. *If* $a \ge b \ge c > 0$*, then*

$$a+b+c-3\sqrt[3]{abc} \ge \frac{3(a-c)^2}{4(a+b+c)}.$$

(Vasile C., 2009)

Solution. Due to homogeneity, assume that $a+b+c=3$. Let

$$x=\left(\frac{a+c}{2}\right)^2, \quad y=ac, \quad x \ge y.$$

We have

$$x=\left(\frac{3-b}{2}\right)^2, \quad x-y=\left(\frac{a-c}{2}\right)^2.$$

The desired inequality is equivalent to

$$3-3\sqrt[3]{by} \ge x-y.$$

There are two cases to consider.

Case 1: $b \le 1$. By the AM-GM inequality, we have

$$y+2\sqrt{b} \ge 3\sqrt[3]{by}.$$

Thus, it suffices to show that

$$3-2\sqrt{b} \ge x.$$

Indeed,

$$3-2\sqrt{b}-x=3-2\sqrt{b}-\left(\frac{3-b}{2}\right)^2=\frac{1}{4}\left(1-\sqrt{b}\right)^3\left(3+\sqrt{b}\right) \ge 0.$$

Case 2: $b \ge 1$. From

$$a+b+c=b+\frac{a+c}{2}+\frac{a+c}{2} \ge 3\sqrt[3]{b\left(\frac{a+c}{2}\right)^2},$$

we get

$$3 \ge 3\sqrt[3]{bx}.$$

Therefore, it suffices to prove that

$$3\sqrt[3]{bx} - 3\sqrt[3]{by} \ge x - y,$$

which is equivalent to

$$\left(\sqrt[3]{x} - \sqrt[3]{y}\right)\left(3\sqrt[3]{b} - \sqrt[3]{x^2} - \sqrt[3]{xy} - \sqrt[3]{y^2}\right) \ge 0.$$

Since

$$y \le x = \left(\frac{3-b}{2}\right)^2 \le 1 \le b,$$

the inequality is clearly true. The equality holds for $a = b = c$

□

P 2.81. *If $a \ge b \ge c > 0$, then*

(a) $$a^6 + b^6 + c^6 - 3a^2b^2c^2 \ge 12a^2c^2(b-c)^2;$$

(b) $$a^6 + b^6 + c^6 - 3a^2b^2c^2 \ge 10a^3c(b-c)^2.$$

(Vasile C., 2014)

Solution. (a) Let us denote

$$E(a,b,c) = a^6 + b^6 + c^6 - 3a^2b^2c^2 - 12a^2c^2(b-c)^2.$$

We will show that

$$E(a,b,c) \ge E(b,b,c) \ge 0.$$

We have

$$\begin{aligned} E(a,b,c) - E(b,b,c) &= (a^2-b^2)[a^4 + a^2b^2 + b^4 - 3b^2c^2 - 12c^2(b-c)^2] \\ &\ge (a^2-b^2)[3b^2(b^2-c^2) - 12c^2(b-c)^2] \\ &= 3(a^2-b^2)(b-c)[b^3 + c(b-2c)^2] \ge 0. \end{aligned}$$

Also,

$$\begin{aligned} E(b,b,c) &= 2b^6 + c^6 - 3b^4c^2 - 12b^2c^2(b-c)^2 \\ &= (b^2-c^2)^2(2b^2+c^2) - 12b^2c^2(b-c)^2 \\ &= (b-c)^2(2b^4 + 4b^3c - 9b^2c^2 + 2bc^3 + c^4) \\ &= (b-c)^3(2b^3 + 6b^2c^2 - 3bc^2 - c^3) \ge 0. \end{aligned}$$

The equality holds for $a = b = c$.

(b) Let

$$E(a,b,c)=a^6+b^6+c^6-3a^2b^2c^2-12a^2c^2(b-c)^2.$$

We will show that

$$E(a,b,c)\geq E(b,b,c)\geq 0.$$

To prove the left inequality, it suffices to show that for fixed b and c, the function

$$f(a)=E(a,b,c)$$

is increasing on $[b,\infty)$; that is, $f'a)\geq 0$. Indeed, we have the derivative

$$\begin{aligned}f'(a)&=6a[a^4-b^2c^2-5ac(b-c)^2]\geq 6a[a^4-a^2c^2-5ac(a-c)^2]\\&=6a^2(a-c)[a(a+c)-5c(a-c)]=6a^2(a-c)[(a-2c)^2+c^2]\geq 0.\end{aligned}$$

With regard to the right inequality, we have

$$\begin{aligned}E(b,b,c)&=2b^6+c^6-3b^4c^2-10b^3c(b-c)^2\\&=(b^2-c^2)^2(2b^2+c^2)-10b^3c(b-c)^2=(b-c)^2g(b,c),\end{aligned}$$

where

$$g(b,c)=2b^4-6b^3c+3b^2c^2+2bc^3+c^4.$$

Since

$$g(b,c)=2b(b-c)(b-2c)^2+c\cdot h(b,c),\quad h(b,c)=4b^3-13b^2c+10bc^2+c^3,$$

it suffices to show that $h(b,c)\geq 0$. For $b\geq 2c$, we have

$$h(b,c)=b(b-2c)(4b-5c)+c^3>0.$$

Also, for $c\leq b\leq 2c$, we have

$$2h(b,c)=(2c-b)(b-c)^2+b(3b-5c)^2\geq 0.$$

Thus, the proof is completed. The equality holds for $a=b=c$.

□

P 2.82. *If $a\geq b\geq c>0$, then*

$$\frac{ab+bc}{a^2+b^2+c^2}\leq\frac{1+\sqrt{3}}{4}.$$

Solution. Denote
$$k = \frac{1+\sqrt{3}}{4} \approx 0.683,$$
and write the inequality as $E(a,b,c) \ge 0$, where
$$E(a,b,c) = k(a^2+b^2+c^2) - ab - bc.$$
We show that
$$E(a,b,c) \ge E(b,b,c) \ge 0.$$
We have
$$E(a,b,c) - E(b,b,c) = (a-b)[ka-(1-k)b] \ge (2k-1)(a-b)b \ge 0$$
and
$$E(b,b,c) = (2k-1)b^2 + kc^2 - bc \ge 2\sqrt{k(2k-1)}bc - bc = 0.$$
The equality holds for $a = b = \dfrac{1+\sqrt{3}}{2}c$.

□

P 2.83. *If $a \ge b \ge c \ge d > 0$, then*
$$\frac{ab+bc+cd}{a^2+b^2+c^2+d^2} \le \frac{2+\sqrt{7}}{6}.$$
Solution. Write the inequality as $E(a,b,c,d) \ge 0$, where
$$E(a,b,c,d) = k(a^2+b^2+c^2+d^2) - ab - bc - cd, \qquad k = \frac{2+\sqrt{7}}{6} \approx 0.774.$$
We show that
$$E(a,b,c,d) \ge E(b,b,c,d) \ge E(c,c,c,d) \ge 0.$$
We have
$$E(a,b,c,d) - E(b,b,c,d) = (a-b)[ka-(1-k)b] \ge (2k-1)(a-b)b \ge 0,$$
$$E(b,b,c,d) - E(c,c,c,d) = (b-c)[(2k-1)b-(2-2k)c] \ge (4k-3)(b-c)c \ge 0$$
and
$$E(c,c,c,d) = (3k-2)c^2 + kd^2 - cd \ge 2\sqrt{k(3k-2)}cd - cd = 0.$$
The equality holds for $a = b = c = \dfrac{2+\sqrt{7}}{3}d$.

□

P 2.84. *If*

$$a \ge 1 \ge b \ge c \ge d \ge 0, \qquad a+b+c+d=4,$$

then

$$ab+bc+cd \le 3.$$

Solution. Write the inequality in the homogeneous form $E(a,b,c,d) \ge 0$, where

$$E(a,b,c,d) = 3(a+b+c+d)^2 - 16(ab+bc+cd).$$

From

$$a+b+c+d = 4 \ge 4b,$$

we get

$$a \ge 3b-c-d.$$

We will show that

$$E(a,b,c,d) \ge E(3b-c-d,b,c,d) \ge 0.$$

We have

$$\begin{aligned} E(a,b,c,d) - E(3b-c-d,b,c,d) &= 3[(a+b+c+d)^2 - (4b)^2] - 16b(a-3b+c+d) \\ &= (a-3b+c+d)(3a-b+3c+3d) \ge 0. \end{aligned}$$

Also,

$$E(3b-c-d,b,c,d) = 48b^2 - 16(3b^2 - bd + cd) = 16d(b-c) \ge 0.$$

The equality holds for

$$a \in [2,3], \quad b=1, \quad c=3-a, \quad d=0.$$

□

P 2.85. *Let k and a,b,c be positive real numbers, and let*

$$E = (ka+b+c)\left(\frac{k}{a}+\frac{1}{b}+\frac{1}{c}\right), \qquad F = (ka^2+b^2+c^2)\left(\frac{k}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

(a) If $k \ge 1$, then

$$\sqrt{\frac{F-(k-2)^2}{2k}} + 2 \ge \frac{E-(k-2)^2}{2k};$$

(b) If $0 < k \le 1$, then

$$\sqrt{\frac{F-k^2}{k+1}} + 2 \ge \frac{E-k^2}{k+1}.$$

(Vasile C., 2007)

Solution. Due to homogeneity, we may assume that $bc = 1$. Under this assumption, if we denote

$$x = a + \frac{1}{a}, \quad y = b + \frac{1}{b} = c + \frac{1}{c}$$

$(x \geq 2,\ y \geq 2)$, then

$$\begin{aligned} E &= \left(ka + b + \frac{1}{b}\right)\left(\frac{k}{a} + b + \frac{1}{b}\right) \\ &= (ka + y)\left(\frac{k}{a} + y\right) \\ &= k^2 + kxy + y^2 \end{aligned}$$

and

$$\begin{aligned} F &= \left(ka^2 + b^2 + \frac{1}{b^2}\right)\left(\frac{k}{a^2} + b^2 + \frac{1}{b^2}\right) \\ &= (ka^2 + y^2 - 2)\left(\frac{k}{a^2} + y^2 - 2\right) \\ &= k^2 + k(x^2 - 2)(y^2 - 2) + (y^2 - 2)^2. \end{aligned}$$

(a) Write the inequality as

$$2kF - 2k(k-2)^2 \geq (E - k^2 - 4)^2.$$

We have

$$E - k^2 - 4 = kxy + y^2 - 4 > 0,$$
$$(E - k^2 - 4)^2 = k^2x^2y^2 + 2kxy(y^2 - 4) + (y^2 - 4)^2,$$

and

$$F - (k-2)^2 = 4k + k(x^2 - 2)(y^2 - 2) + y^2(y^2 - 4),$$
$$2kF - 2k(k-2)^2 = 8k^2 + 2k^2(x^2 - 2)(y^2 - 2) + 2ky^2(y^2 - 4).$$

Therefore,

$$2kF - 2k(k-2)^2 - (E - k^2 - 4)^2 = (y^2 - 4)[k^2(x^2 - 4) - 2ky(x - y) - (y^2 - 4)].$$

Since $y^2 - 4 \geq 0$, we still need to show that

$$k^2(x^2 - 4) - 2ky(x - y) \geq y^2 - 4.$$

We will show that

$$k^2(x^2 - 4) - 2ky(x - y) \geq (x^2 - 4) - 2y(x - y) \geq y^2 - 4.$$

The right inequality reduces to $(x - y)^2 \geq 0$, and the left inequality is equivalent to

$$(k-1)[(k+1)(x^2 - 4) - 2y(x - y)] \geq 0.$$

This is true because

$$(k+1)(x^2-4)-2y(x-y)\ge 2(x^2-4)-2y(x-y)=2(x-y)^2+2(xy-4)\ge 0.$$

The equality holds for $b=c$. If $k=1$, then the equality holds for $a=b$ or $b=c$ or $c=a$.

(b) Write the inequality as

$$(k+1)(F-k^2)\ge (E-k^2-2k-2)^2.$$

We have

$$E-k^2-2k-2=k(xy-2)+y^2-2>0,$$

$$(E-k^2-2k-2)^2=k^2(xy-2)^2+2k(xy-2)(y^2-2)+(y^2-2)^2,$$

and

$$(k+1)(F-k^2)=k^2(x^2-2)(y^2-2)+k(y^2-2)(x^2+y^2-4)+(y^2-2)^2.$$

Thus,

$$(k+1)(F-k^2)-(E-k^2-2k-2)^2=k(x-y)^2(y^2-2k-2)$$

$$\ge k(x-y)^2(y^2-4)\ge 0.$$

If $0<k<1$, then the equality holds for $a=b$ or $a=c$.

□

P 2.86. *If a,b,c are positive real numbers, then*

$$\frac{a}{2b+6c}+\frac{b}{7c+a}+\frac{25c}{9a+8b}>1.$$

Solution. By the Cauchy-Schwarz inequality, we have

$$\frac{a}{2b+6c}+\frac{b}{7c+a}+\frac{25c}{9a+8b}\ge\frac{(a+b+5c)^2}{a(2b+6c)+b(7c+a)+c(9a+8b)}.$$

Therefore, it suffices to show that

$$(a+b+5c)^2\ge 3ab+15bc+15ca,$$

which is equivalent to

$$a^2+b^2+25c^2-ab-5bc-5ca\ge 0.$$

Indeed, we have

$$2(a^2+b^2+25c^2-ab-5bc-5ca)=(a-b)^2+a^2+b^2+50c^2-10bc-10ca$$

$$=(a-b)^2+(a-5c)^2+(b-5c)^2\ge 0.$$

□

P 2.87. *If a,b,c are positive real numbers such that*

$$\frac{1}{a} \ge \frac{1}{b} + \frac{1}{c},$$

then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \ge \frac{55}{12(a+b+c)}.$$

(Vasile C., 2014)

Solution. Denote

$$x = \frac{bc}{b+c}, \quad a \le x,$$

and write the desired inequality as

$$\sum \frac{a+b+c}{b+c} \ge \frac{55}{12},$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \ge \frac{19}{12}.$$

Using the Cauchy-Schwarz inequality

$$\frac{b}{c+a} + \frac{c}{a+b} \ge \frac{(b+c)^2}{b(c+a)+c(a+b)},$$

it suffices to show that

$$F(a,b,c) \ge \frac{19}{12},$$

where

$$F(a,b,c) = \frac{a}{b+c} + \frac{(b+c)^2}{a(b+c)+2bc}.$$

We will show that

$$F(a,b,c) \ge F(x,b,c) \ge \frac{19}{12}.$$

Since

$$F(a,b,c) - F(x,b,c) = (x-a)\left[-\frac{1}{b+c} + \frac{(b+c)^3}{(a(b+c)+2bc)(x(b+c)+2bc)}\right],$$

we need to prove that

$$(b+c)^4 \ge [a(b+c)+2bc][(x(b+c)+2bc].$$

Since

$$a(b+c)+2bc \le x(b+c)+2bc,$$

it is enough to show that

$$(b+c)^2 \ge x(b+c)+2bc,$$

which is equivalent to the obvious inequality

$$(b+c)^2 \geq 3bc.$$

Also, we have

$$F(x,b,c) - \frac{19}{12} = \frac{bc}{(b+c)^2} + \frac{(b+c)^2}{3bc} - \frac{19}{12} = \frac{(b-c)^2(4b^2+5bc+4c^2)}{12bc(b+c)^2} \geq 0.$$

The equality occurs for $2a = b = c$.

□

P 2.88. *If a,b,c are positive real numbers such that*

$$\frac{1}{a} \geq \frac{1}{b} + \frac{1}{c},$$

then

$$\frac{1}{a^2+b^2} + \frac{1}{b^2+c^2} + \frac{1}{c^2+a^2} \geq \frac{189}{40(a^2+b^2+c^2)}.$$

(Vasile C., 2014)

Solution. Denote

$$x = \frac{bc}{b+c}, \quad a \leq x,$$

and write the desired inequality as

$$\sum \frac{a^2+b^2+c^2}{b^2+c^2} \geq \frac{189}{40},$$

$$\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} \geq \frac{69}{40}.$$

Using the Cauchy-Schwarz inequality

$$\frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} \geq \frac{(b^2+c^2)^2}{b^2(c^2+a^2)+c^2(a^2+b^2)},$$

it suffices to show that

$$F(a,b,c) \geq \frac{69}{40},$$

where

$$F(a,b,c) = \frac{a^2}{b^2+c^2} + \frac{(b^2+c^2)^2}{a^2(b^2+c^2)+2b^2c^2}.$$

We will show that

$$F(a,b,c) \geq F(x,b,c) \geq \frac{69}{40}.$$

Since

$$F(a,b,c)-F(x,b,c) = (x^2-a^2)\left[-\frac{1}{b^2+c^2}+\frac{(b^2+c^2)^3}{(a^2(b^2+c^2)+2b^2c^2)(x^2(b^2+c^2)+2b^2c^2)}\right],$$

we need to prove that

$$(b^2+c^2)^4 \ge [a^2(b^2+c^2)+2b^2c^2][x^2(b^2+c^2)+2b^2c^2].$$

Since

$$a^2(b^2+c^2)+2b^2c^2 \le x^2(b^2+c^2)+2b^2c^2,$$

it is enough to show that

$$(b^2+c^2)^2 \ge x^2(b^2+c^2)+2b^2c^2,$$

which is equivalent to

$$(b^4+c^4)(b+c)^2 \ge b^2c^2(b^2+c^2).$$

This inequality follows from $b^4+c^4 > b^2c^2$ and $(b+c)^2 > b^2+c^2$. Also, we have

$$F(x,b,c) = \frac{x^2}{b^2+c^2}+\frac{(b^2+c^2)^2}{x^2(b^2+c^2)+2b^2c^2}.$$

Since

$$2b^2c^2 \le 4x^2(b^2+c^2),$$

we have

$$F(x,b,c) \ge \frac{x^2}{b^2+c^2}+\frac{(b^2+c^2)^2}{5x^2(b^2+c^2)} = \frac{1}{t}+\frac{t}{5},$$

where

$$t = \frac{b^2+c^2}{x^2} \ge 8.$$

Therefore,

$$F(x,b,c)-\frac{69}{40} \ge \frac{1}{t}+\frac{t}{5}-\frac{69}{40} = \frac{(t-8)(8t-5)}{40t} \ge 0.$$

The equality occurs for $2a=b=c$.

□

P 2.89. *Find the best real numbers k,m,n such that*

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})\sqrt{a+b+c} \ge ka+mb+nc$$

for all $a \ge b \ge c \ge 0$.

Solution. For $a = 1$ and $b = c = 0$, for $a = b = 1$ and $c = 0$, and for $a = b = c = 1$, we get respectively

$$k \le 1, \quad k + m \le 2\sqrt{2}, \quad k + m + n \le 3\sqrt{3},$$

which yield

$$\begin{aligned} ka + mb + nc &= k(a-b) + (k+m)(b-c) + (k+m+nz)c \\ &\le a - b + 2\sqrt{2}\,(b-c) + 3\sqrt{3}\,c \\ &= a + (2\sqrt{2}-1)b + (3\sqrt{3}-2\sqrt{2})c. \end{aligned}$$

Therefore, if the following inequality holds

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)\sqrt{a+b+c} \ge a + (2\sqrt{2}-1)b + (3\sqrt{3}-2\sqrt{2})c,$$

then

$$k = 1, \quad m = 2\sqrt{2}-1, \quad n = 3\sqrt{3}-2\sqrt{2}$$

are the best real k, m, n. Since

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2 = a + \left(2\sqrt{ab}+b\right) + \left(2\sqrt{ac}+2\sqrt{bc}+c\right) \ge a + 3b + 5c,$$

it suffices to show that

$$(a+3b+5c)(a+b+c) \ge [a + (2\sqrt{2}-1)b + (3\sqrt{3}-2\sqrt{2})c]^2,$$

which is equivalent to the obvious inequality

$$(3-2\sqrt{2})b(a-b) + (3+2\sqrt{2}-3\sqrt{3})c(a-b) + 3(5-2\sqrt{6})c(b-c) \ge 0.$$

If $k = 1,\ m = 2\sqrt{2}-1,\ n = 3\sqrt{3}-2\sqrt{2}$, then the equality holds for $a = b = c$, for $a = b$ and $c = 0$, and for $b = c = 0$.

□

P 2.90. *Let $a, b \in (0,1]$, $a \le b$.*

(a) If $a \le \dfrac{1}{e}$, then

$$2a^a \ge a^b + b^a;$$

(b) If $b \ge \dfrac{1}{e}$, then

$$2b^b \ge a^b + b^a.$$

(Vasile C., 2012)

Solution. (a) We need to show that $f(a) \geq f(b)$, where

$$f(x) = a^x + x^a, \quad x \in [a, b].$$

This is true if $f(x)$ is decreasing; that is, if $f'(x) \leq 0$ on $[a, b]$. Since the derivative

$$f'(x) = a(x^{a-1} + a^{x-1} \ln a) \leq a(x^{a-1} - a^{x-1}),$$

it suffices to show that

$$x^{a-1} \leq a^{x-1}$$

for $0 < a \leq x \leq 1$. Consider the non-trivial case $0 < a \leq x < 1$, and write the inequality as $g(x) \geq g(a)$, where

$$g(x) = \frac{\ln x}{1-x}.$$

It suffices to show that $g'(x) \geq 0$ for $0 < x < 1$. We have

$$g'(x) = \frac{h(x)}{(1-x)^2}, \quad h(x) = \frac{1}{x} - 1 + \ln x.$$

Since

$$h'(x) = \frac{x-1}{x^2} < 0,$$

$h(x)$ is strictly decreasing, $h(x) > h(1) = 0$, $g'(x) > 0$. This completes the proof. The equality holds for $a = b$.

