This is Volume 1 of the five-volume book Mathematical Inequalities, that introduces and develops the main types of elementary inequalities. The first three volumes are a great opportunity to look into many old and new inequalities, as well as elementary procedures for solving them: Volume 1 - Symmetric Polynomial Inequalities, Volume 2 - Symmetric Rational and Nonrational Inequalities, Volume 3 - Cyclic and Noncyclic Inequalities. As a rule, the inequalities in these volumes are increasingly ordered according to the number of variables: two, three, four, ..., n-variables. The last two volumes (Volume 4 – Extensions and Refinements of Jensen’s Inequality, Volume 5 – Other Recent Methods for Creating and Solving Inequalities) present beautiful and original methods for solving inequalities, such as Half/Partial convex function method, Equal variables method, Arithmetic compensation method, Highest coefficient cancellation method, pqr method etc. The book is intended for a wide audience: advanced middle school students, high school students, college and university students, and teachers. Many problems and methods can be used as group projects for advanced high school students.

The author, Vasile Cirtoaje, is a Professor at the Department of Automatic Control and Computers from University of Ploiesti, Romania. He is the author of many well-known interesting and delightful inequalities, as well as strong methods for creating and proving mathematical inequalities.

978-613-9-88236-6

Symmetric Polynomial Inequalities

Cirtoaje

Vasile Cirtoaje

Mathematical Inequalities Volume 1

Symmetric Polynomial Inequalities

LAP LAMBERT Academic Publishing

Vasile Cîrtoaje

MATHEMATICAL INEQUALITIES

Volume 1

SYMMETRIC POLYNOMIAL INEQUALITIES

LAP LAMBERT Academic Publishing

EDITURA UNIVERSITĂȚII PETROL-GAZE DIN PLOIEȘTI
2021

About the author

The simpler and sharper, the more beautiful.
Vasile Cîrtoaje

Vasile Cîrtoaje is a Professor at the Department of Automatic Control and Computers from Petroleum-Gas University of Ploiesti, Romania, where he teaches university courses such as Control System Theory and Digital Control Systems.

Since 1970, he published many mathematical problems, solutions and articles in the Romanian journals Gazeta Matematica-B, Gazeta Matematica-A and Mathematical Review of Timisoara. In addition, from 2000 to present, Vasile Cîrtoaje has published many interesting problems and articles in *Art of Problem Solving* website, Mathematical Reflections, Crux with Mayhem, Journal of Inequalities and Applications, Journal of Inequalities in Pure and Applied Mathematics, Mathematical Inequalities and Applications, Banach Journal of Mathematical Analysis, Journal of Nonlinear Science and Applications, Journal of Nonlinear Analysis and Application, Australian Journal of Mathematical Analysis and Applications, British Journal of Mathematical and Computer Science,International Journal of Pure and Applied Mathematics, Journal of Inequalities and Special Functions, A.M.M.

He collaborated with Titu Andreescu, Gabriel Dospinescu and Mircea Lascu in writing the book *Old and New Inequalities*, with Vo Quoc Ba Can and Tran Quoc Anh in writing the book *Inequalities with Beautiful Solution*, and he wrote his own books *Algebraic Inequalities - Old and New Methods* and *Mathematical Inequalities* (Volume 1 ... 5).

Notice that Vasile Cîrtoaje is the author of some well-known results and strong methods for proving and creating discrete inequalities, such as:

- Half convex function method (HCF method) for Jensen type discrete inequalities;
- Partial convex function method (PCF method) for Jensen type discrete inequalities;
- Jensen type discrete inequalities with ordered variables;
- Equal variables method (EV method) for real or nonnegative variables;
- Arithmetic compensation method (AC method);
- Best lower and upper bounds for Jensen's inequality;
- Necessary and sufficient conditions for symmetric homogeneous polynomial inequalities of degree six in real variables;

- Necessary and sufficient conditions for symmetric homogeneous polynomial inequalities of degree six in nonnegative variables;
- Highest coefficient cancellation method (HCC method) for symmetric homogeneous polynomial inequalities of degree six and eight in real variables;
- Highest coefficient cancellation method for symmetric homogeneous polynomial inequalities of degree six, seven and eight in nonnegative variables;
- Necessary and sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in real variables;
- Necessary and sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in nonnegative variables;
- Strong sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in real or nonnegative variables;
- Inequalities with power-exponential functions.

Foreword

The author, Vasile Cîrtoaje, professor at University of Ploiesti-Romania, has become well-known for his excellent creations in the mathematical inequality field, ever since the time when he was student in high school (in Breaza city, Prahova Valley). As a student (quite some time ago, oh yes!), I was already familiar with the name of Vasile Cîrtoaje. For me, and many others of my age, it is the name of someone who helped me to grow in mathematics, even though I never met him face to face. It is a name synonymous to hard and beautiful problems involving inequalities. When you say Vasile Cîrtoaje (*Vasc* username on the site Art of Problem Solving), you say inequalities. I remember how happy I was when I could manage to solve one of the problems proposed by professor Cîrtoaje in Gazeta Matematica or Revista Matematica Timisoara.

The first three volumes of this book are a great opportunity to see and know many old and new elementary methods for solving mathematical inequalities: Volume 1 - *Symmetric polynomial inequalities* (in real variables and nonnegative real variables), Volume 2 - *Symmetric rational and nonrational inequalities*, Volume 3 - *Cyclic and noncyclic inequalities*. As a rule, the inequalities from each section of these volumes are increasingly ordered by the number of variables: two, three, four, five, six and n-variables.