(b) We need to show that $f(b) \geq f(a)$, where

$$f(x) = x^b + b^x, \quad x \in [a, b].$$

This is true if $f(x)$ is increasing; that is, if $f'(x) \geq 0$ on $[a, b]$. Since the derivative

$$f'(x) = b(x^{b-1} + b^{x-1} \ln b) \geq b(x^{b-1} - b^{x-1}),$$

it suffices to show that

$$x^{b-1} \geq b^{x-1}$$

for $0 < x \leq b \leq 1$. As we shown at (a), this inequality is true. The equality holds for $a = b$.

□

P 2.91. *If* $0 \leq a \leq b$ *and* $b \geq \frac{1}{2}$, *then*

$$2b^{2b} \geq a^{2b} + b^{2a}.$$

(Vasile C., 2012)

Solution. We need to show that $f(a) \le f(b)$, where

$$f(x) = x^{2b} + b^{2x}, \quad x \in [0, b].$$

From the derivative

$$f''(x) = 2b\left[2b^{2x-1}\ln^2 b + (2b-1)x^{2b-2}\right] > 0, \quad x \in (0, b],$$

it follows that $f(x)$ is convex on $[0, b]$. Therefore, we have

$$f(a) \le \max\{f(0), f(b)\}.$$

From this, it follows that $f(a) \le f(b)$ if $f(0) \le f(b)$. To prove that $f(0) \le f(b)$, we apply Bernoulli's inequality as follows:

$$\begin{aligned} f(b) - f(0) &= 2b^{2b} - 1 = 2[1 + (b-1)]^{2b} - 1 \\ &\ge 2[1 + 2b(b-1)] - 1 = (2b-1)^2 \ge 0. \end{aligned}$$

The equality holds for $a = b \ge \dfrac{1}{2}$, and also for $a = 0$ and $b = \dfrac{1}{2}$.

□

P 2.92. *If $a \ge b \ge 0$, then*

(a) $$a^{b-a} \le 1 + \frac{a-b}{\sqrt{a}};$$

(b) $$a^{a-b} \ge 1 - \frac{3(a-b)}{4\sqrt{a}}.$$

(Vasile C., 2010)

Solution. (a) Write the inequality as

$$(a-b)\ln a + \ln\left(1 + \frac{a-b}{\sqrt{a}}\right) \ge 0,$$

which follows by adding the inequalities

$$\ln\left(1 + \frac{a-b}{\sqrt{a}}\right) - \frac{a-b}{\sqrt{a}} + \frac{(a-b)^2}{2a} \ge 0,$$

$$(a-b)\ln a + \frac{a-b}{\sqrt{a}} - \frac{(a-b)^2}{2a} \ge 0.$$

Denoting

$$x = \frac{a-b}{\sqrt{a}},$$

we can write the first inequality as $f(x) \ge 0$ for $x \ge 0$, where

$$f(x) = \ln(1+x) - x + \frac{x^2}{2}.$$

From the derivative

$$f'(x) = \frac{x^2}{1+x} \ge 0,$$

it follows that f is increasing, hence $f(x) \ge f(0) = 0$.

The second inequality is true if

$$\ln a + \frac{1}{\sqrt{a}} - \frac{a-b}{2a} \ge 0.$$

It suffices to prove that $g(a) \ge 0$, where

$$g(a) = \ln a + \frac{1}{\sqrt{a}} - \frac{1}{2}.$$

From

$$g'(a) = \frac{2\sqrt{a}-1}{2a\sqrt{a}},$$

it follows that g is decreasing on $(0, 1/4]$ and increasing on $[1/4, \infty)$; therefore,

$$g(a) \ge g\left(\frac{1}{4}\right) = \frac{3}{2} - \ln 4 > 0.$$

The equality holds for $a = b$.

(b) Consider the non-trivial case $1 - \dfrac{3(a-b)}{4\sqrt{a}} > 0$, write the inequality as

$$(a-b)\ln a \ge \ln\left(1 - \frac{3a-3b}{4\sqrt{a}}\right),$$

and prove it by adding the inequalities

$$0 \ge \ln\left(1 - \frac{3a-3b}{4\sqrt{a}}\right) + \frac{3(a-b)}{4\sqrt{a}},$$

$$(a-b)\ln a + \frac{3(a-b)}{4\sqrt{a}} \ge 0.$$

Denoting

$$x = \frac{3(a-b)}{4\sqrt{a}}, \quad 0 \le x < 1,$$

we can write the first inequality as $f(x) \le 0$, where

$$f(x) = \ln(1-x) + x.$$

From the derivative
$$f'(x) = \frac{-x}{1-x} \le 0,$$
it follows that f is decreasing, hence $f(x) \le f(0) = 0$.

The second inequality is true if $g(a) \ge 0$, where
$$g(a) = \ln a + \frac{3}{4\sqrt{a}}.$$
From the derivative
$$g'(a) = \frac{8\sqrt{a}-3}{8a\sqrt{a}},$$
it follows that
$$g(a) \ge g\left(\frac{9}{64}\right) = 2\ln\frac{3e}{8} > 0.$$
The equality holds for $a = b$.

□

P 2.93. *If a, b, c are positive real numbers such that*
$$a \ge b \ge c, \qquad ab^2c^3 \ge 1,$$
then
$$a + 2b + 3c \ge \frac{1}{a} + \frac{2}{b} + \frac{3}{c}.$$
(Vasile C., 2018)

Solution. It suffices to prove the homogeneous inequality
$$a + 2b + 3c \ge \sqrt[3]{ab^2c^3}\left(\frac{1}{a} + \frac{2}{b} + \frac{3}{c}\right).$$
Replacing a, b, c with a^3, b^3, c^3, the inequality becomes as follows:
$$a^3 + 2b^3 + 3c^3 \ge \frac{b^2c^3}{a^2} + \frac{2ac^3}{b} + 3ab^2,$$
$$a^3 + 2b^3 - 3ab^2 \ge \frac{c^3}{a^2b}(2a^3 - 3a^2b + b^3),$$
$$(a-b)^2(a+2b) \ge \frac{c^3}{a^2b}(a-b)^2(2a+b).$$
Thus, we need to show that
$$a^2b(a+2b) \ge c^3(2a+b)$$
for $a \ge b \ge c$. Since $c^3 \le ab^2$, we have
$$a^2b(a+2b) - c^3(2a+b) \ge a^2b(a+2b) - ab^2(2a+b) = ab(a^2-b^2) \ge 0.$$
The equality occurs for $a = b = 1/c \ge 1$.

□

P 2.94. *If a,b,c are positive real numbers such that*

$$a+b+c=3, \qquad a \le b \le c,$$

then

$$\frac{1}{a}+\frac{2}{b} \ge a^2+b^2+c^2.$$

(Vasile C., 2020)

Solution. Let

$$f(a,b,c)=\frac{1}{a}+\frac{2}{b}-a^2-b^2-c^2.$$

We will show that

$$f(a,b,c) \ge f(a,x,x) \ge 0,$$

where

$$x=\frac{b+c}{2}=\frac{3-a}{2}.$$

Since

$$f(a,b,c)-f(a,x,x)=\frac{2}{b}-\frac{2}{x}-(b^2+c^2-2x^2)$$

$$=\frac{2(c-b)}{b(b+c)}-\frac{(c-b)^2}{2}=\frac{(c-b)(b^3-bc^2+4)}{2b(b+c)},$$

we need to show that

$$b^3-bc^2+4 \ge 0.$$

Since $b+c<3$, we have

$$b^3-bc^2+4 > b^3-b(3-b)^2+4=6b^2+4-9b \ge (4\sqrt{6}-9)b > 0.$$

Also, since $a \le 1$, we have

$$f(a,x,x)=\frac{1}{a}+\frac{2}{x}-a^2-2x^2=\frac{1}{a}+\frac{4}{3-a}-a^2-\frac{1}{2}(3-a)^2$$

$$=\frac{a^4-5a^3+9a^2-7a+2}{a(3-a)}=\frac{(1-a)^3(2-a)}{a(3-a)} \ge 0.$$

The equality occurs for $a=b=c=1$.

□

P 2.95. *If a,b,c are positive real numbers such that*

$$a+b+c=3, \qquad a \le b \le c,$$

then

$$\frac{2}{a}+\frac{3}{b}+\frac{1}{c} \ge 2(a^2+b^2+c^2).$$

(Vasile C., 2020)

Solution. From

$$a \le b = 3 - a - c,$$

we get

$$a \le \frac{3-c}{2}.$$

For fixed b, write the inequality as $f(a) \ge 0$, where

$$f(a) = \frac{2}{a} + \frac{3}{b} + \frac{1}{c} - 2(a^2 + b^2 + c^2), \quad c = 3 - a - b.$$

We have

$$f'(a) = -\frac{2}{a^2} + \frac{1}{c^2} - 4(a - c) = \frac{1}{c^2} + 4c - 2g(a), \quad g(a) = 2a + \frac{1}{a^2}.$$

Since

$$g'(a) = 2 - \frac{2}{a^3} \le 0,$$

$g(a)$ is decreasing, hence

$$g(a) \ge g\left(\frac{3-c}{2}\right)$$

and

$$f'(a) \le \frac{1}{c^2} + 4c - 2g\left(\frac{3-c}{2}\right) = 6(c-1) - \frac{7c^2 + 6c - 9}{c^2(3-c)^2}$$

$$\le 6(c-1) - \frac{16}{81}(7c^2 + 6c - 9) = \frac{-2}{81}(56c^2 + 171 - 195c)$$

$$\le \frac{-2}{27}(4\sqrt{266} - 65)c < 0.$$

Therefore, $f(a)$ is decreasing. On the other hand, from $a \le b$ and $b \le c = 3 - a - b$, we get

$$a \le b, \quad a \le 3 - 2b.$$

There are two cases to consider: $b \in (0, 1]$ and $b \in [1, 3/2)$.

Case 1: $b \in (0, 1]$. Since $a \le b$, we have

$$f(a) \ge f(b) = \frac{5}{b} + \frac{1}{c} - 2(2b^2 + c^2), \quad c = 3 - 2b,$$

hence

$$f(a) \ge \frac{5}{b} + \frac{1}{3-2b} - 4b^2 - 2(3-2b)^2$$

$$= \frac{3(5-3b)}{b(3-2b)} - 3(4b^2 - 8b + 6)$$

$$= \frac{3(8b^4 - 28b^3 + 36b^2 - 21b + 5)}{b(3-2b)}$$

$$\geq \frac{3(8b^4 - 27b^3 + 35b^2 - 21b + 5)}{b(3-2b)}$$

$$= \frac{3(b-1)^2(8b^2 - 11b + 5)}{b(3-2b)} \geq 0.$$

Case 2: $b \in [1, 3/2)$. Since $a \leq 3 - b$, we have

$$f(a) \geq f(3-b) = \frac{2}{3-2b} + \frac{3}{b} + \frac{1}{c} - 2(3-2b)^2 - 2(b^2 + c^2), \quad c = b,$$

hence

$$f(a) \geq f(3-b) = \frac{2}{3-2b} + \frac{4}{b} - 2(3-2b)^2 - 4b^2$$

$$= \frac{6(2-b)}{b(3-2b)} - 6(2b^2 - 4b + 3)$$

$$= \frac{12(2b^4 - 7b^3 + 9b^2 - 5b + 1)}{b(3-2b)}$$

$$= \frac{12(b-1)^3(2b-1)}{b(3-2b)} \geq 0.$$

The equality occurs for $a = b = c = 1$.

Remark. Since

$$\frac{2}{a} + \frac{3}{b} + \frac{1}{c} \leq 2\left(\frac{1}{a} + \frac{2}{b}\right),$$

the inequality is stronger than the one of P 2.94.

□

P 2.96. *If a, b, c are positive real numbers such that*

$$a + b + c = 3, \qquad a \leq b \leq c,$$

then

$$\frac{31}{a} + \frac{25}{b} + \frac{25}{c} \geq 27(a^2 + b^2 + c^2).$$

(Vasile C., 2020)

Solution. From

$$a \leq b = 3 - a - c,$$

we get

$$a \leq \frac{3-c}{2}.$$

For fixed $c \in [1,3)$,write the inequality as $f(a) \ge 0$, where $a \le \dfrac{3-c}{2}$ and

$$f(a) = \frac{31}{a} + \frac{25}{b} + \frac{25}{c} - 27(a^2+b^2+c^2), \quad b = 3-a-c.$$

We will show that

$$f(a) \ge f\left(\frac{3-c}{2}\right) \ge 0.$$

Since $a+b \le 2$, we have

$$\frac{a+b}{a^2b^2} \ge \frac{16}{(a+b)^3} \ge 2,$$

therefore

$$f'a) = -\frac{31}{a^2} + \frac{25}{b^2} - 27(2a-2b) < -\frac{27}{a^2} + \frac{27}{b^2} - 54(a-b)$$

$$= 27(a-b)\left(\frac{a+b}{a^2b^2} - 2\right) \le 0,$$

$f(a)$ is decreasing, hence $f(a)$ is minimal for $a = \dfrac{3-c}{2}$, when

$$b = 3-a-c = \frac{3-c}{2} = a.$$

So, we have

$$f\left(\frac{3-c}{2}\right) = \frac{56}{a} + \frac{25}{c} - 27(2a^2+c^2)$$

$$= \frac{112}{3-c} + \frac{25}{c} - \frac{27(3-c)^2}{2} - 27c^2$$

$$= \frac{3(27c^4 - 135c^3 + 243c^2 - 185c + 50)}{2c(3-c)}$$

$$= \frac{3(c-1)(3c-2)(3c-5)^2}{2c(3-c)} \ge 0.$$

The equality occurs for $a = b = c = 1$, and also for $a = b = \dfrac{2}{3}$ and $c = \dfrac{5}{3}$.

Remark. Actually, the following stronger inequalities are true:

$$\frac{29}{a} + \frac{27}{b} + \frac{25}{c} \ge 27(a^2+b^2+c^2),$$

$$\frac{28}{a} + \frac{28}{b} + \frac{25}{c} \ge 27(a^2+b^2+c^2). \tag{*}$$

For (*), we have

$$f(a) = \frac{28}{a} + \frac{28}{b} + \frac{25}{c} - 27(a^2+b^2+c^2), \quad b = 3-a-c,$$

$$f'a) = -\frac{28}{a^2} + \frac{28}{b^2} - 27(2a-2b) \le -\frac{27}{a^2} + \frac{27}{b^2} - 54(a-b)$$
$$= 27(a-b)\left(\frac{a+b}{a^2b^2} - 2\right) \le 0$$

and

$$f\left(\frac{3-c}{2}\right) = \frac{56}{a} + \frac{25}{c} - 27(2a^2+c^2)$$
$$= \frac{3(c-1)(3c-2)(3c-5)^2}{2c(3-c)} \ge 0.$$

On the other hand, we can prove the inequality (*) by showing that

$$f(a,b,c) \ge f(x,x,c) \ge 0,$$

where

$$f(a,b,c) = \frac{28}{a} + \frac{28}{b} + \frac{25}{c} - 27(a^2+b^2+c^2), \qquad x = \frac{a+b}{2} = \frac{3-c}{2}.$$

We have

$$f(a,b,c) - f(x,x,c) = 28\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{x}\right) - 27(a^2+b^2-2x^2)$$

$$= \frac{1}{2}(a-b)^2\left[\frac{56}{ab(a+b)} - 27\right] \ge \frac{27}{2}(a-b)^2\left[\frac{2}{ab(a+b)} - 1\right] \ge 0$$

and

$$f(x,x,c) = \frac{56}{x} + \frac{25}{c} - 27(2x^2+c^2) = \frac{3(c-1)(3c-2)(3c-5)^2}{2c(3-c)} \ge 0.$$

□

P 2.97. *If a,b,c are the lengths of the sides of a triangle, then*

$$a^3(b+c) + bc(b^2+c^2) \ge a(b^3+c^3).$$

(Vasile C., 2010)

First Solution. Because the inequality is symmetric in b and c, we may assume that $b \ge c$. Consider the following two cases.

Case 1: $a \ge b$. It suffices to show that

$$a^3(b+c) \ge a(b^3+c^3).$$

We have

$$a^3(b+c) - a(b^3+c^3) \ge ab^2(b+c) - a(b^3+c^3) = ac(b^2-c^2) \ge 0.$$

Case 2: $a \le b$. Write the inequality as

$$c(a^3+b^3)-c^3(a-b)+ab(a^2-b^2)\ge 0.$$

It suffices to show that

$$c(a^3+b^3)+ab(a^2-b^2)\ge 0.$$

We have

$$c(a^3+b^3)+ab(a^2-b^2)\ge c(a^3+b^3)-abc(a+b)=c(a+b)(a-b)^2\ge 0.$$

The equality holds for a degenerate triangle with $a=b$ and $c=0$, or $a=c$ and $b=0$.

Second Solution. Consider two cases.

Case 1: $b^2+c^2\ge a(b+c)$. Write the inequality as

$$bc(b^2+c^2)\ge a(b+c)(b^2+c^2-bc-a^2).$$

It suffices to show that

$$bc\ge b^2+c^2-bc-a^2,$$

which is equivalent to the obvious inequality

$$a^2\ge (b-c)^2.$$

Case 2: $a(b+c)\ge b^2+c^2$. Write the inequality as

$$a(b+c)(a^2+bc)\ge (b^2+c^2)(ab+ac-bc).$$

It suffices to show that

$$a^2+bc\ge ab+ac-bc,$$

which is equivalent to the obvious inequality

$$bc\ge (a-c)(b-a).$$

□

P 2.98. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{(a+b)^2}{2ab+c^2}+\frac{(a+c)^2}{2ac+b^2}\ge \frac{(b+c)^2}{2bc+a^2}.$$

(Vasile C., 2010)

Solution. By the Cauchy-Schwarz inequality, we have

$$\frac{(a+b)^2}{2ab+c^2}+\frac{(a+c)^2}{2ac+b^2}\ge\frac{(2a+b+c)^2}{2a(b+c)+b^2+c^2}.$$

Therefore, it suffices to show that

$$\frac{(2a+b+c)^2}{2a(b+c)+b^2+c^2}\ge\frac{(b+c)^2}{2bc+a^2}.$$

We will show that

$$\frac{(2a+b+c)^2}{2a(b+c)+b^2+c^2}\ge 2\ge\frac{(b+c)^2}{2bc+a^2}.$$

The left inequality reduces to $4a^2\ge(b-c)^2$, and the right inequality reduces to $2a^2\ge(b-c)^2$. These are true because $a^2\ge(b-c)^2$. The equality holds for a degenerate triangle with $a=0$ and $b=c$.

□

P 2.99. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{a+b}{ab+c^2}+\frac{a+c}{ac+b^2}\ge\frac{b+c}{bc+a^2}.$$

(Vasile C., 2010)

Solution. Without loss of generality, assume that $b\ge c$. Since $a+b\ge a+c$ and

$$ab+c^2-(ac+b^2)=(b-c)(a-b-c)\le 0,$$

by Chebyshev's inequality, we have

$$\frac{a+b}{ab+c^2}+\frac{a+c}{ac+b^2}\ge\frac{1}{2}[(a+b)+(a+c)]\left(\frac{1}{ab+c^2}+\frac{1}{ac+b^2}\right)$$
$$\ge\frac{2(2a+b+c)^2}{a(b+c)+b^2+c^2}.$$

On the other hand,

$$\frac{b+c}{bc+a^2}\le\frac{b+c}{\frac{1}{2}(b-c)^2+bc}=\frac{2(b+c)}{b^2+c^2}.$$

Therefore, it suffices to show that

$$\frac{2(2a+b+c)}{a(b+c)+b^2+c^2}\ge\frac{2(b+c)}{b^2+c^2},$$

which is equivalent to $a(b-c)^2\ge 0$. The equality holds for a degenerate triangle with $a=0$ and $b=c$.

□

P 2.100. *If a, b, c are the lengths of the sides of a triangle, then*

$$\frac{b(a+c)}{ac+b^2} + \frac{c(a+b)}{ab+c^2} \ge \frac{a(b+c)}{bc+a^2}.$$

(Vo Quoc Ba Can, 2010)

Solution. Without loss of generality, assume that $b \ge c$. Since

$$ab + c^2 - (ac + b^2) = (b-c)(a-b-c) \le 0,$$

it suffices to prove that

$$\frac{b(a+c)}{ac+b^2} + \frac{c(a+b)}{ac+b^2} \ge \frac{a(b+c)}{bc+a^2},$$

which is equivalent to

$$\frac{2bc + a(b+c)}{ac+b^2} \ge \frac{a(b+c)}{bc+a^2},$$

$$\frac{2bc}{ac+b^2} \ge a(b+c)\left(\frac{1}{bc+a^2} - \frac{1}{ac+b^2}\right),$$

$$2bc(bc+a^2) \ge a(b+c)(b-a)(a+b-c).$$

Consider the nontrivial case $b \ge a$. Since $c \ge b-a$, it suffices to show that

$$2b(bc+a^2) \ge a(b+c)(a+b-c).$$

We have

$$2b(bc+a^2) - a(b+c)(a+b-c) = ab(a-b) + c(2b^2 - a^2 + ac)$$

$$\ge -abc + c(2b^2 - a^2 + ac) = ac(b+c-a) + 2bc(b-a) \ge 0.$$

The equality holds for a degenerate triangle with $a = b$ and $c = 0$, or $a = c$ and $b = 0$.