The last two volumes (Volume 4 - *Extensions and refinements of Jensen's inequality*, Volume 5 - *Other recent methods for creating and solving inequalities*) contain beautiful and efficient original methods for creating and solving inequalities, such as *half or partially convex function method* - for Jensen's type inequalities, *Popoviciu's method* for convex function, *equal variables method* and *arithmetic compensation method* - for symmetric inequalities, *the highest coefficient cancellation method* - for symmetric homogeneous polynomial inequalities of degree six, seven and eight, *pqr method* - for cyclic homogeneous polynomial inequalities of degree four, et al.

Many problems, the majority I would say, are made up by the author himself. The chapters and volumes are relatively independent, and you can open the book somewhere to solve an inequality or only read its solution. If you carefully make a thorough study of the book, then you will find that your skills in solving inequalities are considerably improved.

The book contains more than 1000 beautiful inequalities, hints, solutions and methods, some of them being posted in the last ten years by the author and other inventive mathematicians on *Art of Problem Solving* website (Vo Quoc Ba Can, Pham Kim Hung, Michael Rozenberg, Nguyen Van Quy, Gabriel Dospinescu, Darij Grinberg, Pham Huu Duc, Tran Quoc Anh, Le Huu Dien Khue, Nguyen Anh Tuan, Pham Van Thuan, Bin Zhao, Ji Chen etc.)

Most inequalities and methods are old and recent own creations of the author. Among these, I would like to point out the following inequalities:

$$(a^2+b^2+c^2)^2 \ge 3(a^3b+b^3c+c^3a), \quad a,b,c \in \mathbb{R};$$

$$\sum(a-kb)(a-kc)(a-b)(a-c) \ge 0^{1}, \quad a,b,c,k \in \mathbb{R};$$

$$\left(\frac{a}{a+b}\right)^2+\left(\frac{b}{b+c}\right)^2+\left(\frac{c}{c+d}\right)^2+\left(\frac{d}{d+a}\right)^2 \ge 1, \quad a,b,c,d \ge 0;$$

$$\sum_{i=1}^{4}\frac{1}{1+a_i+a_i^2+a_i^3} \ge 1, \quad a_1,a_2,a_3,a_4>0, \ \ a_1a_2a_3a_4=1;$$

$$\frac{a_1}{a_1+(n-1)a_2}+\frac{a_2}{a_2+(n-1)a_3}+\cdots+\frac{a_n}{a_n+(n-1)a_1} \ge 1, \quad a_1,a_2,\cdots,a_n \ge 0;$$

$$a^{ea}+b^{eb} \ge a^{eb}+b^{ea}, \qquad a,b>0, \quad e \approx 2.7182818.$$

$$a^{3b}+b^{3a} \le 2, \qquad a,b \ge 0, \quad a+b=2.$$

The book represents a rich source of beautiful, serious and profound mathematics, dealing with classical and new approaches and techniques which help the reader to develop his inequality-solving skills, intuition and creativity. As a result, it is suitable for a wide audience: advanced middle school students, high school students, college and university students, and teachers. Each problem has a hint, and many problems have multiple solutions, almost all of which are, not surprisingly, quite ingenious. Almost all inequalities require careful thought and analysis, making the book a rewarding source for anyone interested in Olympiad-type problems and in the development of the inequality field. Many problems and methods can be used as group projects for advanced high school students.

What makes this book so attractive? The answer is simple: the great number of inequalities, their quality and freshness, as well as the new approaches and methods for solving mathematical inequalities. Nevertheless, you will find this book to be delightful, inspired, original and enjoyable. Any interested reader will remark the tenacity, enthusiasm and ability of the author in creating and solving nice and difficult inequalities. This book is neither more, nor less than a Work of a Master. I highly recommend it.

Marian Tetiva
National College "Gheorghe Rosca Codreanu" Bârlad, Romania

[1]Throughout the book, the math symbol $\sum$ means $\sum\limits_{cyc}$;

$$\sum_{cyc} f(a_1,a_2,\dots,a_n)=f(a_1,a_2,\dots,a_n)+f(a_2,a_3,\dots,a_1)+\cdots+f(a_n,a_1,\dots,a_{n-1}).$$

Contents

About the author **i**

Foreword **iii**

1 Some Classic and New Inequalities and Methods **1**

2 Symmetric Polynomial Inequalities in Real Variables **21**
2.1 Application 21
2.2 Solutions 45

3 Symmetric Polynomial Inequalities in Nonnegative Variables **197**
3.1 Applications 197
3.2 Solutions 231

A Bibliography **479**

Chapter 1

Some Classic and New Inequalities and Methods

1. AM-GM (ARITHMETIC MEAN-GEOMETRIC MEAN) INEQUALITY

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$a_1 + a_2 + \cdots + a_n \geq n\sqrt[n]{a_1 a_2 \cdots a_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

2. WEIGHTED AM-GM INEQUALITY

Let $p_1, p_2, \ldots, p_n$ be positive real numbers satisfying

$$p_1 + p_2 + \cdots + p_n = 1.$$

If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers, then

$$p_1 a_1 + p_2 a_2 + \cdots + p_n a_n \geq a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n},$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

3. AM-HM (ARITHMETIC MEAN-HARMONIC MEAN) INEQUALITY

If $a_1, a_2, \ldots, a_n$ are positive real numbers, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2,$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$.

4. POWER MEAN INEQUALITY

The power mean of order k of positive real numbers $a_1, a_2, \ldots, a_n$, that is

$$M_k = \begin{cases} \left(\frac{a_1^k + a_2^k + \cdots + a_n^k}{n}\right)^{\frac{1}{k}}, & k \neq 0 \\ \sqrt[n]{a_1 a_2 \cdots a_n}, & k = 0 \end{cases},$$

is an increasing function with respect to $k \in \mathbb{R}$. For instant, $M_2 \geq M_1 \geq M_0 \geq M_{-1}$ is equivalent to

$$\sqrt{\frac{a_1^2 + a_2^2 + \cdots + a_n^2}{n}} \geq \frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}.$$

5. BERNOULLI'S INEQUALITY

For any real number $x \geq -1$, we have

a) $(1+x)^r \geq 1 + rx$ for $r \geq 1$ and $r \leq 0$;

b) $(1+x)^r \leq 1 + rx$ for $0 \leq r \leq 1$.