□

P 2.101. *If a, b, c, d are positive real numbers such that*

$$a \ge b \ge c \ge d, \qquad ab^2c^3d^6 \ge 1,$$

then

$$a + 2b + 3c + 6d \ge \frac{1}{a} + \frac{2}{b} + \frac{3}{c} + \frac{6}{d}.$$

(Vasile C., 2018)

Solution. It suffices to prove the homogeneous inequality

$$a+2b+3c+6d \ge \sqrt[6]{ab^2c^3d^6}\left(\frac{1}{a}+\frac{2}{b}+\frac{3}{c}+\frac{6}{d}\right).$$

Replacing a,b,c,d with a^6,b^6,c^6,d^6, we need to show that

$$a^6+2b^6+3c^6 \ge \left(\frac{b^2c^3}{a^5}+\frac{2ac^3}{b^4}+\frac{3ab^2}{c^3}-6\right)d^6+6ab^2c^3$$

for $a \ge b \ge c \ge d$. By the AM-GM inequality, we have

$$\frac{b^2c^3}{a^5}+\frac{2ac^3}{b^4}+\frac{3ab^2}{c^3}-6 \ge 6\sqrt[6]{\frac{b^2c^3}{a^5}\cdot\left(\frac{ac^3}{b^4}\cdot\right)^2\left(\frac{ab^2}{c^3}\right)^3}-6=0.$$

Since $d^6 \le ab^2c^3$, it suffices to show that

$$a^6+2b^6+3c^6 \ge \left(\frac{b^2c^3}{a^5}+\frac{2ac^3}{b^4}+\frac{3ab^2}{c^3}-6\right)ab^2c^3+6ab^2c^3,$$

which is equivalent to

$$a^6+2b^6+3c^6 \ge \frac{b^4c^6}{a^4}+\frac{2a^2c^6}{b^2}+3a^2b^4,$$

$$a^6+2b^6-3a^2b^4 \ge \left(\frac{b^4}{a^4}+\frac{2a^2}{b^2}-3\right)c^6,$$

$$(a^2-b^2)^2(a^2+2b^2) \ge \frac{(a^2-b^2)^2(2a^2+b^2)c^6}{a^4b^2}.$$

We need to show that

$$a^4b^2(a^2+2b^2) \ge (2a^2+b^2)c^6.$$

Since $c^6 \le a^2b^4$, we have

$$a^4b^2(a^2+2b^2)-(2a^2+b^2)c^6 \ge a^4b^2(a^2+2b^2)-(2a^2+b^2)a^2b^4 = a^2b^2(a^4-b^4) \ge 0.$$

The equality occurs for $a=b=c=d=1$.

Remark. By induction method, we can prove the following generalization.

- *If $a_1,a_2,\ldots,a_n$ $(n \ge 3)$ are positive real numbers such that*

$$a_1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1a_2^2a_3^3a_4^6\cdots a_n^{k_n} \ge 1, \qquad k_n = 3\cdot 2^{n-3},$$

then

$$a_1+2a_2+3a_3+6a_4+\cdots+k_na_n \ge \frac{1}{a_1}+\frac{2}{a_2}+\frac{3}{a_3}+\frac{6}{a_4}+\cdots+\frac{k_n}{a_n},$$

with equality for $a_1=a_2=\cdots=a_n$.

For $n=3$ and $n=4$, we get the inequalities in P 2.93 and P 2.101.

□

P 2.102. *If a,b,c,d are positive real numbers such that*

$$a \ge b \ge c \ge d, \qquad abc^2d^4 \ge 1,$$

then

$$a+b+2c+4d \ge \frac{1}{a}+\frac{1}{b}+\frac{2}{c}+\frac{4}{d}.$$

(Vasile C., 2018)

Solution. It suffices to prove the homogeneous inequality

$$a+b+2c+4d \ge \sqrt[4]{abc^2d^4}\left(\frac{1}{a}+\frac{1}{b}+\frac{2}{c}+\frac{4}{d}\right).$$

Replacing a,b,c,d with a^4,b^4,c^4,d^4, we need to show that

$$a^4+b^4+2c^4 \ge \left(\frac{bc^2}{a^3}+\frac{ac^2}{b^3}+\frac{2ab}{c^2}-4\right)d^4+4abc^2$$

for $a \ge b \ge c \ge d$. By the AM-GM inequality, we have

$$\frac{bc^2}{a^3}+\frac{ac^2}{b^3}+\frac{2ab}{c^2}-4 \ge 4\sqrt[4]{\frac{bc^2}{a^3}\cdot\frac{ac^2}{b^3}\cdot\left(\frac{ab}{c^2}\right)^2}-4=0.$$

Since $d^4 \le abc^2$, it suffices to show that

$$a^4+b^4+2c^4 \ge \left(\frac{bc^2}{a^3}+\frac{ac^2}{b^3}+\frac{2ab}{c^2}-4\right)abc^2+4abc^2$$

which is equivalent to

$$a^4+b^4+2c^4 \ge \frac{b^2c^4}{a^2}+\frac{a^2c^4}{b^2}+2a^2b^2,$$

$$(a^2-b^2)^2 \ge \frac{(a^2-b^2)^2c^4}{a^2b^2},$$

$$(a^2-b^2)^2\left(1-\frac{c^4}{a^2b^2}\right) \ge 0.$$

The equality occurs for $a=b=c=d=1$.

Remark. By induction method, we can prove the following generalization.

- *If $a_1,a_2,\ldots,a_n$ $(n \ge 3)$ are positive real numbers such that*

$$a_1 \ge a_2 \ge \cdots \ge a_n, \qquad a_1a_2a_3^2a_4^4\cdots a_n^{2^{n-2}} \ge 1,$$

then

$$a_1+a_2+2a_3+4a_4+\cdots+2^{n-2}a_n \ge \frac{1}{a_1}+\frac{1}{a_2}+\frac{2}{a_3}+\frac{4}{a_4}+\cdots+\frac{2^{n-2}}{a_n},$$

with equality for $a_1=a_2=\cdots=a_n$.

For $n=4$, we get the inequalities in P 2.102.

□

P 2.103. *If a,b,c,d are positive real numbers such that*

$$abcd \geq 1, \quad a \geq b \geq c \geq d, \quad ad \geq bc,$$

then

$$a+b+c+d \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

(Vasile C., 2018)

Solution. It suffices to prove the homogeneous inequality

$$a+b+c+d \geq \sqrt{abcd}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

Replacing a,b,c,d with a^2,b^2,c^2,d^2, we need to show that

$$a^2+b^2+c^2+d^2 \geq \frac{bc}{ad}(a^2+d^2)+\frac{ad}{bc}(b^2+c^2)$$

for $a \geq b \geq c \geq d$ and $ad \geq bc$. Write the inequality as follows:

$$(a^2+d^2)\left(1-\frac{bc}{ad}\right)+(b^2+c^2)\left(1-\frac{ad}{bc}\right) \geq 0,$$

$$(ad-bc)\left(\frac{a}{d}+\frac{d}{a}-\frac{b}{c}-\frac{c}{b}\right) \geq 0,$$

$$(ad-bc)\left(\frac{ac-bd}{cd}+\frac{bd-ac}{ab}\right) \geq 0.$$

$$\frac{(ad-bc)(ac-bd)(ab-cd)}{abcd} \geq 0.$$

Clearly, the last inequality is true. The equality occurs for $ad = bc = 1$.

Remark. The following extension is valid.

- *If a,b,c,d,e are positive real numbers such that*

$$abcde \geq 1, \quad a \geq b \geq c \geq d \geq e, \quad ae \geq bd \geq c^2,$$

then

$$a+b+c+d+e \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e},$$

with equality for $af = c^2 = cd = 1$

□

P 2.104. *If a,b,c,d,e,f are positive real numbers such that*

$$abcdef \ge 1, \quad a \ge b \ge c \ge d \ge e \ge f, \quad af \ge be \ge cd,$$

then

$$a+b+c+d+e+f \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{1}{f}.$$

(Vasile C., 2018)

Solution. Write the inequality as

$$(a+f)\left(1-\frac{1}{af}\right)+(b+e)\left(1-\frac{1}{be}\right)+(c+d)\left(1-\frac{1}{cd}\right) \ge 0.$$

For

$$af = k = constant,$$

we claim that the sum $a+f$ is minimum for $a=\frac{k}{e} \ge b$ and $f=e$. Indeed, we have

$$a+f-\frac{k}{e}-e = a+f-\frac{af}{e}-e = a-e-\left(\frac{a}{e}-1\right)f = \frac{(a-e)(e-f)}{e} \ge 0.$$

In addition, for

$$cd = k = constant,$$

we claim that the sum $c+d$ is maximum for $c=\frac{k}{e} \le b$ and $d=e$. Indeed, we have

$$c+d-\frac{k}{e}-e = c+d-\frac{cd}{e}-e = c-e-\left(\frac{c}{e}-1\right)d = \frac{-(c-e)(d-e)}{e} \le 0.$$

Thus, it suffices to prove the inequality for $f=e$ and $d=e$, that is for $d=e=f$. So, we need to show that

$$a+b+c+3d \ge \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{3}{d}$$

for

$$a \ge b \ge c \ge d, \quad abcd^3 \ge 1.$$

It suffices to prove the homogeneous inequality

$$a+b+c+3d \ge \sqrt[3]{abcd^3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{3}{d}\right).$$

Replacing a,b,c,d with a^3,b^3,c^3,d^3, we need to show that

$$a^3+b^3+c^3 \ge \left(\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}-3\right)d^3+3abc$$

for $a \ge b \ge c \ge d$. By the AM-GM inequality, we have

$$\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}-3 \ge 0.$$

Since $d^3 \le c^3$, it suffices to show that

$$a^3+b^3+c^3 \ge \left(\frac{bc}{a^2}+\frac{ca}{b^2}+\frac{ab}{c^2}-3\right)c^3+3abc,$$

which can be written as follows:

$$a^3+b^3+4c^3 \ge \frac{bc^4}{a^2}+\frac{ac^4}{c^2}+4abc,$$

$$(a^3+b^3)\left(1-\frac{c^4}{a^2b^2}\right)-4c(ab-c^2) \ge 0,$$

$$(ab-c^2)[(a^3+b^3)(ab+c^2)-4a^2b^2c] \ge 0.$$

It is true since

$$(a^3+b^3)(ab+c^2)-4a^2b^2c \ge 2ab\sqrt{ab}\left(ab+c^2\right)-4a^2b^2c = 2ab\sqrt{ab}\left(\sqrt{ab}-c\right)^2 \ge 0.$$

The equality occurs for $af = be = cd = 1$.

□

P 2.105. *Let a,b,c,d be nonnegative real numbers such that*

$$a^2-ab+b^2 = c^2-cd+d^2.$$

Prove that

$$(a+b)(c+d) \ge 2(ab+cd).$$

(Vasile C., 2000)

Solution. Let

$$x = a^2-ab+b^2 = c^2-cd+d^2.$$

Without loss of generality, assume that $ab \ge cd$. Then,

$$x \ge ab \ge cd, \quad (a+b)^2 = x+3ab, \quad (c+d)^2 = x+3cd.$$

By squaring, the desired inequality can be restated as

$$(x+3ab)(x+3cd) \ge 4(ab+cd)^2.$$

It is true since

$$\begin{aligned}(x+3ab)(x+3cd)-4(ab+cd)^2 &\ge (ab+3ab)(ab+3cd)-4(ab+cd)^2\\ &= 4cd(ab-cd) \ge 0.\end{aligned}$$

The equality occurs for $a=b=c=d$, and also for $a=b=c$ and $d=0$ (or any cyclic permutation).

□

P 2.106. *Let a,b,c,d be nonnegative real numbers such that*

$$a^2-ab+b^2=c^2-cd+d^2.$$

Prove that

$$\frac{1}{a^2+b^2}+\frac{1}{c^2+d^2}\le\frac{8}{(a+b)^2+(c+d)^2}.$$

(Vasile C. and Relic-93, 2021)

Solution. Let

$$x=a^2-ab+b^2=c^2-cd+d^2.$$

Without loss of generality, assume that $ab\ge cd$. Then, $x\ge ab\ge cd$ and

$$a^2+b^2=x+ab,\quad c^2+d^2=x+cd,\quad (a+b)^2=x+3ab,\quad (c+d)^2=x+3cd.$$

The required inequality can be rewritten as

$$\frac{1}{x+ab}+\frac{1}{x+cd}\le\frac{8}{2x+3(ab+cd)},$$

$$3(a^2b^2+c^2d^2)\le 4x^2+2abcd.$$

It is true if

$$3(a^2b^2+c^2d^2)\le 4a^2b^2+2abcd,$$

which is equivalent to

$$(ab-cd)(ab+3cd)\ge 0.$$

The equality occurs for $a=b=c=d$.

□

P 2.107. *Let a,b,c,d be nonnegative real numbers such that*

$$a^2-ab+b^2=c^2-cd+d^2.$$

Prove that

$$\frac{1}{a^2+ab+b^2}+\frac{1}{c^2+cd+d^2}\le\frac{8}{3(a+b)(c+d)}.$$

(Anhduy98, 2021)

Solution. Without loss of generality, assume that $ab\ge cd$. Let

$$x=a^2-ab+b^2=c^2-cd+d^2,\quad y=ab,\quad z=cd.$$

Then, $x\ge y\ge z$ and

$$a^2+ab+b^2=x+2y,\quad c^2+cd+d^2=x+2z,\quad (a+b)^2=x+3y,\quad (c+d)^2=x+3z.$$

The required inequality can be rewritten as $F(x,y,z)\le 0$, where

$$F(x,y,z)=\frac{1}{x+2y}+\frac{1}{x+2z}-\frac{8}{3\sqrt{(x+3y)(x+3z)}}.$$

We will show that

$$F(x,y,z)\le F(x,x,z)\le 0.$$

The left inequality is equivalent to

$$\frac{4}{\sqrt{x+3z}}\left(\frac{1}{\sqrt{x+3y}}-\frac{1}{2\sqrt{x}}\right)\ge\frac{x-y}{x(x+2y)},$$

$$\frac{6(x-y)}{\sqrt{x(x+3y)(x+3z)}\left(2\sqrt{x}+\sqrt{x+3z}\right)}\ge\frac{x-y}{x(x+2y)}.$$

It is true if

$$\frac{6}{\sqrt{(x+3y)(x+3z)}\left(2\sqrt{x}+\sqrt{x+3z}\right)}\ge\frac{1}{(x+2y)\sqrt{x}}.$$

Since $x\ge y\ge z$, we only need to show that

$$\frac{6}{(x+3y)\left(2\sqrt{x}+\sqrt{4x}\right)}\ge\frac{1}{(x+2y)\sqrt{x}},$$

which is clearly true.

The right inequality $F(x,x,z)\le 0$ is equivalent to

$$\frac{1}{3x}+\frac{1}{x+2z}\le\frac{4}{3\sqrt{x(x+3z)}},$$

$$(2x+z)^2(x+3z)\le 4x(x+2z)^2.$$

It is true because

$$4x(x+2z)^2-(2x+z)^2(x+3z)=3(x-z)z^2\ge 0.$$

The equality occurs for $a=b=c=d$, and also for $a=b=c$ and $d=0$ (or any cyclic permutation).

□

P 2.108. *Let a,b,c,d be nonnegative real numbers such that*

$$a^2-ab+b^2=c^2-cd+d^2.$$

Prove that

$$\frac{1}{(ac+bd)^4}+\frac{1}{(ad+bc)^4}\le\frac{2}{(ab+cd)^4}.$$

(Vasile C., 2021)

Solution. Due to homogeneity, we may set

$$a^2-ab+b^2=c^2-cd+d^2=1.$$

Let

$$x=ab, \quad y=cd, \quad s=x+y, \quad p=xy.$$

From $1=a^2-ab+b^2\ge ab$, we get $x\le 1$. Similarly, $y\le 1$, hence $p\le 1$. In addition, from

$$(1-x)1-y)\ge 0,$$

we get

$$s\le 1+p.$$

Since

$$(ac+bd)(ad+bc)=ab(c^2+d^2)+cd(a^2+b^2)=x(1+y)+y(1+x)=s+2p,$$

$$(ac+bd)^2+(ad+bc)^2=(a^2+b^2)(c^2+d^2)+4abcd=(1+x)(1+y)+4xy=1+s+5p,$$

$$(ac+bd)^4+(ad+bc)^4=\left[(ac+bd)^2+(ad+bc)^2\right]^2-2(ac+bd)^2(ad+bc)^2$$
$$=(1+s+5p)^2-2(s+2p)^2,$$

we need to show that

$$\frac{(1+s+5p)^2-2(s+2p)^2}{(s+2p)^4}\le\frac{2}{s^4},$$

that is equivalent to $f(s,p)\ge g(s,p)$, where

$$f(s,p)=2\left(1+\frac{2p}{s}\right)^4, \qquad g(s,p)=(1+s+5p)^2-2(s+2p)^2.$$

Since

$$f(s,p)\ge f(1+p,p)$$

and

$$g(s,p)-g(1+p,p)=(s-1-p)(3+s+11p)-2(s-1-p)(1+s+5p)=-(s-1-p)^2\le 0,$$

it is enough to show that

$$f(1+p,p)\ge g(1+p,p),$$

that is

$$\frac{2(1+3p)^4}{(1+p)^4}\ge 2(1+3p)^2,$$

$$p(1-p)(1+3p)^2(2+5p+p^2)\ge 0.$$

The equality occurs for $a=b=c=d$, and also for $a=b=c$ and $d=0$ (or any cyclic permutation).

□

P 2.109. *Let a, b, c, d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$a + b + c + d = 13, \qquad a^2 + b^2 + c^2 + d^2 = 43.$$

Prove that

$$ab \ge cd + 3.$$

(PMO, 2021)

Solution *(by Doxuantrong)*. From

$$43 - a^2 = b^2 + c^2 + d^2 \ge \frac{1}{3}(b + c + d)^2 = \frac{1}{3}(13 - a)^2,$$

we get

$$(a - 4)(2a - 5) \le 0,$$

hence $\frac{5}{2} \le a, b, c, d \le 4$. On the other hand, we write the required inequality as follows:

$$2ab \ge 2cd + 6,$$

$$(a + b)^2 - (a^2 + b^2) \ge (c + d)^2 - (c^2 + d^2) + 6,$$

$$(13 - c - d)^2 - (43 - c^2 - d^2) \ge (c + d)^2 - (c^2 + d^2) + 6,$$

$$c^2 + d^2 + 60 \ge 13(c + d),$$

$$(c - d)^2 + (c + d)^2 + 120 \ge 26(c + d),$$

$$(c - d)^2 \ge (c + d - 6)(20 - c - d).$$

Thus, it suffices to show that $c + d \le 6$, that is equivalent to $a + b \ge 7$. If $a = 4$, then

$$a + b \ge a + \frac{b + c + d}{3} = a + \frac{13 - a}{3} = 7.$$

Consider further that $a < 4$. From

$$(b - c)(b - d) \ge 0,$$

we get

$$b^2 - (c + d)b + cd \ge 0,$$

that is equivalent to

$$2b^2 - 2(c + d)b + (c + d)^2 - (c^2 + d^2) \ge 0,$$

$$b^2 + (b - c - d)^2 - (c^2 + d^2) \ge 0,$$

$$b^2 + (a + 2b - 13)^2 - (43 - a^2 - b^2) \ge 0,$$

$$3b^2 - 2(13 - a)b + a^2 - 13a + 63 \ge 0,$$

$$3b \ge 13 - a + \sqrt{(4 - a)(2a - 5)}.$$

Note that we cannot have $3b \le 13-a-\sqrt{(4-a)(2a-5)}$ because this involves a contradiction:

$$13-a=b+c+d \le 3b \le 13-a-\sqrt{(4-a)(2a-5)} < 13-a.$$

From

$$3a \ge 3b \ge 13-a+\sqrt{(4-a)(2a-5)},$$

we get

$$4a-13 \ge \sqrt{(4-a)(2a-5)},$$

$$(2a-7)(a-3) \ge 0,$$

hence $a \ge 7/2$. As a consequence, we have

$$3(a+b-7)=3(a-7)+3b \ge 3(a-7)+13-a+\sqrt{(4-a)(2a-5)}$$

$$=\sqrt{4-a}\left(\sqrt{2a-5}-2\sqrt{4-a}\right)=\frac{3\sqrt{4-a}\,(2a-7)}{\sqrt{2a-5}+2\sqrt{4-a}} \ge 0.$$

The equality occurs for $a=4$ and $b=c=d=3$.

Second solution (by *KaiRain*) To show that $a+b \ge 7$, the key is

$$a^2+b^2+c^2+d^2+6(ab+cd)=(a+b+c+d)^2+2(a-c)(b-d)+2(a-d)(b-c)$$

$$\ge (a+b+c+d)^2,$$

which gives

$$ab+cd \ge 21,$$

$$(a+b)^2+(c+d)^2 \ge a^2+b^2+c^2+d^2+42,$$

$$(a+b)^2+(13-a-b)^2 \ge 85,$$

$$(a+b-6)(a+b-7) \ge 0,$$

$$a+b \ge 7.$$

Hence,

$$ab-cd \ge ab-\frac{c^2+d^2}{2}=ab+\frac{a^2+b^2-43}{2}=\frac{(a+b)^2-43}{2} \ge 3.$$

□

P 2.110. *Let a, b, c, d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$a+b+c+d=13, \qquad a^2+b^2+c^2+d^2=43.$$

Prove that

$$ab-cd \ge \frac{7}{2}$$

(a) *for* $a \le \frac{39}{10}$;

(b) *for* $d \le \frac{31}{11}$.

(Vasile C., 2021)

Solution. (a) As shown at the preceding P 2.109, we have

$$\frac{7}{2} \le a \le 4$$

and

$$b \ge B, \qquad B = \frac{13-a+\sqrt{(4-a)(2a-5)}}{3} .$$

Write the required inequality as follows:

$$2ab \ge 2cd+7,$$

$$(a+b)^2-(a^2+b^2) \ge (c+d)^2-(c^2+d^2)+7,$$

$$(a+b)^2-(a^2+b^2) \ge (13-a-b)^2-(43-a^2-b^2)+7,$$

$$13a-a^2+13b-b^2 \ge \frac{133}{2} .$$

Since

$$13b-b^2-(13B-B^2)=(b-B)(13-b-B) \ge 0,$$

it suffices to show that

$$13a-a^2+13B-B^2 \ge \frac{133}{2},$$

which is equivalent to

$$2(2a+13)\sqrt{(4-a)(2a-5)} \ge 16a^2-182a+481. \tag{*}$$

Write this inequality in the form

$$2(2a+13)(4-a)(2a-5) \ge (16a^2-182a+481)\sqrt{(4-a)(2a-5)} .$$

Since

$$2\sqrt{(4-a)(2a-5)} = 2\sqrt{2(4-a)\cdot\frac{2a-5}{2}} \le 2(4-a)+\frac{2a-5}{2} = \frac{11-2a}{2} ,$$

it suffices to show that

$$4(2a+13)(4-a)(2a-5) \geq (16a^2-182a+481)(11-2a),$$

which is equivalent to the obvious inequality

$$(2a-7)(39-10a) \geq 0.$$

The equality occurs for $a=b=c=\frac{7}{2}$ and $d=\frac{5}{2}$.

Remark 1. Actually, the inequality is true for $a \leq k$, where

$$k \approx 3.980572$$

is a root of the equation

$$16k^3-256k^2+1742k-3887=0.$$

Indeed, by squaring, the equation (*) becomes

$$(2a-7)(16a^3-256a^2+1742a-3887) \leq 0.$$

It is easy to show that his inequality holds for

$$a \leq \frac{613}{154} \approx 3.980519.$$

Indeed, we have

$$16a^3-256a^2+1742a-3887 = 16(a-4)^3-64(a-4)^2+3(154a-613)$$

$$< 3(154a-613) < 0.$$

(b) As shown at the preceding P 2.109, we have

$$d \geq \frac{5}{2}.$$

Write the required inequality as follows:

$$2ab \geq 2cd+7,$$

$$(a+b)^2-(a^2+b^2) \geq (c+d)^2-(c^2+d^2)+7,$$

$$(13-c-d)^2-(43-c^2-d^2) \geq (c+d)^2-(c^2+d^2)+7,$$

$$2c^2-26c+2d^2-26d+119 \geq 0.$$

If $d=\frac{5}{2}$, then

$$c \leq \frac{a+b+c}{3} = \frac{13-d}{3} = \frac{7}{2},$$

hence

$$2c^2-26c+2d^2-26d+119=\frac{(7-2c)(19-2c)}{2}\geq 0.$$

Consider further that $d>\frac{5}{2}$. From

$$(c-a)(c-b)\geq 0,$$

we get

$$c^2-(a+b)c+ab\geq 0,$$

that is equivalent to

$$c^2+(a+b-c)^2-a^2-b^2\geq 0,$$

$$c^2+(13-2c-d)^2+c^2+d^2-43\geq 0,$$

$$3c^2-2(13-d)c+d^2-13d+63\geq 0,$$

$$c\leq C,\qquad C=\frac{13-d-\sqrt{(4-d)(2d-5)}}{3}\ .$$

Note that we cannot have $3c\geq 13-d+\sqrt{(4-d)(2d-5)}$ because this involves a contradiction:

$$13-d=a+b+c\geq 3c\geq 13-d+\sqrt{(4-d)(2d-5)}>13-d.$$

From $d\leq c\leq C$, we get

$$\sqrt{(4-d)(2d-5)}\leq 13-4d,$$

$$(7-2d)(d-3)\leq 0,$$

hence

$$d\leq 3.$$

Since

$$2c^2-26c-(2C^2-26C)=2(c-C)(c+C-26)\geq 0,$$

it suffices to show that

$$2C^2-26C+2d^2-26d+119\geq 0,$$

which is equivalent to

$$2(2d+13)\sqrt{(4-d)(2d-5)}\geq (2d-5)(71-8d)\ .$$

This is true if

$$2(2d+13)\geq (71-8d)\sqrt{\frac{2d-5}{4-d}}\ . \qquad (**)$$

Since

$$2\sqrt{\frac{2d-5}{4-d}} \le \frac{2d-5}{4-d} + 1 = \frac{d-1}{4-d},$$

it is enough to show that

$$4(2d+13)(4-d) \ge (71-8d)(d-1),$$

which is equivalent to the obvious inequality

$$31 - 11d \ge 0.$$

The equality occurs for $a = b = c = \frac{7}{2}$ and $d = \frac{5}{2}$.

Remark 2. The inequality is true for $d \le k$, where

$$k \approx 2.84647$$

is a root of the equation

$$16k^3 - 272k^2 + 1734k - 3101 = 0.$$

Indeed, by squaring, the equation (**) becomes

$$16d^3 - 272d^2 + 1734d - 3101 \le 0.$$

It is easy to show that his inequality holds for

$$d \le \frac{1517}{534} \approx 2.84082.$$

Indeed, we have

$$16d^3 - 272d^2 + 1734d - 3101 = 16(d-3)^3 - 128(d-3)^2 + 534d - 1517$$

$$< 534d - 1517 < 0.$$

□

P 2.111. *Let a, b, c, d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$a + b + c + d = 13, \qquad a^2 + b^2 + c^2 + d^2 = 43.$$

Prove that

$$\frac{83}{4} \le ac + bd \le \frac{169}{8}.$$

(Vasile C., 2021)

Solution. As shown at P 2.109, we have

$$\frac{5}{2} \le a, b, c, d \le 4.$$

Since

$$2(ac+bd) = (a+c)^2 + (b+d)^2 - (a^2+b^2+c^2+d^2) = (a+c)^2 + (13-a-c)^2 - 43$$
$$= 2(a+c)^2 - 26(a+c) + 126,$$

the left required inequality is equivalent to

$$\left(a+c-\frac{13}{2}\right)^2 \ge 0 ,$$

and the right required inequality is equivalent to

$$8(a+c)^2 - 104(a+c) + 335 \ge 0.$$

Since

$$a+c \ge \frac{a+b+c+d}{2} = \frac{13}{2} ,$$

we only need yo show that

$$a+c \le \frac{26+\sqrt{6}}{4} .$$

From

$$(c-b)(c-d) \le 0,$$

we get

$$c^2 - (b+d)b + bd \le 0,$$

that is equivalent to

$$c^2 + (b+d-c)^2 - b^2 - d^2 \le 0 ,$$
$$c^2 + (13-a-2c)^2 + a^2 + c^2 - 43 \le 0 ,$$
$$3c^2 - 2(13-a)c + a^2 - 13a + 63 \le 0 ,$$
$$c \le C, \qquad C = \frac{13-a+\sqrt{(4-a)(2a-5)}}{3} .$$

So, it suffices to show that

$$a+C \le \frac{26+\sqrt{6}}{4},$$

which is equivalent to

$$26 + 3\sqrt{6} - 8a \ge 4\sqrt{(4-a)(2a-5)},$$

$$(\sqrt{6}+2)(4-a) + \frac{\sqrt{6}-2}{2}(2a-5) \ge 4\sqrt{(4-a)(2a-5)} .$$

Clearly, the last inequality is true (by the AM-GM inequality).

The left inequality is an equality for $a+c=b+d=\dfrac{13}{2}$ and $ac+bc=\dfrac{83}{4}$, while the right inequality is an equality for $a=\dfrac{13+\sqrt{6}}{4}$, $b=c=\dfrac{13}{4}$ and $d=\dfrac{13-\sqrt{6}}{4}$.

□

P 2.112. *If a,b,c,d are positive real numbers such that*

$$a+b+c+d=4, \qquad a\le b\le 1\le c\le d,$$

then

$$9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\ge 4+8(a^2+b^2+c^2+d^2).$$

(Vasile C., 2021)

Solution. For fixed b and d, write the required inequality as $f(c)\ge 0$, where

$$f(c)=9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)-4-8(a^2+b^2+c^2+d^2), \qquad a=4-b-c-d.$$

We will show that

$$f(c)\ge f(1)\ge 0.$$

Since

$$a+c\le \frac{a+b}{2}+\frac{c+d}{2}=2,$$

$$\frac{a+c}{a^2c^2}\ge \frac{16}{(a+c)^3}\ge 2,$$

we have

$$f'c)=\frac{9}{a^2}-\frac{9}{c^2}+16(a-c)=9(c-a)\left(\frac{a+c}{a^2c^2}-\frac{16}{9}\right)$$

$$\ge 9(c-a)\left(\frac{a+c}{a^2c^2}-2\right)\ge 0,$$

$f(c)$ is increasing, hence $f(c)\ge f(1)$. The inequality $f(1)\ge 0$ has the form

$$9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{d}\right)-3-8(a^2+b^2+d^2)\ge 0,$$

where

$$a=3-b-d.$$

We may write this inequality as $g(a,b)\ge 0$, where

$$g(a,b)=9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{d}\right)-3-8(a^2+b^2+d^2), \qquad d=3-a-b.$$

We will show that

$$g(a,b) \ge g(x,x) \ge 0,$$

where

$$x = \frac{a+b}{2}, \qquad 0 < x \le 1.$$

We have

$$g(a,b) - g(x,x) = 9\left(\frac{1}{a} + \frac{1}{b} - \frac{2}{x}\right) - 8(a^2 + b^2 - 2x^2)$$

$$= \frac{9(a-b)^2}{2abx} - 4(a-b)^2 = \frac{(a-b)^2(9-8abx)}{2abx} \ge 0$$

and

$$g(x,x) = 9\left(\frac{2}{x} + \frac{1}{d}\right) - 3 - 8(2x^2 + d^2), \qquad d = 3 - 2x,$$

$$g(x,x) = 9\left(\frac{2}{x} + \frac{1}{3-2x}\right) - 3 - 16x^2 - 8(3-2x)^2$$

$$= \frac{6(16x^4 - 56x^3 + 73x^2 - 42x + 9)}{x(3-2x)},$$

$$= \frac{6(x-1)^2(4x-3)^2}{x(3-2x)} \ge 0.$$

The equality holds for $a = b = c = d = 1$, and also for $a = b = \frac{3}{4}$, $c = 1$, $d = \frac{3}{2}$.

□

P 2.113. *If a, b, c, d are positive real numbers such that*

$$a^2 + b^2 + c^2 + d^2 = 4, \qquad a \le b \le c \le d,$$

then

$$\frac{1}{a} + a + b + c + d \ge 5.$$

(Vasile C., 2021)

Solution. Write the inequality in the homogeneous form

$$\frac{a^2 + b^2 + c^2 + d^2}{4} + a(a+b+c+d) \ge 5a\sqrt{\frac{a^2 + b^2 + c^2 + d^2}{4}}.$$

For fixed a, b, d, we need to prove that $f(c) \ge 0$, where

$$f(c) = 5a^2 + b^2 + c^2 + d^2 + 4a(b+c+d) - 10a\sqrt{a^2 + b^2 + c^2 + d^2}, \qquad c \in [b,d].$$

From

$$f'(c) = 2c + 4a - \frac{10ac}{\sqrt{a^2+b^2+c^2+d^2}} \geq 4a + 2c - \frac{10ac}{\sqrt{2(a^2+c^2)}}$$

$$\geq 4\sqrt{2ac} - 5\sqrt{ac} = (4\sqrt{2}-5)\sqrt{ac} > 0,$$

it follows that $f(c)$ is increasing, hence $f(c) \geq f(b)$. The inequality $f(b) \geq 0$ is equivalent to

$$5a^2 + 2b^2 + d^2 + 4a(2b+d) - 10a\sqrt{a^2+2b^2+d^2} \geq 0.$$

For fixed a and d, we need to show that $g(b) \geq 0$, where

$$g(b) = 5a^2 + 2b^2 + d^2 + 4a(2b+d) - 10a\sqrt{a^2+2b^2+d^2}, \quad b \in [a,d].$$

From

$$g'(b) = 4b + 8a - \frac{20ab}{\sqrt{a^2+2b^2+d^2}} \geq 4b + 8a - \frac{20ab}{\sqrt{a^2+3b^2}}$$

$$\geq 8\sqrt{2ab} - \frac{20\sqrt{ab}}{\sqrt{2\sqrt{3}}} = 4\left(2\sqrt{2} - \frac{5}{\sqrt{2\sqrt{3}}}\right)\sqrt{ab} > 0,$$

it follows that $g(b)$ is increasing, hence $g(b) \geq g(a)$, that is

$$g(b) \geq 15a^2 + 4ad + d^2 - 10a\sqrt{3a^2+d^2}.$$

Thus, we only need to show that

$$15a^2 + 4ad + d^2 \geq 10a\sqrt{3a^2+d^2}.$$

Due to homogeneity, we may set $a = 1$, hence $d \geq 1$. We need to show that

$$(15 + 4d + d^2)^2 \geq 100(3 + d^2),$$

which is equivalent to

$$d^4 + 8d^3 - 54d^2 + 120d - 75 \geq 0,$$

$$(d-1)(d^3 + 9d^2 - 45d + 75) \geq 0.$$

This is true because

$$d^3 + 9d^2 - 45d + 75 > 9d^2 - 45d + 63 = 9(d^2 - 5d + 7) > 0.$$

The equality holds for $a = b = c = d = 1$.

Remark. Similarly, we can prove the following stronger inequality

$$\frac{3}{4a} + a + b + c + d \geq \frac{19}{4}.$$

□

P 2.114. *If a, b, c, d are real numbers, then*

$$6(a^2+b^2+c^2+d^2)+(a+b+c+d)^2 \ge 12(ab+bc+cd).$$

(Vasile C., 2005)

Solution. Let

$$E(a,b,c,d) = 6(a^2+b^2+c^2+d^2)+(a+b+c+d)^2-12(ab+bc+cd).$$

First Solution. We have

$$E(x+a,x+b,x+c,x+d) =$$

$$= 4x^2+4(2a-b-c+2d)x+7(a^2+b^2+c^2+d^2)+2(ac+ad+bd)-10(ab+bc+cd)$$

$$= (2x+2a-b-c+2d)^2+3(a^2+2b^2+2c^2+d^2-2ab+2ac-2ad-4bc+2bd-2cd)$$

$$= (2x+2a-b-c+2d)^2+3(b-c)^2+3(a-b+c-d)^2.$$

For $x=0$, we get

$$E(a,b,c,d) = (2a-b-c+2d)^2+3(b-c)^2+3(a-b+c-d)^2 \ge 0.$$

The equality holds for $2a=b=c=2d$.

Second Solution. Let

$$x = a-b, \qquad y = c-d.$$

We have

$$\begin{aligned}
E &= 6[(a-b)^2+(c-d)^2]+(a+b+c+d)^2-12bc\\
&= 6(x^2+y^2)+[x+y+2(b+c)]^2-12bc\\
&= 3(x-y)^2+3(x+y)^2+[x+y+2(b+c)]^2-12bc\\
&= 3(x-y)^2+4(x+y)^2+4(x+y)(b+c)+(b+c)^2+3(b-c)^2\\
&= 3(x-y)^2+(2x+2y+b+c)^2+3(b-c)^2 \ge 0.
\end{aligned}$$

□

P 2.115. *If a, b, c, d are positive real numbers, then*

$$\frac{1}{a^2+ab}+\frac{1}{b^2+bc}+\frac{1}{c^2+cd}+\frac{1}{d^2+da} \ge \frac{4}{ac+bd}.$$

Solution. Write the inequality as follows:

$$\sum\left(\frac{ac+bd}{a^2+ab}+1\right)\ge 8,$$

$$\sum\frac{a(c+a)+b(d+a)}{a(a+b)}\ge 8,$$

$$\sum\frac{c+a}{a+b}+\sum\frac{b(d+a)}{a(a+b)}\ge 8.$$

By the AM-GM inequality, we have

$$\sum\frac{b(d+a)}{a(a+b)}\ge 4\sqrt[4]{\prod\frac{b(d+a)}{a(a+b)}}=4.$$

Therefore, it suffices to prove the inequality

$$\sum\frac{c+a}{a+b}\ge 4,$$

which is equivalent to

$$(a+c)\left(\frac{1}{a+b}+\frac{1}{c+d}\right)+(b+d)\left(\frac{1}{b+c}+\frac{1}{d+a}\right)\ge 4.$$

This inequality follows immediately from

$$\frac{1}{a+b}+\frac{1}{c+d}\ge\frac{4}{(a+b)+(c+d)}$$

and

$$\frac{1}{b+c}+\frac{1}{d+a}\ge\frac{4}{(b+c)+(d+a)}.$$

The equality occurs for $a=b=c=d$.

□

P 2.116. *If a,b,c,d are positive real numbers, then*

$$\frac{1}{a(1+b)}+\frac{1}{b(1+a)}+\frac{1}{c(1+d)}+\frac{1}{d(1+c)}\ge\frac{16}{1+8\sqrt{abcd}}.$$

(Vasile C., 2007)

Solution. Let

$$x=\sqrt{ab},\quad y=\sqrt{cd}.$$

Write the inequality as

$$\frac{a+b+2ab}{ab(1+a)(1+b)}+\frac{c+d+2cd}{cd(1+c)(1+d)}\ge\frac{16}{1+8\sqrt{abcd}}.$$

We claim that

$$x \ge 1 \quad \Longrightarrow \quad \frac{a+b+2ab}{ab(1+a)(1+b)} \ge \frac{1}{ab},$$

and

$$x \le 1 \quad \Longrightarrow \quad \frac{a+b+2ab}{ab(1+a)(1+b)} \ge \frac{2}{\sqrt{ab}+ab}.$$

The first inequality is equivalent to $ab \ge 1$, while the second inequality is equivalent to

$$\left(1-\sqrt{ab}\right)\left(\sqrt{a}-\sqrt{b}\right)^2 \ge 0.$$

Similarly, we have

$$y \ge 1 \quad \Longrightarrow \quad \frac{c+d+2cd}{cd(1+d)(1+d)} \ge \frac{1}{cd}$$

and

$$y \le 1 \quad \Longrightarrow \quad \frac{c+d+2cd}{cd(1+d)(1+d)} \ge \frac{2}{\sqrt{cd}+cd}.$$

There are four cases to consider.

Case 1: $x \ge 1$, $y \ge 1$. It suffices to show that

$$\frac{1}{x^2}+\frac{1}{y^2} \ge \frac{16}{1+8xy}.$$

Indeed, we have

$$\frac{1}{x^2}+\frac{1}{y^2} \ge \frac{2}{xy} > \frac{16}{1+8xy}.$$

Case 2: $x \le 1$, $y \le 1$. It suffices to show that

$$\frac{2}{x+x^2}+\frac{2}{y+y^2} \ge \frac{16}{1+8xy}.$$

Putting $s = x+y$ and $p = \sqrt{xy}$, this inequality becomes

$$\frac{s^2+s-2p^2}{p^2(s+p^2+1)} \ge \frac{8}{1+8p^2},$$

$$(1+8p^2)s^2+s-24p^4-10p^2 \ge 0.$$

Since $s \ge 2p$, we get

$$\begin{aligned}(1+8p^2)s^2+s-24p^4-10p^2 &\ge 4(1+8p^2)p^2+2p-24p^4-10p^2\\ &= 2p(p+1)(2p-1)^2 \ge 0.\end{aligned}$$

Case 3: $x \ge 1$, $y \le 1$. It suffices to show that

$$\frac{1}{x^2}+\frac{2}{y+y^2} \ge \frac{16}{1+8xy}.$$

This inequality is equivalent in succession to

$$(1+8xy)(2x^2+y^2+y) \ge 16x^2y(1+y),$$

$$(1+8xy)(x-y)^2+8x^3y+8xy^2-16x^2y+2xy+x^2+y \ge 0,$$

$$(1+8xy)(x-y)^2+8xy(x-1)^2+8xy^2+x^2+y \ge 6xy.$$

The last inequality is true since the AM-GM inequality yields

$$8xy^2+x^2+y \ge 3\sqrt[3]{8xy^2\cdot x^2\cdot y} = 3\sqrt[3]{8x^3y^3} = 6xy.$$

Case 4: $x \le 1$, $y \ge 1$. It suffices to show that

$$\frac{2}{x+x^2}+\frac{1}{y^2} \ge \frac{16}{1+8xy},$$

which is equivalent to

$$(1+8xy)(x-y)^2+8xy(y-1)^2+8x^2y+y^2+x \ge 6xy.$$

As in the case 3, we have

$$8x^2y+y^2+x \ge 3\sqrt[3]{8x^2y\cdot y^2\cdot x} = 3\sqrt[3]{8x^3y^3} = 6xy.$$

The proof is completed. The equality holds for $a=b=c=d=\frac{1}{2}$.

□

P 2.117. *If a,b,c,d are positive real numbers such that $a \ge b \ge c \ge d$ and*

$$a+b+c+d=4,$$

then

$$ac+bd \le 2.$$

(Vasile C., 2019)

Solution. Write the inequality in the homogeneous form

$$(a+b+c+d)^2 \ge 8(ac+bd).$$

We have

$$(a+b+c+d)^2-8(ac+bd) = a^2+2(b+d-3c)a+(b+c+d)^2-8bd$$

$$=(a+b+d-3c)^2-(b+d-3c)^2+(b+d+c)^2-8bd$$

$$=(a+b+d-3c)^2+8(b-c)(c-d) \ge 0.$$

The equality holds for $b=c=1$ and $a+d=2$.