If $a_1, a_2, \ldots, a_n$ are real numbers such that either $a_1, a_2, \ldots, a_n \geq 0$ or

$$-1 \leq a_1, a_2, \ldots, a_n \leq 0,$$

then

$$(1+a_1)(1+a_2)\cdots(1+a_n) \geq 1 + a_1 + a_2 + \cdots + a_n.$$

6. SCHUR'S INEQUALITY

For any nonnegative real numbers a, b, c and any positive number k, the inequality holds

$$a^k(a-b)(a-c) + b^k(b-c)(b-a) + c^k(c-a)(c-b) \geq 0,$$

with equality for $a = b = c$, and for $a = 0$ and $b = c$ (or any cyclic permutation). For $k = 1$, we get the third degree Schur's inequality, which can be rewritten as follows

$$a^3 + b^3 + c^3 + 3abc \geq ab(a+b) + bc(b+c) + ca(c+a),$$

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

$$a^2 + b^2 + c^2 + \frac{9abc}{a+b+c} \geq 2(ab+bc+ca),$$

$$(b-c)^2(b+c-a) + (c-a)^2(c+a-b) + (a-b)^2(a+b-c) \geq 0.$$

For $k = 2$, we get the fourth degree Schur's inequality, which holds for any real numbers a, b, c, and can be rewritten as follows

$$a^4 + b^4 + c^4 + abc(a+b+c) \geq ab(a^2+b^2) + bc(b^2+c^2) + ca(c^2+a^2),$$

$$a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2 \geq (ab+bc+ca)(a^2+b^2+c^2-ab-bc-ca),$$

$$(b-c)^2(b+c-a)^2 + (c-a)^2(c+a-b)^2 + (a-b)^2(a+b-c)^2 \geq 0,$$

$$6abcp \geq (p^2-q)(4q-p^2), \quad p = a+b+c, \ \ q = ab+bc+ca.$$

A generalization of the fourth degree Schur's inequality for any real numbers a, b, c and any real number m, is the following (see [11]):

$$\sum (a-mb)(a-mc)(a-b)(a-c) \geq 0,$$

where the equality holds for $a = b = c$, and for $a/m = b = c$ (or any cyclic permutation). This inequality is equivalent to

$$\sum a^4 + m(m+2)\sum a^2b^2 + (1-m^2)abc\sum a \geq (m+1)\sum ab(a^2+b^2),$$

$$\sum (b-c)^2(b+c-a-ma)^2 \geq 0.$$

A more general result is given by the following theorem (see [17]).

Theorem. *Let*

$$f_4(a,b,c) = \sum a^4 + \alpha \sum a^2b^2 + \beta abc \sum a - \gamma \sum ab(a^2+b^2),$$

where α, β, γ are real constants such that $1 + \alpha + \beta = 2\gamma$. Then,

(a) $f_4(a,b,c) \geq 0$ *for all $a, b, c \in \mathbb{R}$ if and only if*

$$1 + \alpha \geq \gamma^2;$$

(b) $f_4(a,b,c) \geq 0$ *for all $a, b, c \geq 0$ if and only if*

$$\alpha \geq (\gamma - 1)\max\{2, \gamma+1\}.$$

7. CAUCHY-SCHWARZ INEQUALITY

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$(a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2) \geq (a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2,$$

with equality for

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \cdots = \frac{a_n}{b_n}.$$

Notice that the equality conditions are also valid for $a_i = b_i = 0$, where $1 \leq i \leq n$.

8. HÖLDER'S INEQUALITY

If x_{ij} $(i=1,2,\cdots,m; j=1,2,\cdots n)$ are nonnegative real numbers, then

$$\prod_{i=1}^{m}\left(\sum_{j=1}^{n}x_{ij}\right) \ge \left(\sum_{j=1}^{n}\sqrt[m]{\prod_{i=1}^{m}x_{ij}}\right)^{m}.$$

9. CHEBYSHEV'S INEQUALITY

Let $a_1 \ge a_2 \ge \cdots \ge a_n$ be real numbers.

a) If $b_1 \ge b_2 \ge \cdots b_n$, then

$$n\sum_{i=1}^{n}a_ib_i \ge \left(\sum_{i=1}^{n}a_i\right)\left(\sum_{i=1}^{n}b_i\right);$$

b) If $b_1 \le b_2 \le \cdots \le b_n$, then

$$n\sum_{i=1}^{n}a_ib_i \le \left(\sum_{i=1}^{n}a_i\right)\left(\sum_{i=1}^{n}b_i\right).$$

10. MINKOWSKI'S INEQUALITY

For any real number $k \ge 1$ and any positive real numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, the following inequalities hold:

$$\sum_{i=1}^{n}\left(a_i^k + b_i^k\right)^{\frac{1}{k}} \ge \left[\left(\sum_{i=1}^{n}a_i\right)^k + \left(\sum_{i=1}^{n}b_i\right)^k\right]^{\frac{1}{k}};$$

$$\sum_{i=1}^{n}\left(a_i^k + b_i^k + c_i^k\right)^{\frac{1}{k}} \ge \left[\left(\sum_{i=1}^{n}a_i\right)^k + \left(\sum_{i=1}^{n}b_i\right)^k + \left(\sum_{i=1}^{n}c_i\right)^k\right]^{\frac{1}{k}}.$$

11. REARRANGEMENT INEQUALITY

(1) If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are two increasing (or decreasing) real sequences, and $(i_1, i_2, \cdots, i_n)$ is an arbitrary permutation of $(1, 2, \cdots, n)$, then

$$a_1b_1 + a_2b_2 + \cdots + a_nb_n \ge a_1b_{i_1} + a_2b_{i_2} + \cdots + a_nb_{i_n}$$

and

$$n(a_1b_1 + a_2b_2 + \cdots + a_nb_n) \ge (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).$$

(2) If $a_1, a_2, \ldots, a_n$ is decreasing and $b_1, b_2, \ldots, b_n$ is increasing, then

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \le a_1 b_{i_1} + a_2 b_{i_2} + \cdots + a_n b_{i_n}$$

and

$$n(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n) \le (a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n).$$

(3) Let $b_1, b_2, \ldots, b_n$ and $c_1, c_2, \ldots, c_n$ be two real sequences such that

$$b_1 + \cdots + b_k \ge c_1 + \cdots + c_k, \quad k = 1, 2, \cdots, n.$$

If $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$, then

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \ge a_1 c_1 + a_2 c_2 + \cdots + a_n c_n.$$

Notice that all these inequalities follow immediately from the identity

$$\sum_{i=1}^{n} a_i(b_i - c_i) = \sum_{i=1}^{n} (a_i - a_{i+1}) \left(\sum_{j=1}^{i} b_j - \sum_{j=1}^{i} c_j \right),$$

where $a_{n+1} = 0$.

12. MACLAURIN'S INEQUALITY and NEWTON'S INEQUALITY

If a_1, a_2, ..., a_n are nonnegative real numbers, then

$$S_1 \ge S_2 \ge \cdots \ge S_n \qquad (Maclaurin)$$

and

$$S_k^2 \ge S_{k-1} S_{k+1}, \qquad (Newton)$$

where

$$S_k = \sqrt[k]{\frac{\displaystyle\sum_{1 \le i_1 < \cdots < i_k \le n} a_{i_1} a_{i_2} \cdots a_{i_k}}{\dbinom{n}{k}}}.$$

13. CONVEX FUNCTIONS

A function f defined on a real interval $\mathbb{I}$ is said to be *convex* if

$$f(\alpha x + \beta y) \le \alpha f(x) + \beta f(y)$$

for all x, $y \in \mathbb{I}$ and any α, $\beta \ge 0$ with $\alpha + \beta = 1$. If the inequality is reversed, then f is said to be concave.

If f is differentiable on $\mathbb{I}$, then f is (strictly) convex if and only if the derivative f' is (strictly) increasing. If $f'' \ge 0$ on $\mathbb{I}$, then f is convex on $\mathbb{I}$.
Let $\mathbb{I}$ be a real interval, s an interior point of $\mathbb{I}$ and

$$\mathbb{I}_{\ge s} = \{u|u \in \mathbb{I}, u \ge s\}, \qquad \mathbb{I}_{\le s} = \{u|u \in \mathbb{I}, u \le s\}.$$

A function $f : \mathbb{I} \to \mathbb{R}$ is *half convex* if there exists an interior point $s \in \mathbb{I}$ such that f is convex on $\mathbb{I}_{\le s}$ or $\mathbb{I}_{\ge s}$.
A function $f : \mathbb{I} \to \mathbb{R}$ is *right partially convex* related to an interior point $s \in \mathbb{I}$ if there exists an interior point $s_0 \in \mathbb{I}$, $s_0 > s$, such that f is convex on $[s, s_0]$. Also, a function $f : \mathbb{I} \to \mathbb{R}$ is *left partially convex* related to an interior point $s \in \mathbb{I}$ if there exists an interior point $s_0 \in \mathbb{I}$, $s_0 < s$, such that f is convex on $[s_0, s]$.

Jensen's inequality. *Let $p_1, p_2, \dots, p_n$ be positive real numbers. If f is a convex function on a real interval $\mathbb{I}$, then for any $a_1, a_2, \dots, a_n \in \mathbb{I}$, the inequality holds*

$$\frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} \ge f\left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right).$$

For $p_1 = p_2 = \cdots = p_n$, Jensen's inequality becomes

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge n f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right).$$

We can extend Jensen's inequality for convex functions to half or partially convex functions (see [8], [13], [19], [29], [32], [33]).

Half Convex Function-Theorem. *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\ge s}$ or $\mathbb{I}_{\le s}$, where $s \in int(\mathbb{I})$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge n f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying $a_1 + a_2 + \cdots + a_n = ns$ if and only if

$$f(x) + (n-1) f(y) \ge n f(s)$$

for all $x, y \in \mathbb{I}$ such that $x + (n-1)y = ns$.

Right Half Convex Function Theorem for Ordered Variables. *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\ge s}$, where $s \in int(\mathbb{I})$. The inequality*

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge n f\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

holds for all $a_1, a_2, \dots, a_n \in \mathbb{I}$ satisfying $a_1 + a_2 + \cdots + a_n = ns$ and

$$a_1 \le a_2 \le \cdots \le a_m \le s, \quad m \in \{1, 2, \dots, n-1\},$$

if and only if

$$f(x)+(n-m)f(y) \geq (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ such that

$$x \leq s \leq y, \quad x+(n-m)y = (1+n-m)s.$$

Left Half Convex Function Theorem for Ordered Variables. *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $\mathbb{I}_{\leq s}$, where $s \in int(\mathbb{I})$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1, a_2, \ldots, a_n \in \mathbb{I}$ satisfying $a_1+a_2+\cdots+a_n = ns$ and

$$a_1 \geq a_2 \geq \cdots \geq a_m \geq s, \quad m \in \{1, 2, \ldots, n-1\},$$

if and only if

$$f(x)+(n-m)f(y) \geq (1+n-m)f(s)$$

for all $x, y \in \mathbb{I}$ such tht

$$x \geq s \geq y, \quad x+(n-m)y = (1+n-m)s.$$

Right Partially Convex Function-Theorem. *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s, s_0]$, where $s, s_0 \in int(\mathbb{I})$, $s < s_0$. In addition, f is decreasing on $\mathbb{I}_{\leq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}_{\geq s_0}$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1, a_2, \cdots, a_n \in \mathbb{I}$ satisfying $a_1+a_2+\cdots+a_n = ns$ if and only if

$$f(x)+(n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ such that $x \leq s \leq y$ and $x+(n-1)y = ns$.