□

P 2.118. *If a,b,c,d are positive real numbers such that $a \ge b \ge c \ge d$ and*

$$a+b+c+d=4,$$

then

$$2\left(\frac{1}{b}+\frac{1}{d}\right) \ge a^2+b^2+c^2+d^2.$$

(Vasile C., 2019)

Solution. Write the inequality in the homogeneous form

$$(a+b+c+d)^3\left(\frac{1}{b}+\frac{1}{d}\right)-32(a^2+b^2+c^2+d^2) \ge 0.$$

For fixed b,c,d, the inequality becomes $f(a) \ge 0$, with

$$f'(a)=3(a+b+c+d)^2\left(\frac{1}{b}+\frac{1}{d}\right)-64a.$$

For $a+b+c+d=4$, when $a=4-b-c-d \le 4-b-2d$, we have

$$\frac{1}{16}f'(a) \ge 3\left(\frac{1}{b}+\frac{1}{d}\right)-4(4-b-2d)$$

$$=\left(\frac{3}{b}+4b\right)+\left(\frac{3}{d}+8d\right)-16 \ge 4(\sqrt{3}+\sqrt{(6)}-4)>0.$$

Therefore, $f(a)$ is increasing, hence $f(a) \ge f(b)$. Similarly, for fixed a,b,d, the inequality becomes $g(c) \ge 0$, with

$$g'(c)=3(a+b+c+d)^2\left(\frac{1}{b}+\frac{1}{d}\right)-64c \ge f'(a)>0.$$

Therefore, $g(c)$ is increasing, hence $g(c) \ge g(d)$. As a consequence, it suffices to prove the original inequality for $a=b$ and $c=d$. So, we only need to show that $b+d=2$ involves

$$\frac{1}{b}+\frac{1}{d} \ge b^2+d^2,$$

which is equivalent to

$$(bd-1)^2 \ge 0.$$

The equality holds for $a=b=c=d=1$.

□

P 2.119. *Let a,b,c,d be positive real numbers such that $a \ge b \ge c \ge d$ and*

$$ab+bc+cd+da=3.$$

Prove that

$$a^3bcd<4.$$

(Vasile C., 2012)

Solution. Write the desired inequality as

$$4(ab+bc+cd+da)^3 > 27a^3bcd,$$

$$4\left(b+d+\frac{bc+cd}{a}\right)^3 > 27bcd.$$

It suffices to show that

$$4(b+d)^3 \ge 27bcd.$$

Indeed, by the AM-GM inequality, we have

$$(b+d)^3 = \left(\frac{b}{2}+\frac{b}{2}+d\right)^3 \ge 27\left(\frac{b}{2}\right)^2 d \ge \frac{27bcd}{4}.$$

□

P 2.120. *Let a,b,c,d be positive real numbers such that $a \ge b \ge c \ge d$ and*

$$ab+bc+cd+da = 6.$$

Prove that

$$acd \le 2.$$

(Vasile C., 2012)

Solution. Write the desired inequality in the homogeneous form

$$(a+c)^3(b+d)^3 \ge 54a^2c^2d^2.$$

Since $b \ge c$, we only need to show that

$$(a+c)^3(c+d)^3 \ge 54a^2c^2d^2.$$

By the AM-GM inequality, we have

$$(a+c)^3 = \left(\frac{a}{2}+\frac{a}{2}+c\right)^3 \ge 27\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)c = \frac{27}{4}a^2c.$$

Thus, it suffices to show that

$$(c+d)^3 \ge 8cd^2.$$

Indeed,

$$(c+d)^3 - 8cd^2 = (c-d)(c^2+4cd-d^2) \ge 0.$$

The equality holds for $a=2$ and $b=c=d=1$.

□

P 2.121. *Let a, b, c, d be positive real numbers such that $a \ge b \ge c \ge d$ and*

$$ab + bc + cd + da = 9.$$

Prove that

$$abd \le 4.$$

(Vasile C., 2012)

Solution. Write the desired inequality in the homogeneous form

$$(a+c)^3(b+d)^3 \ge \frac{729}{16}a^2b^2d^2.$$

Since $c \ge d$, we only need to show that

$$(a+d)^3(b+d)^3 \ge \frac{729}{16}a^2b^2d^2.$$

By the AM-GM inequality, we have

$$(a+d)^3 = \left(\frac{a}{2} + \frac{a}{2} + d\right)^3 \ge 27\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)d = \frac{27}{4}a^2d$$

and, similarly,

$$(b+d)^3 \ge \frac{27}{4}b^2d$$

Multiplying these inequalities, the desired inequality holds. The equality occurs for $a = b = 2$ and $c = d = 1$.

□

P 2.122. *Let a, b, c, d be positive real numbers such that $a \ge b \ge c \ge d$ and*

$$a^2 + b^2 + c^2 + d^2 = 10.$$

Prove that

$$2b + 4d \le 3c + 5.$$

(Vasile C., 2012)

Solution. Write the desired inequality in the homogeneous form

$$2b - 3c + 4d \le \sqrt{\frac{5}{2}(a^2 + b^2 + c^2 + d^2)}.$$

It is true if

$$5(a^2 + b^2 + c^2 + d^2) \ge 2(2b - 3c + 4d)^2.$$

Since $a \ge b$, it remains to show that

$$5(2b^2+c^2+d^2) \ge 2(2b-3c+4d)^2,$$

which is equivalent to

$$2b^2+24bc+48cd \ge 13c^2+27d^2+32bd.$$

Since $d^2 \le cd$, it suffices to prove that

$$2b^2+24bc+48cd \ge 13c^2+27cd+32bd,$$

which is equivalent to

$$2b^2+24bc \ge 13c^2+(32b-21c)d.$$

Since $32b-21c>0$ and $c \ge d$, it is enough to show that

$$2b^2+24bc \ge 13c^2+(32b-21c)c.$$

This reduces to the obvious inequality

$$2(b-2c)^2 \ge 0.$$

The equality holds for $a=b=2$ and $c=d=1$.

□

P 2.123. *Let a,b,c,d be positive real numbers such that $a \le b \le c \le d$ and*

$$abcd=1.$$

Prove that

$$4+\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a} \ge 2(a+b)(c+d).$$

Solution. Since

$$\frac{b}{c}+\frac{d}{a}-\frac{b}{a}-\frac{d}{c}=\frac{(d-b)(c-a)}{ca} \ge 0,$$

we only need to prove that

$$4+\frac{a}{b}+\frac{b}{a}+\frac{c}{d}+\frac{d}{c} \ge 2(a+b)(c+d),$$

which is equivalent to

$$\frac{(a+b)^2}{ab}+\frac{(c+d)^2}{cd} \ge 2(a+b)(c+d),$$

$$\left(\frac{a+b}{\sqrt{ab}}-\frac{c+d}{\sqrt{cd}}\right)^2 \ge 0.$$

The proof is completed. The equality holds for $a=b=c=d=1$.

□

P 2.124. *Let a, b, c, d be positive real numbers such that $a \ge b \ge c \ge d$ and*

$$3(a^2+b^2+c^2+d^2) = (a+b+c+d)^2.$$

Prove that

(a) $$\frac{a+d}{b+c} \le 2;$$

(b) $$\frac{a+c}{b+d} \le \frac{7+2\sqrt{6}}{5};$$

(c) $$\frac{a+c}{c+d} \le \frac{3+\sqrt{5}}{2}.$$

(Vasile C., 2010)

Solution. (a) Since

$$(a+d)(b+c) - 2(ad+bc) = (a-b)(c-d) + (a-c)(b-d) \ge 0,$$

we have

$$\begin{aligned} a^2+b^2+c^2+d^2 &= (a+d)^2 + (b+c)^2 - 2(ad+bc) \\ &\ge (a+d)^2 + (b+c)^2 - (a+d)(b+c), \end{aligned}$$

hence

$$\frac{1}{3}(a+b+c+d)^2 \ge (a+d)^2 + (b+c)^2 - (a+d)(b+c),$$

$$\left(\frac{a+d}{b+c} - 2\right)\left(\frac{a+d}{b+c} - \frac{1}{2}\right) \le 0,$$

from where the desired result follows. The equality holds for $a/3 = b = c = d$.

(b) From $(a-d)(b-c) \ge 0$ and the AM-GM inequality, we have

$$2(ac+bc) \le (a+d)(b+c) \le \frac{(a+b+c+d)^2}{4},$$

hence

$$\begin{aligned} a^2+b^2+c^2+d^2 &= (a+c)^2 + (b+d)^2 - 2(ac+bd) \\ &\ge (a+c)^2 + (b+d)^2 - \frac{(a+b+c+d)^2}{4}, \end{aligned}$$

$$\frac{1}{3}(a+b+c+d)^2 \ge (a+c)^2 + (b+d)^2 - \frac{(a+b+c+d)^2}{4},$$

$$\left(\frac{a+c}{b+d} - \frac{7+2\sqrt{6}}{2}\right)\left(\frac{a+c}{b+d} - \frac{7-2\sqrt{6}}{2}\right) \le 0,$$

from where the desired result follows. The equality holds for

$$(3-\sqrt{6})a = b = c = (3+\sqrt{6})d.$$

(c) Writing the hypothesis $3(a^2+b^2+c^2+d^2) = (a+b+c+d)^2$ as

$$b^2-(a+c+d)b+a^2+c^2+d^2-ac-cd-da = 0,$$

$$(2b-a-c-d)^2 = 3(2ac+2cd+2da-a^2-c^2-d^2),$$

it follows that

$$2ac+2cd+2da \ge a^2+c^2+d^2,$$

$$a^2-2(c+d)a+(c-d)^2 \le 0,$$

$$a \le c+d+2\sqrt{cd}.$$

Thus, it suffices to prove that

$$\frac{2c+d+2\sqrt{cd}}{c+d} \le \frac{3+\sqrt{5}}{2},$$

which is equivalent to

$$(\sqrt{5}-1)c+(\sqrt{5}+1)d \ge 4\sqrt{cd}.$$

This inequality follows immediately from the AM-GM inequality. The equality holds for

$$\frac{a}{3+\sqrt{5}} = \frac{b}{4} = \frac{c}{2} = \frac{d}{3-\sqrt{5}}.$$

□

P 2.125. *Let a,b,c,d be nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$2(a^2+b^2+c^2+d^2) = (a+b+c+d)^2.$$

Prove that

$$a \ge b+3c+(2\sqrt{3}-1)d.$$

(Vasile C., 2010)

First Solution. For $c = d = 0$, the desired inequality is an equality. Assume further that $c > 0$. From the hypothesis $2(a^2+b^2+c^2+d^2) = (a+b+c+d)^2$, we get

$$a = b+c+d \pm 2\sqrt{bc+cd+db}.$$

It is not possible to have

$$a = b+c+d-2\sqrt{bc+cd+db},$$

because this equality and $a \ge b$ involve

$$c + d \ge 2\sqrt{bc + cd + db},$$
$$(c - d)^2 \ge 4b(c + d),$$
$$(c - d)^2 \ge 4c(c + d),$$
$$d^2 \ge 3c(c + 2d),$$

which is not true. Thus, we have

$$a = b + c + d + 2\sqrt{bc + cd + db}.$$

Using this equality, we can rewrite the desired inequality as

$$b + c + d - 2\sqrt{bc + cd + db} \ge b + 3c + (2\sqrt{3} - 1)d,$$
$$\sqrt{b(c + d) + cd} \ge c + (\sqrt{3} - 1)d.$$

Since $b \ge c$, it suffices to show that

$$\sqrt{c(c + d) + cd} \ge c + (\sqrt{3} - 1)d.$$

By squaring, we get the obvious inequality $d(c - d) \ge 0$. The equality holds for $a = b$ and $c = d = 0$, for $\frac{a}{4} = b = c$ and $d = 0$, and for $\frac{a}{3 + 2\sqrt{3}} = b = c = d$.

Second Solution (by *Vo Quoc Ba Can*). Write the hypothesis $2(a^2 + b^2 + c^2 + d^2) = (a + b + c + d)^2$ as

$$(a - b)^2 + (c - d)^2 \ge 2(a + b)(c + d).$$

Since

$$a + b \ge (a - b) + 2c,$$

we get

$$(a - b)^2 + (c - d)^2 \ge 2[(a - b) + 2c](c + d),$$

which is equivalent to

$$(a - b)^2 - 2(c + d)(a - b) - 3c^2 - 6cd + d^2 \ge 0.$$

From this, we get

$$a - b \ge c + d + 2\sqrt{c^2 + 2cd}.$$

Thus, the desired inequality

$$a - b \ge 3c + (2\sqrt{3} - 1)d$$

is true if

$$c + d + 2\sqrt{c^2 + 2cd} \ge 3c + (2\sqrt{3} - 1)d,$$

that is,

$$\sqrt{c^2 + 2cd} \ge c + (\sqrt{3} - 1)d.$$

By squaring, we get the obvious inequality $d(c - d) \ge 0$.

□

P 2.126. *If $a \ge b \ge c \ge d \ge 0$, then*

(a) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{3}{2}\left(\sqrt{b}-2\sqrt{c}+\sqrt{d}\right)^2;$$

(b) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{2}{9}\left(3\sqrt{b}-2\sqrt{c}-\sqrt{d}\right)^2;$$

(c) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{4}{19}\left(3\sqrt{b}-\sqrt{c}-2\sqrt{d}\right)^2;$$

(d) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{3}{8}\left(\sqrt{b}-3\sqrt{c}+2\sqrt{d}\right)^2;$$

(e) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{1}{2}\left(2\sqrt{b}-3\sqrt{c}+\sqrt{d}\right)^2;$$

(f) $$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{1}{6}\left(2\sqrt{b}+\sqrt{c}-3\sqrt{d}\right)^2.$$

(Vasile C., 2010)

Solution. First, we show that

$$a-4\sqrt[4]{abcd} \ge b-4\sqrt[4]{b^2cd}.$$

Write this inequality as

$$a-b \ge 4\sqrt[4]{bcd}\left(\sqrt[4]{a}-\sqrt[4]{b}\right),$$

and prove then the following sharper inequality

$$a-b \ge 4\sqrt[4]{b^3}\left(\sqrt[4]{a}-\sqrt[4]{b}\right).$$

Indeed,

$$a-b-4\sqrt[4]{b^3}\left(\sqrt[4]{a}-\sqrt[4]{b}\right)=\left(\sqrt[4]{a}-\sqrt[4]{b}\right)\left(\sqrt[4]{a^3}+\sqrt[4]{a^2b}+\sqrt[4]{ab^2}-3\sqrt[4]{b^3}\right)\ge 0.$$

Thus, we have

$$a+b+c+d-4\sqrt[4]{abcd} \ge 2b+c+d-4\sqrt[4]{b^2cd},$$

which is equivalent to

$$a+b+c+d-4\sqrt[4]{abcd} \ge 2\left(\sqrt{b}-\sqrt[4]{cd}\right)^2+\left(\sqrt{c}-\sqrt{d}\right)^2.$$

Since

$$\sqrt{b}-\sqrt[4]{cd} \ge \sqrt{b}-\frac{\sqrt{c}+\sqrt{d}}{2} \ge 0,$$

we have

$$a+b+c+d-4\sqrt[4]{abcd} \ge \frac{1}{2}\left(2\sqrt{b}-\sqrt{c}-\sqrt{d}\right)^2+\left(\sqrt{c}-\sqrt{d}\right)^2.$$

Using the substitution

$$x = \sqrt{b} - \sqrt{c}, \quad y = \sqrt{c} - \sqrt{d}, \qquad x, y \ge 0,$$

we get

$$a + b + c + d - 4\sqrt[4]{abcd} \ge \frac{1}{2}(2x + y)^2 + y^2,$$

that is

$$a + b + c + d - 4\sqrt[4]{abcd} \ge \frac{1}{2}(4x^2 + 4xy + 3y^2). \tag{*}$$

The inequality (*) is an equality for $a = b$ and $c = d$.

(a) According to (*), it suffices to show that

$$4x^2 + 4xy + 3y^2 \ge 3(x - y)^2,$$

which is equivalent to

$$x(x + 10y) \ge 0.$$

The equality holds for $a = b = c = d$.

(b) According to (*), it suffices to show that

$$9(4x^2 + 4xy + 3y^2) \ge 4(3x + y)^2,$$

which is equivalent to

$$y(12x + 23y) \ge 0.$$

The equality holds for $a = b$ and $c = d$.

(c) According to (*), it suffices to show that

$$19(4x^2 + 4xy + 3y^2) \ge 8(3x + 2y)^2,$$

which is equivalent to

$$(2x - 5y)^2 \ge 0.$$

The equality holds for $a = b = c = d$.

(d) According to (*), it suffices to show that

$$4(4x^2 + 4xy + 3y^2) \ge 3(x - 2y)^2,$$

which is equivalent to

$$x(13x + 28y) \ge 0.$$

The equality holds for $a = b = c = d$.

(e) According to (*), it suffices to show that

$$4x^2 + 4xy + 3y^2 \ge (2x - y)^2,$$

which is equivalent to

$$y(4x+y) \geq 0.$$

The equality holds for $a=b$ and $c=d$.

(f) According to (*), it suffices to show that

$$3(4x^2+4xy+3y^2) \geq (2x+3y)^2,$$

which is equivalent to

$$x^2 \geq 0.$$

The equality holds for $a=b=c=d$.

□

P 2.127. *If $a \geq b \geq c \geq d \geq 0$, then*

(a) $$a+b+c+d-4\sqrt[4]{abcd} \geq \left(\sqrt{a}-\sqrt{d}\right)^2;$$

(b) $$a+b+c+d-4\sqrt[4]{abcd} \geq 2\left(\sqrt{b}-\sqrt{c}\right)^2;$$

(c) $$a+b+c+d-4\sqrt[4]{abcd} \geq \frac{4}{3}\left(\sqrt{b}-\sqrt{d}\right)^2;$$

(d) $$a+b+c+d-4\sqrt[4]{abcd} \geq \frac{3}{2}\left(\sqrt{c}-\sqrt{d}\right)^2.$$

(Vasile C., 2010)

Solution. (a) Write the inequality as

$$b+c+2\sqrt{ad} \geq 4\sqrt[4]{abcd},$$

which follows immediately from the AM-GM inequality. The equality holds for

$$b=c=\sqrt{ad}.$$

(b) ***First Solution.*** Since

$$a+b+c+d-4\sqrt[4]{abcd} \geq 2\sqrt{ab}+2\sqrt{cd}-4\sqrt[4]{abcd} = 2\left(\sqrt[4]{ab}-\sqrt[4]{cd}\right)^2,$$

we only need to show that

$$\sqrt[4]{ab}-\sqrt[4]{cd} \geq \sqrt{b}-\sqrt{c},$$

which is equivalent to the obvious inequality

$$\sqrt[4]{b}\left(\sqrt[4]{a}-\sqrt[4]{b}\right)+\sqrt[4]{c}\left(\sqrt[4]{c}-\sqrt[4]{d}\right) \geq 0.$$

The equality holds for $a = b$ and $c = d$.

Second Solution. According to the inequality (*) from the proof of the preceding P 2.126, it suffices to show that

$$4x^2 + 4xy + 3y^2 \geq 4x^2,$$

which is obvious.

(c) According to the inequality (*) from the proof of the preceding P 2.126, it suffices to show that

$$3(4x^2 + 4xy + 3y^2) \geq 8(x + y)^2,$$

which is equivalent to

$$(2x - y)^2 \geq 0.$$

The equality holds for $a = b = c = d$.

(d) According to the inequality (*) from the proof of the preceding P 2.126, it suffices to show that

$$4x^2 + 4xy + 3y^2 \geq 3y^2,$$

which is obvious. The equality holds for $a = b = c = d$.

□

P 2.128. *If $a \geq b \geq c \geq d \geq e \geq 0$, then*

$$a + b + c + d + e - 5\sqrt[5]{abcde} \geq 2\left(\sqrt{b} - \sqrt{d}\right)^2.$$

(Vasile C., 2010)

Solution. From the AM-GM inequality, we have

$$c + 4\sqrt[4]{abde} \geq 5\sqrt[5]{abcde},$$

which can be rewritten as

$$c - 5\sqrt[5]{abcde} \geq -4\sqrt[4]{abde}.$$

Thus, it suffices to show that

$$a + b + d + e - 4\sqrt[4]{abde} \geq 2\left(\sqrt{b} - \sqrt{d}\right)^2.$$

Since

$$a + b + d + e - 4\sqrt[4]{abde} \geq 2\sqrt{ab} + 2\sqrt{de} - 4\sqrt[4]{abde} = 2\left(\sqrt[4]{ab} - \sqrt[4]{de}\right)^2,$$

we only need to prove that

$$\sqrt[4]{ab}-\sqrt[4]{de}\ge \sqrt{b}-\sqrt{d},$$

which is equivalent to the obvious inequality

$$\sqrt[4]{b}\left(\sqrt[4]{a}-\sqrt[4]{b}\right)+\sqrt[4]{d}\left(\sqrt[4]{d}-\sqrt[4]{e}\right)\ge 0.$$

The equality holds for

$$a=b,\quad d=e,\quad c^2=ad.$$

□

P 2.129. *If a,b,c,d,e are real numbers, then*

$$\frac{ab+bc+cd+de}{a^2+b^2+c^2+d^2+e^2}\le\frac{\sqrt{3}}{2}.$$

Solution. Using the AM-GM inequality, we have

$$a^2+b^2+c^2+d^2+e^2=\left(a^2+\frac{1}{3}b^2\right)+\left(\frac{2}{3}b^2+\frac{1}{2}c^2\right)+\left(\frac{1}{2}c^2+\frac{2}{3}d^2\right)+\left(\frac{1}{3}d^2+e^2\right)$$

$$\ge 2\sqrt{a^2\cdot\frac{1}{3}b^2}+2\sqrt{\frac{2}{3}b^2\cdot\frac{1}{2}c^2}+2\sqrt{\frac{1}{2}c^2\cdot\frac{2}{3}d^2}+2\sqrt{\frac{1}{3}d^2\cdot e^2}$$

$$\ge\frac{2}{\sqrt{3}}(ab+bc+cd+da).$$

The equality holds for

$$a=\frac{b}{\sqrt{3}}=\frac{c}{2}=\frac{d}{\sqrt{3}}=e.$$

Remark. The following more general inequality holds

$$\frac{a_1a_2+a_2a_3+\cdots+a_{n-1}a_n}{a_1^2+a_2^2+\cdots+a_n^2}\le\cos\frac{\pi}{n+1},$$

with equality for

$$\frac{a_1}{\sin\frac{\pi}{n+1}}=\frac{a_2}{\sin\frac{2\pi}{n+1}}=\cdots=\frac{a_n}{\sin\frac{n\pi}{n+1}}.$$

Denoting

$$c_i=\frac{\sin\frac{(i+1)\pi}{n+1}}{2\sin\frac{i\pi}{n+1}},\quad i=1,2,\cdots,n-1,$$

we have

$$c_1 = \cos\frac{\pi}{n+1}, \qquad 4c_{n-1} = \frac{1}{\cos\frac{\pi}{n+1}},$$

$$\frac{1}{4c_i} + c_{i+1} = \cos\frac{\pi}{n+1}, \qquad i = 1, 2, \cdots, n-2,$$

hence

$$(a_1^2 + a_2^2 + \cdots + a_n^2)\cos\frac{\pi}{n+1} =$$

$$= c_1a_1^2 + \left(\frac{1}{4c_1} + c_2\right)a_2^2 + \cdots + \left(\frac{1}{4c_{n-2}} + c_{n-1}\right)a_{n-1}^2 + \frac{1}{4c_{n-1}}a_n^2$$

$$= \left(c_1a_1^2 + \frac{1}{4c_1}a_2^2\right) + \left(c_2a_2^2 + \frac{1}{4c_2}a_3^2\right) + \cdots + \left(c_{n-1}a_{n-1}^2 + \frac{1}{4c_{n-1}}a_n^2\right)$$

$$\geq 2\sqrt{c_1a_1^2 \cdot \frac{1}{4c_1}a_2^2} + 2\sqrt{c_2a_2^2 \cdot \frac{1}{4c_2}a_3^2} + \cdots + 2\sqrt{c_{n-1}a_{n-1}^2 \cdot \frac{1}{4c_{n-1}}a_n^2}$$

$$\geq a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n.$$

□

P 2.130. *If a, b, c, d, e are positive real numbers, then*

$$\frac{a^2b^2}{bd+ce} + \frac{b^2c^2}{cd+ae} + \frac{c^2a^2}{ad+be} \geq \frac{3abc}{d+e}.$$

Solution. Using the Cauchy-Schwarz inequality

$$\frac{a^2b^2}{bd+ce} + \frac{b^2c^2}{cd+ae} + \frac{c^2a^2}{ad+be} \geq \frac{(ab+bc+ca)^2}{(bd+ce)+(cd+ae)+(ad+be)},$$

it suffices to show that

$$\frac{(ab+bc+ca)^2}{(bd+ce)+(cd+ae)+(ad+be)} \geq \frac{3abc}{d+e},$$

which is equivalent to

$$\frac{(ab+bc+ca)^2}{a+b+c} \geq 3abc,$$

$$a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2 \geq 0.$$

The equality holds for $a = b = c$.

□

P 2.131. *If a,b,c,d,e,f are nonnegative real numbers such that*

$$a \ge b \ge c \ge d \ge e \ge f,$$

then

$$(a+b+c+d+e+f)^2 \ge 8(ac+bd+ce+df).$$

(Vasile C., 2005)

First Solution. Let us denote

$$x = b+c+d+e+f,$$

and write the inequality as follows:

$$(a+x)^2 - 8(ac+bd+ce+df) \ge 0,$$

$$(a+x-4c)^2 + 8(a+x)c - 16c^2 - 8(ac+bd+ce+df) \ge 0,$$

$$(a+x-4c)^2 - 8[c^2 - (b+d+f)c + d(b+f)] \ge 0,$$

$$(a+x-4c)^2 - 8(c-d)(c-b-f) \ge 0,$$

$$(a+x-4c)^2 + 8(c-d)(b-c+f) \ge 0.$$

The last inequality is clearly true. The equality holds for $c = d = (a+b+e+f)/2$, and for $c = b+f = (a+d+e)/2$; that is, for

$$a = b = c = d, \quad e = f = 0,$$

and for

$$a \ge d+e, \quad b = c = \frac{a+d+e}{2}, \quad f = 0.$$

□

P 2.132. *If $a \ge b \ge c \ge d \ge e \ge f \ge 0$, then*

$$a+b+c+d+e+f - 6\sqrt[6]{abcdef} \ge 2\left(\sqrt{b}-\sqrt{e}\right)^2.$$

(Vasile C., 2010)

Solution. Since

$$a+b \ge 2\sqrt{ab}, \quad c+d \ge 2\sqrt{cd}, \quad e+f \ge 2\sqrt{ef},$$

it suffices to show that

$$\sqrt{ab}+\sqrt{cd}+\sqrt{ef} - 3\sqrt[6]{abcdef} \ge \left(\sqrt{b}-\sqrt{e}\right)^2.$$

By the AM-GM inequality, we have

$$\sqrt{cd}+2\sqrt[4]{abef}\ge 3\sqrt[6]{abcdef},$$

which can be rewritten as

$$\sqrt{cd}-3\sqrt[6]{abcdef}\ge -2\sqrt[4]{abef}.$$

Thus, it suffices to show that

$$\sqrt{ab}+\sqrt{ef}-2\sqrt[4]{abef}\ge\left(\sqrt{b}-\sqrt{e}\right)^2.$$

Since

$$\sqrt{ab}+\sqrt{ef}-2\sqrt[4]{abef}=\left(\sqrt[4]{ab}-\sqrt[4]{ef}\right)^2,$$

we only need to prove that

$$\sqrt[4]{ab}-\sqrt[4]{ef}\ge\sqrt{b}-\sqrt{e},$$

which is equivalent to the obvious inequality

$$\sqrt[4]{b}\left(\sqrt[4]{a}-\sqrt[4]{b}\right)+\sqrt[4]{e}\left(\sqrt[4]{e}-\sqrt[4]{f}\right)\ge 0.$$

The equality holds for

$$a=b,\quad c=d,\quad e=f,\quad c^2=ae.$$

□

P 2.133. *Let a,b,c and x,y,z be positive real numbers such that*

$$x+y+z=a+b+c.$$

Prove that

$$ax^2+by^2+cz^2+xyz\ge 4abc.$$

(Vasile C., 1989)

First Solution. Write the inequality as $E\ge 0$, where

$$E=ax^2+by^2+cz^2+xyz-4abc.$$

Among the numbers

$$a-\frac{y+z}{2},\quad b-\frac{z+x}{2},\quad c-\frac{x+y}{2},$$

there are two of them with the same sign; let

$$pq \ge 0,$$

where

$$p = b - \frac{z+x}{2}, \quad q = c - \frac{x+y}{2}.$$

We have

$$b = p + \frac{x+z}{2}, \quad c = q + \frac{x+y}{2}, \quad a = x+y+z-b-c = \frac{y+z}{2} - p - q.$$

Then,

$$\begin{aligned} E =& \left(\frac{y+z}{2} - p - q\right)x^2 + \left(p + \frac{x+z}{2}\right)y^2 + \left(q + \frac{x+y}{2}\right)z^2 \\ &+ xyz - 4\left(\frac{y+z}{2} - p - q\right)\left(p + \frac{x+z}{2}\right)\left(q + \frac{x+y}{2}\right) \\ =& 4pq(p+q) + 2p^2(x+y) + 2q^2(x+z) + 4pqx \\ =& 4q^2\left(p + \frac{x+z}{2}\right) + 4p^2\left(q + \frac{x+y}{2}\right) + 4pqx \\ =& 4(q^2b + p^2c + pqx) \ge 0. \end{aligned}$$

The equality holds for $a = \dfrac{y+z}{2}$, $b = \dfrac{z+x}{2}$, $c = \dfrac{x+y}{2}$.

Second Solution. Consider the following two cases.

Case 1: $x^2 \ge 4bc$. We have

$$ax^2 + by^2 + cz^2 + xyz - 4abc > ax^2 - 4abc \ge 0.$$

Case 2: $x^2 \le 4bc$. Let

$$u = x + y + z = a + b + c.$$

Substituting

$$z = u - x - y, \quad a = u - b - c,$$

the inequality can be restated as

$$Au^2 + Bu + C \ge 0,$$

where

$$A = c,$$

$$B = (x^2 - 4bc) - 2c(x+y) + xy,$$

$$C = -(b+c)(x^2 - 4bc) + by^2 + c(x+y)^2 - xy(x+y).$$

Since the quadratic function $Au^2 + Bu + C$ has the discriminant

$$D = (x^2 - 4bc)(2c - x - y)^2 \le 0,$$

the conclusion follows.

□

P 2.134. *Let a,b,c and x,y,z be positive real numbers such that*

$$x+y+z=a+b+c.$$

Prove that

$$\frac{x(3x+a)}{bc}+\frac{y(3y+b)}{ca}+\frac{z(3z+c)}{ab}\ge 12.$$

(Vasile C., 1990)

Solution. Write the inequality as

$$ax^2+by^2+cz^2+\frac{1}{3}(a^2x+b^2y+c^2z)\ge 4abc.$$

Applying the Cauchy-Schwarz inequality, we have

$$a^2x+b^2y+c^2z\ge \frac{(a+b+c)^2}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}=\frac{xyz(x+y+z)^2}{xy+yz+zx}\ge 3xyz.$$

Therefore, it suffices to show that

$$ax^2+by^2+cz^2+xyz\ge 4abc,$$

which is just the inequality in the preceding P 2.133. The equality holds for

$$x=y=z=a=b=c.$$

□

P 2.135. *Let a,b,c be given positive numbers. Find the minimum value $F(a,b,c)$ of*

$$E(x,y,z)=\frac{ax}{y+z}+\frac{by}{z+x}+\frac{cz}{x+y},$$

where x,y,z are nonnegative real numbers, no two of which are zero.

(Vasile C., 2006)

Solution. Assume that

$$a=\max\{a,b,c\}.$$

There are two cases to consider.

Case 1: $\sqrt{a}<\sqrt{b}+\sqrt{c}$. Using the Cauchy-Schwarz inequality, we get

$$E=\sum\frac{a(x+y+z)-a(y+z)}{y+z}=(x+y+z)\sum\frac{a}{y+z}-\sum a$$

$$\ge (x+y+z)\frac{\left(\sum\sqrt{a}\right)^2}{\sum(y+z)}-\sum a=\sqrt{ab}+\sqrt{bc}+\sqrt{ca}-\frac{a+b+c}{2}.$$

The equality holds for
$$\frac{y+z}{\sqrt{a}}=\frac{z+x}{\sqrt{b}}=\frac{x+y}{\sqrt{c}};$$
that is, for
$$\frac{x}{\sqrt{b}+\sqrt{c}-\sqrt{a}}=\frac{y}{\sqrt{c}+\sqrt{a}-\sqrt{b}}=\frac{z}{\sqrt{a}+\sqrt{b}-\sqrt{c}}.$$
Case 2: $\sqrt{a}\ge\sqrt{b}+\sqrt{c}$. Let us denote
$$A=(\sqrt{b}+\sqrt{c})^2,$$
$$X=\frac{y+z}{2},\quad Y=\frac{z+x}{2},\quad Z=\frac{x+y}{2},$$
hence
$$x=Y+Z-X,\quad y=Z+X-Y,\quad z=X+Y-Z.$$
We have
$$\begin{aligned}E&\ge\frac{Ax}{y+z}+\frac{by}{z+x}+\frac{cz}{x+y}\\&=\frac{A(Y+Z-X)}{2X}+\frac{b(Z+X-Y)}{2Y}+\frac{c(X+Y-Z)}{2Z}\\&=\frac{1}{2}\left(A\frac{Y}{X}+b\frac{X}{Y}\right)+\frac{1}{2}\left(b\frac{Z}{Y}+c\frac{Y}{Z}\right)+\frac{1}{2}\left(c\frac{X}{Z}+A\frac{Z}{X}\right)-b-c-\sqrt{bc}\\&\ge\sqrt{Ab}+\sqrt{bc}+\sqrt{cA}-b-c-\sqrt{bc}=2\sqrt{bc}.\end{aligned}$$

The equality holds for $x=0$ and $\frac{y}{z}=\sqrt{\frac{c}{b}}$. Therefore, for $a=\max\{a,b,c\}$, we have
$$F(a,b,c)=\begin{cases}\sqrt{ab}+\sqrt{bc}+\sqrt{ca}-\frac{a+b+c}{2}, & \sqrt{a}<\sqrt{b}+\sqrt{c}\\ 2\sqrt{bc}, & \sqrt{a}\ge\sqrt{b}+\sqrt{c}\end{cases}.$$

□

P 2.136. *Let a,b,c and x,y,z be real numbers.*

(a) If $ab+bc+ca>0$, then
$$[(b+c)x+(c+a)y+(a+b)z]^2\ge 4(ab+bc+ca)(xy+yz+zx);$$
(b) If $a,b,c\ge 0$, then
$$[(b+c)x+(c+a)y+(a+b)z]^2\ge 4(a+b+c)(ayz+bzx+cxy).$$

(Vasile C., 1995)

Solution. (a) ***First Solution.*** The condition $ab + bc + ca > 0$ yields $b + c \neq 0$. Indeed, if $b + c = 0$, then $ab + bc + ca = -b^2 \leq 0$, which is false. The desired inequality is equivalent to $D \geq 0$, where D is the discriminant of the quadratic function

$$f(t) = (at - x)(bt - y) + (bt - y)(ct - z) + (ct - z)(at - x).$$

For the sake of contradiction, assume that $D < 0$ for some real numbers a, b, c and x, y, z. Since the coefficient of t^2 is positive, we have $f(t) > 0$ for all real t. This is not true, because for

$$(bt - y) + (ct - z) = 0,$$

we get

$$t = \frac{y + z}{b + c}$$

and

$$f\left(\frac{y + z}{b + c}\right) = -\left(\frac{bz - cy}{b + c}\right)^2 \leq 0.$$

For $pqr \neq 0$, the equality holds when

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$

Second Solution. If $xy + yz + zx \leq 0$, then the inequality is obviously true. Otherwise, due to homogeneity in x, y, z, we may assume that

$$x + y + z = a + b + c.$$

Then, by the AM-GM inequality, we have

$$2\sqrt{(ab + bc + ca)(xy + yz + zx)} \leq (ab + bc + ca) + (xy + yz + zx)$$

$$= \frac{(a + b + c)^2 - a^2 - b^2 - c^2}{2} + \frac{(x + y + z)^2 - x^2 - y^2 - z^2}{2}$$

$$= (a + b + c)(x + y + z) - \frac{a^2 + x^2}{2} - \frac{b^2 + y^2}{2} - \frac{c^2 + z^2}{2}$$

$$\leq (a + b + c)(x + y + z) - ax - by - cz = (b + c)x + (c + a)y + (a + b)z.$$

(b) Assume that x is between y and z, that is,

$$(x - y)(x - z) \leq 0.$$

Consider the non-trivial case

$$a + b + c > 0.$$

The desired inequality is equivalent to $D \ge 0$, where D is the discriminant of the quadratic function

$$f(t) = a(t-y)(t-z) + b(t-z)(t-x) + c(t-x)(t-y).$$

For the sake of contradiction, assume that $D < 0$ for some $a,b,c \ge 0$ and real numbers x,y,z. Since the coefficient of t^2 is positive, we have $f(t) > 0$ for all real t. This is false, because

$$f(x) = a(x-y)(x-z) \le 0.$$

The equality holds for $x = y = z$, and also for $a = 0$ and $x = \dfrac{cy+bz}{c+b}$, or $b = 0$ and $y = \dfrac{az+cx}{a+c}$, or $c = 0$ and $z = \dfrac{bx+ay}{b+a}$.

Remark 1. For $x = b$, $y = c$, $z = a$, from the inequality in (b), we get the following cyclic inequality:

$$(a^2+b^2+c^2+ab+bc+ca)^2 \ge 4(a+b+c)(ab^2+bc^2+ca^2),$$

where $a,b,c \ge 0$. The equality holds for $a = b = c$, and also for $a = 0$ and $\dfrac{b}{c} = \dfrac{\sqrt{5}-1}{2}$ (or any cyclic permutation). Notice that this inequality is equivalent to

$$a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2 \ge 2(ab^3+bc^3+ca^3-a^3b-b^3c-c^3a),$$

which is the inequality in P 3.95 from Volume 1.

Remark 2. For $x = 1/c$, $y = 1/a$, $z = 1/b$, from the inequality in (b), we get the following cyclic inequality:

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+3\right)^2 \ge 4(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right),$$

which is the inequality in P 1.49-(c).

Remark 3. For $a = x(x-y+z)$, $b = y(y-z+x)$, $c = z(z-x+y)$, the inequality in (b) turns into

$$(x^2y+y^2z+z^2x)^2 \ge xyz(x+y+z)(x^2+y^2+z^2).$$

where x,y,z are the lengths of the sides of a triangle (see P 1.187).

□

P 2.137. *Let a,b,c and x,y,z be positive real numbers such that*

$$\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}=1.$$

Prove that

(a) $$x+y+z\ge\sqrt{4(a+b+c+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\ )+3\sqrt[3]{abc}};$$

(b) $$x+y+z>\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}.$$

Solution. (a) Write the desired inequality in the form

$$\left(\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}\right)(x+y+z)^2\ge 4\left(a+b+c+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)+3\sqrt[3]{abc}.$$

We have

$$\left(\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}\right)(x^2+y^2+z^2)=\sum\frac{ax^2}{yz}+\sum\frac{a(y^2+z^2)}{yz}.$$

In addition, by the AM-GM inequality, we get

$$\sum\frac{ax^2}{yz}\ge 3\sqrt[3]{abc},$$

$$\sum\frac{a(y^2+z^2)}{yz}\ge 2(a+b+c).$$

Therefore,

$$\left(\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}\right)(x^2+y^2+z^2)\ge 3\sqrt[3]{abc}+2(a+b+c).$$

Adding this inequality to the Cauchy-Schwarz inequality

$$2\left(\frac{a}{yz}+\frac{b}{zx}+\frac{c}{xy}\right)(yz+zx+xy)\ge 2\left(\sqrt{a}+\sqrt{a}+\sqrt{c}\right)^2$$

yields the desired inequality. The equality holds for

$$x=y=z=\sqrt{3a}=\sqrt{3b}=\sqrt{3c}.$$

(b) According to the inequality in (a), it suffices to show that

$$4\left(a+b+c+\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)\ge\left(\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}\right)^2.$$

This inequality is equivalent to

$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2\ge\sqrt{(a+b)(b+c)}+\sqrt{(b+c)(c+a)}+\sqrt{(c+a)(a+b)}\ ,$$

which follows immediately from the inequality P 2.24 in Volume 2.

□

P 2.138. *If a,b,c and x,y,z are nonnegative real numbers, then*

$$\frac{2}{(b+c)(y+z)}+\frac{2}{(c+a)(z+x)}+\frac{2}{(a+b)(x+y)}\ge\frac{9}{(b+c)x+(c+a)y+(a+b)z}.$$

(Ji Chen and Vasile Cîrtoaje, 2010)

Solution. Since

$$(b+c)x+(c+a)y+(a+b)z=a(y+z)+(b+c)x+bz+cy,$$

we can write the inequality as

$$\sum\frac{2a(y+z)+2(b+c)x+2(bz+cy)}{(b+c)(y+z)}\ge 9,$$

$$\sum\frac{2a}{b+c}+\sum\frac{2x}{y+z}\ge 9-\sum\frac{2(bz+cy)}{(b+c)(y+z)},$$

$$\sum\frac{2a}{b+c}+\sum\frac{2x}{y+z}\ge 6+\sum\left[1-\frac{2(bz+cy)}{(b+c)(y+z)}\right],$$

$$\sum\frac{2a}{b+c}+\sum\frac{2x}{y+z}\ge 6+\sum\frac{(b-c)(y-z)}{(b+c)(y+z)}.$$

Since

$$\sum\frac{(b-c)(y-z)}{(b+c)(y+z)}\le\frac{1}{2}\sum\left(\frac{b-c}{b+c}\right)^2+\frac{1}{2}\sum\left(\frac{y-z}{y+z}\right)^2,$$

it suffices to show that

$$\sum\frac{2a}{b+c}+\sum\frac{2x}{y+z}\ge 6+\frac{1}{2}\sum\left(\frac{b-c}{b+c}\right)^2+\frac{1}{2}\sum\left(\frac{y-z}{y+z}\right)^2,$$

which is equivalent to

$$\sum\frac{2a}{b+c}+\sum\frac{2x}{y+z}\ge 9-\sum\frac{2bc}{(b+c)^2}-\sum\frac{2yz}{(y+z)^2},$$

$$\sum\left[\frac{2a}{b+c}+\frac{2bc}{(b+c)^2}\right]+\sum\left[\frac{2x}{y+z}+\frac{2yz}{(y+z)^2}\right]\ge 9,$$

$$2(ab+bc+ca)\sum\frac{1}{(b+c)^2}+2(xy+yz+zx)\sum\frac{1}{(y+z)^2}\ge 9.$$

This inequality can be obtained by summing the known inequalities (see P 1.72 in Volume 2, case $k=2$)

$$4(ab+bc+ca)\sum\frac{1}{(b+c)^2}\ge 9,$$

$$4(xy+yz+zx)\sum\frac{1}{(y+z)^2}\ge 9.$$

The equality holds for $a = b = c$ and $x = y = z$, and also for $a = x = 0$, $b = c$ and $y = z$ (or any cyclic permutation).