Left Partially Convex Function-Theorem. *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0, s]$, where $s_0, s \in \mathbb{I}$, $s_0 < s$. In addition, f is increasing on $\mathbb{I}_{\geq s_0}$ and $f(u) \geq f(s_0)$ for $u \in \mathbb{I}_{\leq s_0}$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \geq nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1, a_2, \cdots, a_n \in \mathbb{I}$ satisfying $a_1+a_2+\cdots+a_n = ns$ if and only if

$$f(x)+(n-1)f(y) \geq nf(s)$$

for all $x, y \in \mathbb{I}$ such that $x \geq s \geq y$ and $x+(n-1)y = ns$.

Right Partially Convex Function Theorem for Ordered Variables (*Vasile Cirtoaje*, 2014). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s,s_0]$, where $s,s_0 \in \mathbb{I}$, $s < s_0$. In addition, f is decreasing on $\mathbb{I}_{\le s_0}$ and $f(u) \ge f(s_0)$ for $u \in \mathbb{I}_{\ge s_0}$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1,a_2,\ldots,a_n \in \mathbb{I}$ satisfying $a_1+a_2+\cdots+a_n = ns$ and

$$a_1 \le a_2 \le \cdots \le a_m \le s, \quad m \in \{1,2,\ldots,n-1\},$$

if and only if

$$f(x)+(n-m)f(y) \ge (1+n-m)f(s)$$

for all $x,y \in \mathbb{I}$ such that $x \le s \le y$ and $x+(n-m)y = (1+n-m)s$.

Left Partially Convex Function Theorem for Ordered Variables (*Vasile Cirtoaje*, 2014). *Let f be a real function defined on an interval $\mathbb{I}$ and convex on $[s_0,s]$, where $s_0,s \in \mathbb{I}$, $s_0 < s$. In addition, f is increasing on $\mathbb{I}_{\ge s_0}$ and $f(u) \ge f(s_0)$ for $u \in \mathbb{I}_{\le s_0}$. The inequality*

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

holds for all $a_1,a_2,\cdots,a_n \in \mathbb{I}$ satisfying $a_1+a_2+\cdots+a_n = ns$ and

$$a_1 \ge a_2 \ge \cdots \ge a_m \ge s, \quad m \in \{1,2,\ldots,n-1\},$$

if and only if

$$f(x)+(n-m)f(y) \ge (1+n-m)f(s)$$

for all $x,y \in \mathbb{I}$ such that $x \ge s \ge y$ and $x+(n-m)y = (1+n-m)s$.

In all these theorems, we may replace the hypothesis condition

$$f(x)+(n-m)f(y) \ge (1+n-m)f(s),$$

by the equivalent condition

$$h(x,y) \ge 0 \text{ for all } x,y \in \mathbb{I} \text{ such that } x+(n-m)y = (1+n-m),$$

where

$$h(x,y) = \frac{g(x)-g(y)}{x-y}, \qquad g(u) = \frac{f(u)-f(s)}{u-s}.$$

The following theorem is also useful to prove some symmetric inequalities.

Left Convex-Right Concave Function Theorem (see [11]). *Let $a < c$ be real numbers, let f be a continuous function on $\mathbb{I} = [a,\infty)$, strictly convex on $[a,c]$ and strictly concave on $[c,\infty)$, and let*

$$E(a_1,a_2,\ldots,a_n) = f(a_1)+f(a_2)+\cdots+f(a_n).$$

If $a_1, a_2, \ldots, a_n \in \mathbb{I}$ *such that*

$$a_1 + a_2 + \cdots + a_n = S = constant,$$

then

(a) *E is minimal for* $a_1 = a_2 = \cdots = a_{n-1} \le a_n$;
(b) *E is maximal for either* $a_1 = a$ or $a < a_1 \le a_2 = \cdots = a_n$.

On the other hand, it is known the following result concerning the best upper bound of Jensen's difference.

Best Upper Bound of Jensen's Difference-Theorem (see [6], [21]). *Let* $p_1, p_2, \ldots, p_n$ *be fixed positive real numbers, and let* f *be a convex function on a closed interval* $\mathbb{I} = [a, b]$. *If* $a_1, a_2, \ldots, a_n \in \mathbb{I}$, *then Jensen's difference*

$$D = \frac{p_1 f(a_1) + p_2 f(a_2) + \cdots + p_n f(a_n)}{p_1 + p_2 + \cdots + p_n} - f\left(\frac{p_1 a_1 + p_2 a_2 + \cdots + p_n a_n}{p_1 + p_2 + \cdots + p_n}\right)$$

is maximal when some of a_i *are equal to* a *and the others* a_i *are equal to* b; *that is, when all* $a_i \in \{a, b\}$.

14. KARAMATA'S MAJORIZATION INEQUALITY

Karamata's inequality is also called the H-L-P inequality (Hardy-Littlewood-Polya inequality).

Let f be a convex function on a real interval $\mathbb{I}$. If a decreasingly ordered sequence

$$A = (a_1, a_2, \ldots, a_n), \quad a_i \in \mathbb{I},$$

majorizes a decreasingly ordered sequence

$$B = (b_1, b_2, \ldots, b_n), \quad b_i \in \mathbb{I},$$

then

$$f(a_1) + f(a_2) + \cdots + f(a_n) \ge f(b_1) + f(b_2) + \cdots + f(b_n).$$

We say that a sequence $A = (a_1, a_2, \ldots, a_n)$ with $a_1 \ge a_2 \ge \cdots \ge a_n$ majorizes a sequence $B = (b_1, b_2, \ldots, b_n)$ with $b_1 \ge b_2 \ge \cdots \ge b_n$, and write it as

$$A \succ B,$$

if

$$\begin{gathered} a_1 \ge b_1, \\ a_1 + a_2 \ge b_1 + b_2, \\ \cdots\cdots\cdots\cdots \\ a_1 + a_2 + \cdots + a_{n-1} \ge b_1 + b_2 + \cdots + b_{n-1}, \\ a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n. \end{gathered}$$

15. POPOVICIU'S INEQUALITY

Theorem (see [7], [11]). *If f is a convex function on a real interval $\mathbb{I}$ and $a_1, a_2, \ldots, a_n \in \mathbb{I}$, then*

$$f(a_1)+f(a_2)+\cdots+f(a_n)+n(n-2)f\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)\geq$$

$$\geq (n-1)[f(b_1)+f(b_2)+\cdots+f(b_n)],$$

where

$$b_i=\frac{1}{n-1}\sum_{j\neq i}a_j, \quad i=1,2,\cdots,n.$$

In the same conditions, the following similar inequality holds:

$$f(a_1)+f(a_2)+\cdots+f(a_n)+\frac{n}{n-2}f\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)\geq$$

$$\geq \frac{2}{n-2}\sum_{1\leq i<j\leq n}f\left(\frac{a_i+a_j}{2}\right).$$

16. SQUARE PRODUCT INEQUALITY

Let a,b,c be real numbers, and let

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc,$$

$$s=\sqrt{p^2-3q}=\sqrt{a^2+b^2+c^2-ab-bc-ca}.$$

From the identity

$$(a-b)^2(b-c)^2(c-a)^2=-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3$$
$$=\frac{4(p^2-3q)^3-(2p^3-9pq+27r)^2}{27},$$

it follows that

$$\frac{-2p^3+9pq-2(p^2-3q)\sqrt{p^2-3q}}{27}\leq r\leq\frac{-2p^3+9pq+2(p^2-3q)\sqrt{p^2-3q}}{27},$$

which is equivalent to

$$\frac{p^3-3ps^2-2s^3}{27}\leq r\leq\frac{p^3-3ps^2+2s^3}{27}.$$

Therefore, for constant p and q, the product r is minimal and maximal when two of a,b,c are equal.

17. SYMMETRIC INEQUALITIES OF DEGREE THREE, FOUR OR FIVE

Theorem (see [26], [27]). *Let $f_n(a,b,c)$ be a symmetric homogeneous polynomial of degree n.*

(a) *The inequality $f_4(a,b,c) \ge 0$ holds for all real numbers a,b,c if and only if $f_4(a,1,1) \ge 0$ for all real a;*

(b) *For $n \in \{3,4,5\}$, the inequality $f_n(a,b,c) \ge 0$ holds for all $a,b,c \ge 0$ if and only if $f_n(a,1,1) \ge 0$ and $f_n(0,b,c) \ge 0$ for all $a,b,c \ge 0$.*

18. SYMMETRIC HOMOGENEOUS INEQUALITIES OF DEGREE SIX

Any sixth degree symmetric homogeneous polynomial $f_6(a,b,c)$ can be written in the form

$$f_6(a,b,c) = Ar^2 + B(p,q)r + C(p,q),$$

where A is called *the highest coefficient* of f_6, and

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

Theorem (see [26], [27]). *Let $f_6(a,b,c)$ be a sixth degree symmetric homogeneous polynomial having the highest coefficient $A \le 0$.*

(a) *The inequality $f_6(a,b,c) \ge 0$ holds for all real numbers a,b,c if and only if $f_6(a,1,1) \ge 0$ for all real a;*

(b) *The inequality $f_6(a,b,c) \ge 0$ holds for all $a,b,c \ge 0$ if and only if $f_6(a,1,1) \ge 0$ and $f_6(0,b,c) \ge 0$ for all $a,b,c \ge 0$.*

This theorem is also valid for the case where $B(p,q)$ and $C(p,q)$ are homogeneous rational functions.

For $A > 0$, we can use the *highest coefficient cancellation method* (see [30]). This method consists in finding some suitable real numbers B, C and D such that the following sharper inequality holds

$$f_6(a,b,c) \ge A\left(r + Bp^3 + Cpq + D\frac{q^2}{p}\right)^2.$$

Because the function g_6 defined by

$$g_6(a,b,c) = f_6(a,b,c) - A\left(r + Bp^3 + Cpq + D\frac{q^2}{p}\right)^2$$

has the highest coefficient equal to zero, we can prove the inequality $g_6(a,b,c) \ge 0$ using Theorem above.

Notice that sometimes it is useful to break the problem into two parts, $p^2 \le \xi q$ and $p^2 > \xi q$, where ξ is a suitable real number.