Remark. For $x = a$, $y = b$ and $z = c$, we get the known inequality (Iran 1996):

$$\frac{1}{(a+b)^2} + \frac{1}{(a+c)^2} + \frac{1}{(b+c)^2} \ge \frac{9}{4(ab+bc+ca)}.$$

□

P 2.139. *Let a, b, c be the lengths of the sides of a triangle. If x, y, z are real numbers, then*

$$(ya^2 + zb^2 + xc^2)(za^2 + xb^2 + yc^2) \ge (xy + yz + zx)(a^2b^2 + b^2c^2 + c^2a^2).$$

(Vasile C., 2001)

First Solution. Write the inequality as follows:

$$x^2b^2c^2 + y^2c^2a^2 + z^2a^2b^2 \ge \sum yza^2(b^2 + c^2 - a^2),$$

$$x^2b^2c^2 + y^2c^2a^2 + z^2a^2b^2 \ge 2abc \sum yza \cos A,$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \ge \frac{2yz \cos A}{bc} + \frac{2zx \cos B}{ca} + \frac{2xy \cos C}{ab},$$

$$\left(\frac{x}{a} - \frac{y}{b}\cos C - \frac{z}{c}\cos B\right)^2 + \left(\frac{y}{b}\sin C - \frac{z}{c}\sin B\right)^2 \ge 0.$$

The equality holds for

$$\frac{x}{a^2} = \frac{y}{b^2} = \frac{z}{c^2}.$$

Second Solution. Write the inequality as

$$b^2c^2x^2 - Bx + C \ge 0,$$

where

$$B = c^2(a^2 + b^2 - c^2)y + b^2(a^2 - b^2 + c^2)z,$$

$$C = a^2[c^2y^2 - (b^2 + c^2 - a^2)yz + b^2z^2].$$

It suffices to show that

$$B^2 - 4b^2c^2C \le 0,$$

which is equivalent to

$$A(c^2y - b^2z)^2 \ge 0,$$

where

$$A = 2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4.$$

This inequality is true since

$$A = (a+b+c)(a+b-c)(b+c-a)(c+a-b) \ge 0.$$

Remark 1. For $x = 1/b$, $y = 1/c$ and $z = 1/a$, we get the well-known inequality from P 1.189-(a):

$$a^3b + b^3c + c^3a \ge a^2b^2 + b^2c^2 + c^2a^2.$$

Remark 2. For $x = 1/c^2$, $y = 1/a^2$ and $z = 1/b^2$, we get the elegant cyclic inequality of *Walker*:

$$3\left(\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2}\right) \ge (a^2+b^2+c^2)\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right).$$

□

P 2.140. *If* $a_1 \ge a_2 \ge \cdots \ge a_8 \ge 0$*, then*

$$a_1 + a_2 + \cdots + a_8 - 8\sqrt[8]{a_1a_2\cdots a_8} \ge 3\left(\sqrt{a_6} - \sqrt{a_7}\right)^2.$$

Solution. Let us denote

$$x = \sqrt[6]{a_1a_2\cdots a_6}, \quad y = \sqrt{a_7a_8}, \quad x \ge a_6 \ge a_7 \ge y.$$

By the AM-GM inequality, we have

$$a_1 + a_2 + \cdots + a_6 \ge 6x, \quad a_7 + a_8 \ge 2y.$$

Also, we have

$$\sqrt{a_6} - \sqrt{a_7} \le \sqrt{x} - \sqrt{y}.$$

Thus, it suffices to show that

$$6x + 2y - 8\sqrt[8]{x^6y^2} \ge 3(\sqrt{x} - \sqrt{y})^2.$$

For the nontrivial case $y \ne 0$, we can set $y = 1$ (due to homogeneity) and $x = t^4$, $t \ge 1$. The inequality can be restated as

$$6t^4 + 2 - 8t^3 \ge 3(t^2-1)^2,$$

which is equivalent to

$$(t-1)^3(3t+1) \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_8$.

□

P 2.141. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be real numbers. Prove that*

$$a_1b_1 + \cdots + a_nb_n + \sqrt{(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)} \ge \frac{2}{n}(a_1 + \cdots + a_n)(b_1 + \cdots + b_n).$$

(Vasile C., 1989)

First Solution. Write the inequality as

$$\sqrt{(a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)} \ge a_1(2b - b_1) + \cdots + a_n(2b - b_n),$$

where

$$b = \frac{1}{n}(b_1 + \cdots + b_n).$$

Using the substitution

$$x_i = 2b - b_i, \quad i = 1, 2, \ldots, n,$$

we have

$$\sum_{i=1}^n x_i = 2nb - \sum_{i=1}^n b_i = 2nb - nb = nb,$$

$$\sum_{i=1}^n b_i^2 = \sum_{i=1}^n (2b - x_i)^2 = 4nb^2 - 4b\sum_{i=1}^n x_i + \sum_{i=1}^n x_i^2 = \sum_{i=1}^n x_i^2.$$

Therefore, the desired inequality can be restated as

$$\sqrt{(a_1^2 + \cdots + a_n^2)(x_1^2 + \cdots + x_n^2)} \ge a_1x_1 + \cdots + a_nx_n,$$

which is just the Cauchy-Schwarz inequality. If $a_1a_2\cdots a_n \ne 0$, then the equality holds for

$$\frac{2b - b_1}{a_1} = \frac{2b - b_2}{a_2} = \cdots = \frac{2b - b_n}{a_n} \ge 0.$$

Second Solution. Consider the nontrivial case where $a_1^2 + \cdots + a_n^2 \ne 0$ and $b_1^2 + \cdots + b_n^2 \ne 0$, denote

$$p = \sqrt{\frac{b_1^2 + \cdots + b_n^2}{a_1^2 + \cdots + a_n^2}},$$

and use the substitution

$$b_i = px_i, \quad i = 1, 2, \ldots, n$$

to have

$$a_1^2 + \cdots + a_n^2 = x_1^2 + \cdots + x_n^2.$$

The desired inequality becomes

$$(a_1x_1 + \cdots + a_nx_n) + (a_1^2 + \cdots + a_n^2) \ge \frac{2}{n}(a_1 + \cdots + a_n)(x_1 + \cdots + x_n),$$

$$(a_1+x_1)^2+\cdots+(a_n+x_n)^2 \ge \frac{4}{n}(a_1+\cdots+a_n)(x_1+\cdots+x_n).$$

Since

$$4(a_1+\cdots+a_n)(x_1+\cdots+x_n) \le [(a_1+\cdots+a_n)+(x_1+\cdots+x_n)]^2,$$

it suffices to show that

$$(a_1+x_1)^2+\cdots+(a_n+x_n)^2 \ge \frac{1}{n}[(a_1+x_1)+\cdots+(a_n+x_n)]^2.$$

This follows immediately from the Cauchy-Schwarz inequality.

Remark. Substituting $b_i = 1/a_i$ for all i, we get the following inequality

$$n^2+n\sqrt{(a_1^2+\cdots+a_n^2)\left(\frac{1}{a_1^2}+\cdots+\frac{1}{a_n^2}\right)} \ge 2(a_1+\cdots+a_n)\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right).$$

If $a_1 \le a_2 \le \cdots \le a_n$ and n is even, $n=2k$, then the equality holds for

$$a_1=a_2=\cdots=a_k, \quad a_{k+1}=a_{k+2}=\cdots=a_{2k}.$$

If n is odd, then the equality holds only if $a_1=a_2=\cdots=a_n$.

Conjecture. *If $a_1,a_2,\ldots,a_n$ are positive real numbers and n is odd, then*

$$n^2+1+\sqrt{(n^2-1)(a_1^2+\cdots+a_n^2)\left(\frac{1}{a_1^2}+\cdots+\frac{1}{a_n^2}\right)-n^2+1} \ge$$

$$\ge 2(a_1+\cdots+a_n)\left(\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right).$$

If $a_1 \le a_2 \le \cdots \le a_n$ and n is odd, $n=2k+1$, then the equality holds for

$$a_1=a_2=\cdots=a_k, \quad a_{k+1}=a_{k+2}=\cdots=a_{2k+1},$$

and for

$$a_1=a_2=\cdots=a_{k+1}, \quad a_{k+2}=a_{k+3}=\cdots=a_{2k+1}.$$

□

P 2.142. *Let $a_1, a_2, \ldots, a_n$ be positive real numbers such that $a_1 \ge 2a_2$. Prove that*

$$(5n-1)(a_1^2+a_2^2+\cdots+a_n^2) \ge 5(a_1+a_2+\cdots+a_n)^2.$$

(Vasile C., 2009)

Solution. Let

$$a_1 = ka_2, \quad k \geq 2.$$

By the Cauchy-Schwarz inequality, we have

$$a_1^2 + a_2^2 + \cdots + a_n^2 = (k^2+1)a_2^2 + a_3^2 + \cdots + a_n^2$$

$$\geq \frac{[(k+1)a_2 + a_3 + \cdots + a_n]^2}{\frac{(k+1)^2}{k^2+1} + n - 2} = \frac{(a_1 + a_2 + \cdots + a_n)^2}{\frac{2k}{k^2+1} + n - 1}.$$

Therefore, it suffices to show that

$$\frac{5n-1}{5} \geq \frac{2k}{k^2+1} + n - 1,$$

which is equivalent to the obvious inequality

$$(k-2)(2k-1) \geq 0.$$

The equality holds if and only if $k = 2$ and

$$5a_2^2 + a_3^2 + \cdots + a_n^2 = \frac{(3a_2 + a_3 + \cdots + a_n)^2}{\frac{9}{5} + n - 2};$$

that is, if and only if

$$\frac{5a_1}{6} = \frac{5a_2}{3} = a_3 = \cdots = a_n.$$

□

P 2.143. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 \geq 4a_2$, then*

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq \left(n + \frac{1}{2}\right)^2.$$

Solution. Setting

$$a_1 = ka_2, \quad k \geq 4,$$

the inequality becomes

$$[(1+k)a_2 + a_3 + \cdots + a_n]\left(\frac{1+k}{ka_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}\right) \geq \left(n + \frac{1}{2}\right)^2.$$

By the Cauchy-Schwarz inequality, we have

$$[(1+k)a_2 + a_3 + \cdots + a_n]\left(\frac{1+k}{ka_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}\right) \geq \left(\frac{1+k}{\sqrt{k}} + n - 2\right)^2.$$

Thus, we only need to show that

$$\frac{1+k}{\sqrt{k}}+n-2 \geq n+\frac{1}{2},$$

which reduces to

$$\left(\sqrt{k}-2\right)\left(2\sqrt{k}-1\right) \geq 0.$$

The equality holds if and only if $k=4$ and

$$\frac{a_1}{2}=2a_2=a_3=\cdots=a_n.$$

□

P 2.144. *If* $a_1 \geq a_2 \geq \cdots \geq a_n > 0$ *such that* $a_1+a_2+\cdots+a_n=n$, *then*

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n \geq \frac{4(n-1)^2}{n^3}(a_1-a_2)^2.$$

(Vasile C., 2009)

Solution. Since

$$\frac{1}{a_2}+\frac{1}{a_3}+\cdots+\frac{1}{a_n} \geq \frac{(n-1)^2}{a_2+a_3+\cdots+a_n}=\frac{(n-1)^2}{n-a_1}$$

and

$$a_1-a_2 \leq a_1-\frac{a_2+a_3+\cdots+a_n}{n-1}=a_1-\frac{n-a_1}{n-1}=\frac{n(a_1-1)}{n-1},$$

it suffices to show that

$$\frac{1}{a_1}+\frac{(n-1)^2}{n-a_1}-n \geq \frac{4}{n}(a_1-1)^2.$$

This is equivalent to the obvious inequality

$$(a_1-1)^2(2a_1-n)^2 \geq 0.$$

The equality holds for

$$a_1=a_2=\cdots=a_n=1,$$

and also for

$$a_1=\frac{n}{2}, \quad a_2=a_3=\cdots=a_n=\frac{n}{2(n-1)}.$$

□

P 2.145. *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are real numbers such that*

$$a_1 \le a_2 \le \cdots \le a_n, \quad a_1 + a_2 + \cdots + a_n = 0,$$

then

$$a_1^2 + a_2^2 + \cdots + a_n^2 + na_1a_n \le 0.$$

(Vasile C., 2009)

Solution. For the nontrivial case $a_1^2+a_2^2+\cdots+a_n^2 \ne 0$, let $a_1 = a < 0$ and $a_n = b > 0$ be fixed. We claim that for

$$a \le a_2 \le \cdots \le a_{n-1} \le b, \quad a_2 + \cdots + a_{n-1} = -a - b,$$

the sum $S = a_2^2 + \cdots + a_{n-1}^2$ is maximum when at least $n-3$ of the numbers $a_2, \ldots, a_{n-1}$ are equal to a or b. In the contrary case, if $a < a_i \le a_j < b$, then

$$a_i^2 + a_j^2 < c_i^2 + c_j^2$$

for all c_i and c_j such that

$$a \le c_i < a_i \le a_j < c_j \le b, \quad c_i + c_j = a_i + a_j;$$

indeed,

$$a_i^2 + a_j^2 - c_i^2 - c_j^2 = (a_i - c_i)(a_i + c_i) + (a_j - c_j)(a_j + c_j) = (a_i - c_i)(a_i + c_i - a_j - c_j) < 0.$$

This result confirms our claim. Therefore, it suffices to consider the case where at least $n-3$ of the numbers $a_2, \ldots, a_{n-1}$ are equal to a or b. More precisely, assume that k of $a_2, \ldots, a_{n-1}$ are equal to a and m of $a_2, \ldots, a_{n-1}$ are equal to b, where

$$k + m = n - 3, \quad k, m \ge 0.$$

Therefore, it suffices to show that

$$(k+1)a^2 + c^2 + (m+1)b^2 + (k+m+3)ab \le 0,$$

where

$$a \le c \le b, \quad (k+1)a + c + (m+1)b = 0.$$

We have

$$\begin{aligned}(k+1)a^2 + c^2 + (m+1)b^2 + (k+m+3)ab &= c^2 + (a+b)[(k+1)a + (m+1)b] + ab \\ &= c^2 - (a+b)c + ab = (c-a)(c-b) \le 0.\end{aligned}$$

The equality holds if and only if

$$a_1, a_2, \ldots, a_n \in \{a_1, a_n\}, \quad a_1 + a_2 + \cdots + a_n = 0.$$

□

P 2.146. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 4)$ be nonnegative real numbers such that*

$$a_1 \ge a_2 \ge \cdots \ge a_n$$

and

$$(a_1 + a_2 + \cdots + a_n)^2 = 4(a_1^2 + a_2^2 + \cdots + a_n^2).$$

Prove that

$$1 \le \frac{a_1 + a_2}{a_3 + a_4 + \cdots + a_n} \le 1 + \sqrt{\frac{2n-8}{n-2}}.$$

(Vasile C., 2007)

Solution. Denote

$$A = a_1 + a_2, \qquad B = a_3 + a_4 + \cdots + a_n.$$

Since

$$2(a_1^2 + a_2^2) \ge A^2, \qquad (n-2)(a_3^2 + a_4^2 + \cdots + a_n^2) \ge B^2,$$

from the hypothesis

$$(a_1 + a_2 + \cdots + a_n)^2 = 4(a_1^2 + a_2^2) + 4(a_3^2 + \cdots + a_n^2),$$

we get

$$(A+B)^2 \ge 2A^2 + \frac{4}{n-2}B^2,$$

$$A \le \left(1 + \sqrt{\frac{2n-8}{n-2}}\right)B.$$

The right inequality is an equality for

$$a_1 = a_2 = ka_3 = \cdots = ka_n, \qquad k = \frac{n-2+\sqrt{2(n-2)(n-4)}}{2}.$$

To prove the left inequality, let

$$a_1 \ge a_2 \ge x \ge a_3 \ge \cdots \ge a_n.$$

From

$$\frac{A}{a_1a_2} = \frac{1}{a_1} + \frac{1}{a_2} \le \frac{1}{x} + \frac{1}{x} = \frac{2}{x},$$

we get

$$2a_1a_2 \ge Ax,$$

hence

$$a_1^2 + a_2^2 = A^2 - 2a_1a_2 \le A^2 - Ax.$$

In addition,

$$a_3^2 + \cdots + a_n^2 \le a_3x + \cdots + a_nx = Bx.$$

Therefore, from the hypothesis

$$(a_1 + a_2 + \cdots + a_n)^2 = 4(a_1^2 + a_2^2) + 4(a_3^2 + \cdots + a_n^2),$$

we get

$$(A+B)^2 \le 4(A^2 - Ax) + 4Bx,$$

$$4(A-B)x - 3A^2 + 2AB + B^2 \le 0,$$

$$(A-B)(3A+B-4x) \ge 0.$$

Since

$$3A+B-4x \ge 3A-4x \ge 6x-4x \ge 0,$$

it follows that $A-B \ge 0$. The left inequality is an equality only for $n=4$ and $a_1 = a_2 = a_3 = a_4$.

□

P 2.147. *If* $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$, *then*

(a) $$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \ge \frac{1}{3}\left(\sqrt{a_1} + \sqrt{a_2} - 2\sqrt{a_n}\right)^2;$$

(b) $$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \ge \frac{1}{4}\left(2\sqrt{a_1} - \sqrt{a_{n-1}} - \sqrt{a_n}\right)^2.$$

(Vasile C., 2010)

Solution. (a) For $n=2$, the inequality is equivalent to $\left(\sqrt{a_1} - \sqrt{a_2}\right)^2 \ge 0$. Consider further $n \ge 3$. By the AM-GM inequality, we have

$$a_3 + \cdots + a_{n-1} + 3\sqrt[3]{a_1 a_2 a_n} \ge n\sqrt[n]{a_1 a_2 \cdots a_n}.$$

Therefore, it suffices to prove that

$$a_1 + a_2 + a_n - 3\sqrt[3]{a_1 a_2 a_n} \ge \frac{1}{3}\left(\sqrt{a_1} + \sqrt{a_2} - 2\sqrt{a_n}\right)^2.$$

Setting

$$x = \left(\frac{\sqrt{a_1} + \sqrt{a_2}}{2}\right)^2, \quad x \ge a_n,$$

since $a_1 + a_2 \ge 2x$ and $a_1 a_2 \le x^2$, it suffices to show that

$$2x + a_n - 3\sqrt[3]{x^2 a_n} \ge \frac{4}{3}\left(\sqrt{x} - \sqrt{a_n}\right)^2.$$

For the nontrivial case $a_n \ne 0$, we may consider $a_n = 1$ (due to homogeneity). In addition, substituting $x = y^6$, $y \ge 1$, the inequality can be restated as

$$2y^6 + 1 - 3y^4 \ge \frac{4}{3}(y^3 - 1)^2,$$

$$(y-1)^2[3(y+1)^2(2y^2+1)-4(y^2+y+1)^2]\geq 0.$$

This inequality is true if

$$(y+1)\sqrt{3(2y^2+1)}\geq 2(y^2+y+1).$$

Since

$$\sqrt{3(2y^2+1)}\geq 2y+1,$$

we have

$$(y+1)\sqrt{3(2y^2+1)}-2(y^2+y+1)\geq (y+1)(2y+1)-2(y^2+y+1)=y-1\geq 0.$$

This completes the proof. The equality holds for $a_1=a_2=\cdots=a_n$.

(b) For $n=2$, the inequality is equivalent to $\left(\sqrt{a_1}-\sqrt{a_2}\right)^2\geq 0$. Consider further $n\geq 3$. By the AM-GM inequality, we have

$$a_2+a_3+\cdots+a_{n-2}+3\sqrt[3]{a_1a_{n-1}a_n}\geq n\sqrt[n]{a_1a_2\cdots a_n}.$$

Therefore, it suffices to prove that

$$a_1+a_{n-1}+a_n-3\sqrt[3]{a_1a_{n-1}a_n}\geq \frac{1}{4}\left(2\sqrt{a_1}-\sqrt{a_{n-1}}-\sqrt{a_n}\right)^2.$$

Setting

$$x=\sqrt{a_{n-1}a_n},\quad x\leq a_1,$$

since $a_{n-1}+a_n\geq 2x$ and $\sqrt{a_{n-1}}+\sqrt{a_n}\geq 2\sqrt{x}$, it suffices to show that

$$a_1+2x-3\sqrt[3]{a_1x^2}\geq \left(\sqrt{a_1}-\sqrt{x}\right)^2.$$

Due to homogeneity, we may consider $a_1=1$. In addition, substituting $x=y^6$, $y\leq 1$, the inequality becomes

$$1+2y^6-3y^4\geq (1-y^3)^2,$$

which is equivalent to the obvious inequality

$$y^3(y-1)^2(y+2)\geq 0.$$

The equality holds for $a_1=a_2=\cdots=a_n$. If $n\geq 3$, then the equality holds also for $a_2=\cdots=a_n=0$.

□

P 2.148. *If* $a_1\geq a_2\geq\cdots\geq a_n\geq 0,\ n\geq 3$, *then*

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n}\geq \frac{n-1}{2n}\left(\sqrt{a_{n-2}}+\sqrt{a_{n-1}}-2\sqrt{a_n}\right)^2.$$

(Vasile C., 2010)

Solution. Let us denote

$$x = \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}, \quad x \ge a_n.$$

By the AM-GM inequality, we have

$$a_1 a_2 \cdots a_{n-1} \le x^{n-1}.$$

Also,

$$\frac{\sqrt{a_{n-2}} + \sqrt{a_{n-1}}}{2} \le \sqrt{\frac{a_{n-2} + a_{n-1}}{2}} \le \sqrt{\frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}} = \sqrt{x}.$$

Then, it suffices to show that

$$(n-1)x + a_n - n\sqrt[n]{x^{n-1}a_n} \ge \frac{2(n-1)}{n}\left(\sqrt{x} - \sqrt{a_n}\right)^2.$$

For the nontrivial case $a_n \ne 0$, we may consider $a_n = 1$ (due to homogeneity). In addition, substituting $x = t^{2n}$, $t \ge 1$, the inequality becomes $g(t) \ge 0$, where

$$g(t) = (n-1)t^{2n} + 1 - nt^{2n-2} - \frac{2(n-1)}{n}(t^n - 1)^2.$$

We have

$$g'(t) = 2(n-1)t^{n-1}h(t),$$

where

$$h(t) = n(t^n - t^{n-2}) - 2(t^n - 1).$$

Since

$$h'(t) = n(n-2)t^{n-3}(t^2 - 1) \ge 0,$$

$h(t)$ is increasing, $h(t) \ge h(1) = 0$, $g'(t) \ge 0$, $g(t)$ is increasing, hence $g(t) \ge g(1) = 0$. This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n$.

□

P 2.149. *Let* $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$. *If*

$$\frac{n}{2} \le k \le n-1,$$

then

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n} \ge \frac{2k(n-k)}{n}\left(\sqrt{a_k} - \sqrt{a_{k+1}}\right)^2.$$

(Vasile C., 2010)

Solution. Let us denote

$$x = \sqrt[k]{a_1a_2\cdots a_k}, \quad y = \sqrt[n-k]{a_{k+1}a_{k+2}\cdots a_n}, \quad x \ge a_k \ge a_{k+1} \ge y.$$

By the AM-GM inequality, we have

$$a_1 + a_2 + \cdots + a_k \ge kx, \quad a_{k+1} + a_{k+2} + \cdots + a_n \ge (n-k)y.$$

Also, we have

$$\sqrt{a_k} - \sqrt{a_{k+1}} \le \sqrt{x} - \sqrt{y}.$$

Thus, it suffices to show that

$$kx + (n-k)y - n\sqrt[n]{x^ky^{n-k}} \ge \frac{2k(n-k)}{n}(\sqrt{x} - \sqrt{y})^2.$$

For the nontrivial case $y > 0$, we can set $y = 1$ (due to homogeneity). In addition, setting $x = t^{2n}$, $t \ge 1$, the inequality becomes $f(t) \ge 0$, where

$$f(t) = kt^{2n} + n - k - nt^{2k} - \frac{2k(n-k)}{n}(t^n - 1)^2.$$

We have the derivative

$$f'(t) = 2kt^{n-1}h(t),$$

where

$$h(t) = n(t^n - t^{2k-n}) - 2(n-k)(t^n - 1).$$

Since

$$h'(t) = n(2k-n)(t^{n-1} - t^{2k-n-1}) \ge 0,$$

$h(t)$ is increasing for $t \ge 1$, $h(t) \ge h(1) = 0$, $f'(t) \ge 0$, $f(t)$ is increasing, $f(t) \ge f(1) = 0$. This completes the proof. The equality holds for $a_1 = a_2 = \cdots = a_n$. If n is even and $2k = n$, then the equality holds for $a_1 = a_2 = \cdots = a_k$ and $a_{k+1} = a_{k+2} = \cdots = a_n$.

□

P 2.150. *Let* $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$. *If*

$$1 \le k < j \le n, \qquad k + j \ge n + 1,$$

then

$$a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{2k(n-j+1)}{n+k-j+1}\left(\sqrt{a_k} - \sqrt{a_j}\right)^2.$$

(Vasile C., 2010)

Solution. Let us denote
$$P=\frac{k(n-j+1)}{n+k-j+1}$$
and
$$x=\sqrt[k]{a_1a_2\cdots a_k},\quad y=\sqrt[n-j+1]{a_ja_{j+1}\cdots a_n},\quad x\ge a_k\ge a_j\ge y.$$
By the AM-GM inequality, we have
$$a_1+a_2+\cdots+a_k\ge kx,\quad a_j+a_{j+1}+\cdots+a_n\ge(n-j+1)y$$
and
$$a_{k+1}+\cdots+a_{j-1}\ge(j-k-1)\sqrt[j-k-1]{a_{k+1}\cdots a_{j-1}}.$$
Also, we have
$$\sqrt{a_k}-\sqrt{a_j}\le\sqrt{x}-\sqrt{y}.$$
Thus, it suffices to show that
$$kx+(n-j+1)y+(j-k-1)\sqrt[j-k-1]{a_{k+1}\cdots a_{j-1}}-n\sqrt[n]{a_1a_2\cdots a_n}\ge 2P(\sqrt{x}-\sqrt{y})^2.$$
By the AM-GM inequality, we have
$$(j-k-1)\sqrt[j-k-1]{a_{k+1}\cdots a_{j-1}}+(n-j+k+1)\sqrt[n-j+k+1]{(a_1\cdots a_k)(a_j\cdots a_n)}\ge n\sqrt[n]{a_1a_2\cdots a_n},$$
which is equivalent to
$$\begin{aligned}(j-k-1)\sqrt[j-k-1]{a_{k+1}\cdots a_{j-1}}-n\sqrt[n]{a_1a_2\cdots a_n}&\ge-(n-j+k+1)\sqrt[n-j+k+1]{(a_1\cdots a_k)(a_j\cdots a_n)}\\&=-(n-j+k+1)x^{\frac{k}{n-j+k+1}}y^{\frac{n-j+1}{n-j+k+1}}.\end{aligned}$$
Therefore, we only need to show that
$$kx+(n-j+1)y-(n-j+k+1)x^{\frac{k}{n-j+k+1}}y^{\frac{n-j+1}{n-j+k+1}}\ge 2P\left(\sqrt{x}-\sqrt{y}\right)^2.$$
For the nontrivial case $y\ne 0$, we can set $y=1$ (due to homogeneity). Thus, we need to prove that $f(x)\ge 0$ for $x\ge 1$, where
$$f(x)=kx+n-j+1-(n-j+k+1)x^{\frac{k}{n-j+k+1}}-2P\left(\sqrt{x}-1\right)^2.$$
We have the derivatives
$$f'(x)=k-kx^{\frac{k}{n-j+k+1}-1}+2P\left(\frac{1}{\sqrt{x}}-1\right),$$
$$f''(x)=P\left(x^{\frac{k}{n-j+k+1}-2}-x^{\frac{-3}{2}}\right).$$
Since $f''(x)\ge 0$ for $x\ge 1$, f' is increasing, $f'(x)\ge f'(1)=0$, f is increasing, $f(x)\ge f(1)=0$. This completes the proof. The equality holds for $a_1=a_2=\cdots=a_n$. If n is even, $k=n/2$ and $j=k+1$, then the equality holds for $a_1=a_2=\cdots=a_k$ and $a_{k+1}=a_{k+2}=\cdots=a_n$.

Remark. For $j=k+1$, we get the inequality in P 2.149.

□

P 2.151. *If* $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0,\ n \ge 4$*, then*

(a) $$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{1}{2}\left(1-\frac{1}{n}\right)\left(\sqrt{a_{n-2}}-3\sqrt{a_{n-1}}+2\sqrt{a_n}\right)^2;$$

(b) $$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \left(1-\frac{2}{n}\right)\left(2\sqrt{a_{n-2}}-3\sqrt{a_{n-1}}+\sqrt{a_n}\right)^2.$$

(Vasile C., 2010)

Solution. Let

$$x=\sqrt{a_{n-2}}-\sqrt{a_{n-1}} \ge 0, \quad y=\sqrt{a_{n-1}}-\sqrt{a_n} \ge 0.$$

For $k=n-2$ and $k=n-1$, the inequality in P 2.149 becomes respectively

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{4(n-2)x^2}{n}$$

and

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{2(n-1)y^2}{n}.$$

Therefore,

$$a_1+a_2+\cdots+a_n-n\sqrt[n]{a_1a_2\cdots a_n} \ge \frac{2}{n}\max\{2(n-2)x^2,\ (n-1)y^2\}.$$

(a) It suffices to show that

$$\max\{8(n-2)x^2,\ 4(n-1)y^2\} \ge (n-1)(x-2y)^2.$$

This is true since

$$8(n-2)x^2 \ge (n-1)x^2 \ge (n-1)(x-2y)^2$$

for $x-2y \ge 0$, and

$$4(n-1)y^2 \ge (n-1)(2y-x)^2$$

for $2y-x \ge 0$. The equality holds for $a_1=a_2=\cdots=a_n$.

(b) ***First Solution.*** It suffices to show that

$$\max\{4(n-2)x^2,\ 2(n-1)y^2\} \ge (n-2)(2x-y)^2.$$

This is true since

$$4(n-2)x^2 \ge (n-2)(2x-y)^2$$

for $2x-y \ge 0$, and

$$2(n-1)y^2 \ge (n-2)y^2 \ge (n-2)(y-2x)^2$$

for $y-2x \geq 0$. The equality holds for $a_1 = a_2 = \cdots = a_n$. If $n = 4$, then the equality holds for $a_1 = a_2$ and $a_3 = a_4$.

Second Solution. Let us denote

$$A = \sqrt[n-2]{a_1 a_2 \cdots a_{n-2}}\,, \quad B = \sqrt{a_{n-1} a_n}\,, \quad A \geq a_{n-2} \geq B.$$

By the AM-GM inequality, we have

$$a_1 + a_2 + \cdots + a_{n-2} \geq (n-2)A,$$

$$a_{n-1} + a_n \geq 2B,$$

and

$$\sqrt{a_{n-1}} + \sqrt{a_n} \geq 2\sqrt{B}.$$

Then, it suffices to show that

$$(n-2)A + 2B - n\sqrt[n]{A^{n-2}B^2} \geq \frac{4(n-2)}{n}\left(\sqrt{A} - \sqrt{B}\right)^2.$$

For the nontrivial case $B \neq 0$, we may consider $B = 1$ (due to homogeneity). In addition, substituting $A = t^{2n}$, $t \geq 1$, the inequality becomes $g(t) \geq 0$, where

$$g(t) = (n-2)t^{2n} + 2 - nt^{2n-4} - \frac{4(n-2)}{n}(t^n - 1)^2.$$

We have

$$g'(t) = 2(n-2)t^{n-1}h(t),$$

where

$$h(t) = (n-4)t^n - nt^{n-4} + 4.$$

Since

$$h'(t) = n(n-4)t^{n-5}(t^4 - 1) \geq 0,$$

$h(t)$ is increasing, $h(t) \geq h(1) = 0$, $g'(t) \geq 0$, $g(t)$ is increasing, hence $g(t) \geq g(1) = 0$. This completes the proof.

□

Appendix A

Glosar

1. AM-GM (ARITHMETIC MEAN-GEOMETRIC MEAN) INEQUALITY

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$a_1 + a_2 + \cdots + a_n \geq n\sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

2. WEIGHTED AM-GM INEQUALITY

Let $p_1, p_2, \ldots, p_n$ be positive real numbers satisfying

$$p_1 + p_2 + \cdots + p_n = 1.$$

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n \geq a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

3. AM-HM (ARITHMETIC MEAN-HARMONIC MEAN) INEQUALITY

If $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2,$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

4. POWER MEAN INEQUALITY

The power mean of order k of positive real numbers $a_1, a_2, \ldots, a_n$,

$$M_k = \begin{cases} \left(\frac{a_1^k + a_2^k + \cdots + a_n^k}{n}\right)^{\frac{1}{k}}, & k \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n}, & k = 0 \end{cases},$$

is an increasing function with respect to $k \in \mathbb{R}$. For instant, $M_2 \ge M_1 \ge M_0 \ge M_{-1}$ is equivalent to

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \ge \frac{a_1 + a_2 + \cdots + a_n}{n} \ge \sqrt[n]{a_1 a_2 \cdots a_n} \ge \frac{n}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}}.$$

5. BERNOULLI'S INEQUALITY

For any real number $x \ge -1$, we have

a) $(1+x)^r \ge 1 + rx$ for $r \ge 1$ and $r \le 0$;

b) $(1+x)^r \le 1 + rx$ for $0 \le r \le 1$.

If $a_1, a_2, \ldots, a_n$ are real numbers such that either $a_1, a_2, \ldots, a_n \ge 0$ or

$$-1 \le a_1, a_2, \ldots, a_n \le 0,$$

then

$$(1+a_1)(1+a_2)\cdots(1+a_n) \ge 1 + a_1 + a_2 + \cdots + a_n.$$

6. SCHUR'S INEQUALITY

For any nonnegative real numbers a, b, c and any positive number k, the inequality holds

$$a^k(a-b)(a-c) + b^k(b-c)(b-a) + c^k(c-a)(c-b) \ge 0,$$

with equality for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation). For $k = 1$, we get the third degree Schur's inequality, which can be rewritten as follows

$$a^3 + b^3 + c^3 + 3abc \ge ab(a+b) + bc(b+c) + ca(c+a),$$

$$(a+b+c)^3 + 9abc \ge 4(a+b+c)(ab+bc+ca),$$

$$a^2 + b^2 + c^2 + \frac{9abc}{a+b+c} \ge 2(ab+bc+ca),$$

$$(b-c)^2(b+c-a) + (c-a)^2(c+a-b) + (a-b)^2(a+b-c) \ge 0.$$

For $k = 2$, we get the fourth degree Schur's inequality, which holds for any real numbers a, b, c, and can be rewritten as follows

$$a^4 + b^4 + c^4 + abc(a + b + c) \ge ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2),$$

$$a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 \ge (ab + bc + ca)(a^2 + b^2 + c^2 - ab - bc - ca),$$

$$(b - c)^2(b + c - a)^2 + (c - a)^2(c + a - b)^2 + (a - b)^2(a + b - c)^2 \ge 0,$$

$$6abcp \ge (p^2 - q)(4q - p^2), \quad p = a + b + c, \; q = ab + bc + ca.$$

A generalization of the fourth degree Schur's inequality, which holds for any real numbers a, b, c and any real number m, is the following (*Vasile Cirtoaje*, 2004)

$$\sum (a - mb)(a - mc)(a - b)(a - c) \ge 0,$$

where the equality holds for $a = b = c$, and for $a/m = b = c$ (or any cyclic permutation). This inequality is equivalent to

$$\sum a^4 + m(m + 2)\sum a^2b^2 + (1 - m^2)abc\sum a \ge (m + 1)\sum ab(a^2 + b^2),$$

$$\sum (b - c)^2(b + c - a - ma)^2 \ge 0.$$

A more general result is given by the following theorem (*Vasile Cirtoaje*, 2004).

Theorem. *Let*

$$f_4(a, b, c) = \sum a^4 + \alpha\sum a^2b^2 + \beta abc\sum a - \gamma\sum ab(a^2 + b^2),$$

where α, β, γ *are real constants such that* $1 + \alpha + \beta = 2\gamma$. *Then,*

(a) $f_4(a, b, c) \ge 0$ *for all* $a, b, c \in \mathbb{R}$ *if and only if*

$$1 + \alpha \ge \gamma^2;$$

(b) $f_4(a, b, c) \ge 0$ *for all* $a, b, c \ge 0$ *if and only if*

$$\alpha \ge (\gamma - 1)\max\{2, \gamma + 1\}.$$

7. CAUCHY-SCHWARZ INEQUALITY

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \ge (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2,$$

with equality for

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$

Notice that the equality conditions are also valid for $a_i = b_i = 0$, where $1 \le i \le n$.

8. HÖLDER'S INEQUALITY

If x_{ij} $(i = 1, 2, \cdots, m; j = 1, 2, \cdots n)$ are nonnegative real numbers, then

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n} x_{ij}\right) \geq \left(\sum_{j=1}^{n} \sqrt[m]{\prod_{i=1}^{m} x_{ij}}\right)^m .$$

9. CHEBYSHEV'S INEQUALITY

Let $a_1 \geq a_2 \geq \cdots \geq a_n$ be real numbers.

a) If $b_1 \geq b_2 \geq \cdots b_n$, then

$$n\sum_{i=1}^{n} a_i b_i \geq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right);$$

b) If $b_1 \leq b_2 \leq \cdots \leq b_n$, then

$$n\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right).$$

10. REARRANGEMENT INEQUALITY

(1) If $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ are two increasing (or decreasing) real sequences, and $(i_1, i_2, \cdots, i_n)$ is an arbitrary permutation of $(1, 2, \cdots, n)$, then

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq a_1 b_{i_1} + a_2 b_{i_2} + \cdots + a_n b_{i_n}$$

and

$$n(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) \geq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).$$

(2) If $(a_1, a_2, \dots, a_n)$ is decreasing and $(b_1, b_2, \dots, b_n)$ is increasing, then

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \leq a_1 b_{i_1} + a_2 b_{i_2} + \cdots + a_n b_{i_n}$$

and

$$n(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) \leq (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).$$

(3) Let $b_1, b_2, \dots, b_n)$ and $(c_1, c_2, \dots, c_n)$ be two real sequences such that

$$b_1 + \cdots + b_i \geq c_1 + \cdots + c_i, \quad i = 1, 2, \cdots, n.$$

If $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, then

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \geq a_1 c_1 + a_2 c_2 + \cdots + a_n c_n.$$

Notice that all these inequalities follow immediately from the identity

$$\sum_{i=1}^{n} a_i(b_i - c_i) = \sum_{i=1}^{n}(a_i - a_{i+1})\left(\sum_{j=1}^{i} b_j - \sum_{j=1}^{i} c_j\right), \qquad a_{n+1} = 0.$$

11. CONVEX FUNCTIONS

A function f defined on a real interval $\mathbb{I}$ is said to be *convex* if

$$f(\alpha x + \beta y) \le \alpha f(x) + \beta f(y)$$

for all $x, y \in \mathbb{I}$ and any $\alpha, \beta \ge 0$ with $\alpha + \beta = 1$. If the inequality is reversed, then f is said to be concave.

If f is differentiable on $\mathbb{I}$, then f is (strictly) convex if and only if the derivative f' is (strictly) increasing. If $f'' \ge 0$ on $\mathbb{I}$, then f is convex on $\mathbb{I}$. Also, if $f'' \ge 0$ on (a, b) and f is continuous on $[a, b]$, then f is convex on $[a, b]$.

Jensen's inequality. *Let $p_1, p_2, \dots, p_n$ be positive real numbers. If f is a convex function on a real interval $\mathbb{I}$, then for any $a_1, a_2, \dots, a_n \in \mathbb{I}$, the inequality holds*

$$\frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} \ge f\left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right).$$

For $p_1 = p_2 = \cdots = p_n$, Jensen's inequality becomes

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge n f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right).$$

12. SQUARE PRODUCT INEQUALITY

Let a, b, c be real numbers, and let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc,$$
$$s = \sqrt{p^2 - 3q} = \sqrt{a^2 + b^2 + c^2 - ab - bc - ca}.$$

From the identity

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3,$$

it follows that

$$\frac{-2p^3 + 9pq - 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27} \le r \le \frac{-2p^3 + 9pq + 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27},$$

which is equivalent to

$$\frac{p^3 - 3ps^2 - 2s^3}{27} \le r \le \frac{p^3 - 3ps^2 + 2s^3}{27}.$$

Therefore, for constant p and q, the product r is minimal and maximal when two of a, b, c are equal.

13. KARAMATA'S MAJORIZATION INEQUALITY

Let f be a convex function on a real interval $\mathbb{I}$. If a decreasingly ordered sequence

$$A = (a_1, a_2, \ldots, a_n), \quad a_i \in \mathbb{I},$$

majorizes a decreasingly ordered sequence

$$B = (b_1, b_2, \ldots, b_n), \quad b_i \in \mathbb{I},$$

then

$$f(a_1) + f(a_2) + \cdots + f(a_n) \geq f(b_1) + f(b_2) + \cdots + f(b_n).$$

We say that a sequence $A = (a_1, a_2, \ldots, a_n)$ with $a_1 \geq a_2 \geq \cdots \geq a_n$ majorizes a sequence $B = (b_1,\ b_2,\ \ldots,\ b_n)$ with $b_1 \geq b_2 \geq \cdots \geq b_n$, and write it as

$$A \succ B,$$

if

$$\begin{gathered} a_1 \geq b_1, \\ a_1 + a_2 \geq b_1 + b_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_1 + a_2 + \cdots + a_{n-1} \geq b_1 + b_2 + \cdots + b_{n-1}, \\ a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n. \end{gathered}$$

14. VASC'S EXPONENTIAL INEQUALITY

Let $0 < k \leq e$.

(a) *If* $a, b > 0$*, then (Vasile Cîrtoaje, 2006)*

$$a^{ka} + b^{kb} \geq a^{kb} + b^{ka};$$

(b) *If* $a, b \in (0, 1]$*, then (Vasile Cîrtoaje, 2010)*

$$2\sqrt{a^{ka} b^{kb}} \geq a^{kb} + b^{ka}.$$

Appendix B

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