A symmetric homogeneous polynomial of degree six in three variables has the form

$$f_6(a,b,c) = A_1\sum a^6 + A_2\sum ab(a^4+b^4) + A_3\sum a^2b^2(a^2+b^2)$$

$$+A_4\sum a^3b^3 + A_5abc\sum a^3 + A_6abc\sum ab(a+b) + 3A_7a^2b^2c^2,$$

where $A_1,\ldots,A_7$ are real constants. In order to write this polynomial as a function of p, q and r, the following relations are useful:

$$\sum a^3 = 3r + p^3 - 3pq,$$

$$\sum ab(a+b) = -3r + pq,$$

$$\sum a^3b^3 = 3r^2 - 3pqr + q^3,$$

$$\sum a^2b^2(a^2+b^2) = -3r^2 - 2(p^3-2pq)r + p^2q^2 - 2q^3,$$

$$\sum ab(a^4+b^4) = -3r^2 - 2(p^3-7pq)r + p^4q - 4p^2q^2 + 2q^3,$$

$$\sum a^6 = 3r^2 + 6(p^3-2pq)r + p^6 - 6p^4q + 9p^2q^2 - 2q^3,$$

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2 + 2(9pq-2p^3)r + p^2q^2 - 4q^3.$$

According to these relations, the highest coefficient A of the polynomial $f_6(a,b,c)$ is

$$A = 3(A_1 - A_2 - A_3 + A_4 + A_5 - A_6 + A_7).$$

The polynomials

$$P_1(a,b,c) = \sum(A_1a^2 + A_2bc)(B_1a^2 + B_2bc)(C_1a^2 + C_2bc),$$

$$P_2(a,b,c) = \sum(A_1a^2 + A_2bc)(B_1b^2 + B_2ca)(C_1c^2 + C_2ab)$$

and

$$P_3(a,b,c) = (A_1a^2 + A_2bc)(A_1b^2 + A_2ca)(A_1c^2 + A_2ab)$$

has the highest coefficients

$$P_1(1,1,1), \quad P_2(1,1,1), \quad P_3(1,1,1),$$

respectively. The polynomial

$$P_4(a,b,c) = (a^2 + mab + b^2)(b^2 + mbc + c^2)(c^2 + mca + a^2)$$

has the highest coefficient

$$A = (m-1)^3.$$

19. EQUAL VARIABLES METHOD

The Equal Variables Theorem (EV-Theorem) for nonnegative real variables has the following statement (see [9],[11]).

EV-Theorem (for nonnegative variables). *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be fixed nonnegative real numbers, and let $x_1 \leq x_2 \leq \cdots \leq x_n$ be nonnegative real variables such that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where k is a real number; for $k = 0$, assume that $x_1 x_2 \cdots x_n = a_1 a_2 \cdots a_n > 0$. Let $f : \mathbb{I} \to \mathbb{R}$, where $\mathbb{I} = [0, \infty)$ when f is continuous at $x = 0$, and $\mathbb{I} = (0, \infty)$ when $f(0_+) = \pm\infty$. In addition, f is differentiable on $(0, \infty)$ and the associated function $g : (0, \infty) \to \mathbb{R}$ defined by

$$g(x) = f'\left(x^{\frac{1}{k-1}}\right)$$

is strictly convex on $(0, \infty)$. Let

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n).$$

(1) *If $k \leq 0$, then S_n is maximal for*

$$x_1 = x_2 = \cdots = x_{n-1} \leq x_n,$$

and is minimal for

$$0 < x_1 \leq x_2 = x_3 = \cdots = x_n;$$

(2) *If $k > 0$ and either f is continuous at $x = 0$ or $f(0_+) = -\infty$, then S_n is maximal for*

$$x_1 = x_2 = \cdots = x_{n-1} \leq x_n;$$

(3) *If $k > 0$ and either f is continuous at $x = 0$ or $f(0_+) = \infty$, then S_n is minimal for*

$$x_1 = \cdots = x_{j-1} = 0, \quad x_{j+1} = \cdots = x_n, \qquad j \in \{1, 2, \ldots, n\}.$$

For $f(x) = x^m$, we get the following corollary.

EV-Corollary (for nonnegative variables). *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be fixed nonnegative real numbers, let $x_1 \leq x_2 \leq \cdots \leq x_n$ be nonnegative real variables such that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$
$$x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

and let

$$S_n = x_1^m + x_2^m + \cdots + x_n^m.$$

Case $1:\ k \le 0$ *(for $k=0$, assume that $x_1x_2\cdots x_n = a_1a_2\cdots a_n > 0$).*

(a) *If $m \in (k,0)\cup(1,\infty)$, then S_n is maximal for*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimal for

$$x_1 \le x_2 = x_3 = \cdots = x_n;$$

(b) *If $m \in (-\infty,k)\cup(0,1)$, then S_n is minimal for*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is maximal for

$$x_1 \le x_2 = x_3 = \cdots = x_n.$$

Case $2:\ 0 < k < 1.$

(a) *If $m \in (0,k)\cup(1,\infty)$, then S_n is maximal for*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimal for

$$x_1 = \cdots = x_{j-1} = 0, \quad x_{j+1} = \cdots = x_n, \qquad j \in \{1,2,\ldots,n\};$$

(b) *If $m \in (-\infty,0)\cup(k,1)$, then S_n is minimal for*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n;$$

(c) *If $m \in (k,1)$, then S_n is maximal for*

$$x_1 = \cdots = x_{j-1} = 0, \quad x_{j+1} = \cdots = x_n, \qquad j \in \{1,2,\ldots,n\}.$$

Case $3:\ k > 1.$

(a) *If $m \in (0,1)\cup(k,\infty)$, then S_n is maximal for*

$$x_1 = x_2 = \cdots = x_{n-1} \le x_n,$$

and is minimal for

$$x_1 = \cdots = x_{j-1} = 0, \quad x_{j+1} = \cdots = x_n, \qquad j \in \{1,2,\ldots,n\};$$

(b) *If $m \in (-\infty,0)\cup(1,k)$, then S_n is minimal for*

$$0 \le x_1 = x_2 = \cdots = x_{n-1} \le x_n;$$

(c) *If $m \in (1,k)$, then S_n is maximal for*

$$x_1 = \cdots = x_{j-1} = 0, \quad x_{j+1} = \cdots = x_n, \qquad j \in \{1,2,\ldots,n\}.$$

The Equal Variables Theorem (EV-Theorem) for real variables has the following statement (see [31]).

EV-Theorem (for real variables). *Let $a_1, a_2, \dots, a_n$ $(n \ge 3)$ be fixed real numbers, let $x_1 \le x_2 \le \cdots \le x_n$ be real variables such that*

$$x_1 + x_2 + \cdots + x_n = a_1 + a_2 + \cdots + a_n,$$

$$x_1^k + x_2^k + \cdots + x_n^k = a_1^k + a_2^k + \cdots + a_n^k,$$

where k is an even positive integer, and let f be a differentiable function on $\mathbb{R}$ such that the associated function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = f'\left(\sqrt[k-1]{x} \right)$$

is strictly convex on $\mathbb{R}$. Then, the sum

$$S_n = f(x_1) + f(x_2) + \cdots + f(x_n)$$

is minimal for

$$x_2 = x_3 = \cdots = x_n,$$

and is maximal for

$$x_1 = x_2 = \cdots = x_{n-1}.$$

For $n = 3$, the following results are valid.

Theorem 1. *Let $a \ge b \ge c$ be real numbers such that*

$$a + b + c = p, \quad ab + bc + ca = q,$$

where p and q are fixed real numbers satisfying $p^2 \ge 3q$. The product

$$r = abc$$

is minimal when $a = b$, and is maximal when $b = c$.

Theorem 2. *Let $a \ge b \ge c$ such that*

$$a + b + c = p, \qquad ab + bc + ca = q,$$

where p and q are fixed real numbers satisfying $p^2 \ge 3q$.

(a) *If a, b, c are nonnegative real numbers, then the product $r = abc$ is maximal when $b = c$, and is minimal when $a = b$ or $c = 0$;*

(b) *If a, b, c are the lengths of the sides of a triangle (non-degenerate or degenerate), then the product $r = abc$ is maximal when $b = c \ge \frac{a}{2}$ or $b + c = a$, and is minimal when $a = b \ge c$.*

Theorem 3. *Let $a \ge b \ge c$ be positive real numbers such that*

$$a+b+c=p, \qquad abc=r,$$

where p and r are fixed positive numbers satisfying $p^3 \ge 27r$. Then,

$$q=ab+bc+ca$$

is minimal when $b=c$, and is maximal when $a=b$.

Theorem 4. *Let $a \ge 1 \ge b \ge c \ge 0$ such that*

$$a+b+c=3, \qquad ab+bc+ca=q,$$

where $q \in [0,3]$ is a fixed number. The product $r=abc$ is minimal when $b=1$ or $c=0$, and maximal when $b=c$.

Theorem 5. *Let $a \ge b \ge 1 \ge c \ge 0$ such that*

$$a+b+c=3, \qquad ab+bc+ca=q,$$

where $q \in [0,3]$ is a fixed number. The product $r=abc$ is minimal when $a=b$ or $c=0$, and maximal when $b=1$.

20. ARITHMETIC COMPENSATION METHOD

The Arithmetic Compensation Theorem (AC-Theorem) has the following statement (see [10], [11], [25]).

AC-Theorem. *Let $s>0$ and let F be a symmetric continuous function on the compact set in $\mathbb{R}^n$*

$$S=\{(x_1,x_2,\ldots,x_n): x_1+x_2+\cdots+x_n=s,\ x_i \ge 0,\ i=1,2,\cdots,n\}.$$

If

$$F(x_1,x_2,x_3,\ldots,x_n) \ge$$
$$\ge \min\left\{F\left(\frac{x_1+x_2}{2},\frac{x_1+x_2}{2},x_3,\ldots,x_n\right),F(0,x_1+x_2,x_3,\ldots,x_n)\right\}$$

for all $(x_1,x_2,\ldots,x_n) \in S$, then $F(x_1,x_2,x_3,\ldots,x_n)$ is minimal when

$$x_1=x_2=\cdots=x_k=\frac{s}{k}, \qquad x_{k+1}=\cdots=x_n=0;$$

that is,

$$F(x_1,x_2,x_3,\ldots,x_n) \ge \min_{1\le k\le n} F\left(\frac{s}{k},\cdots,\frac{s}{k},0,\cdots,0\right)$$

for all $(x_1,x_2,\ldots,x_n) \in S$.

Notice that if

$$F(x_1, x_2, x_3, \ldots, x_n) < F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \ldots, x_n\right)$$

involves

$$F(x_1, x_2, x_3, \ldots, x_n) \geq F(0, x_1+x_2, x_3, \ldots, x_n),$$

then the hypothesis

$$F(x_1, x_2, x_3, \ldots, x_n) \geq$$
$$\geq \min\left\{F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \ldots, x_n\right), F(0, x_1+x_2, x_3, \ldots, x_n)\right\}$$

is satisfied.

21. VASC'S CYCLIC INEQUALITY

The following theorem gives Vasc's cyclic inequality (*Vasile Cirtoaje*, 1991).

Theorem 1. *If a, b, c are real numbers, then*

$$(a^2+b^2+c^2)^2 \geq 3(a^3b+b^3c+c^3a),$$

with equality for $a=b=c$, and also for

$$\frac{a}{\sin^2\frac{4\pi}{7}} = \frac{b}{\sin^2\frac{2\pi}{7}} = \frac{c}{\sin^2\frac{\pi}{7}}$$

(or any cyclic permutation).

A generalization of Vasc's inequality is given in [17].

Theorem 2. *Let*

$$f_4(a,b,c) = \sum a^4 + A\sum a^2b^2 + Babc\sum a + C\sum a^3b + D\sum ab^3,$$

where A, B, C, D are real constants such that

$$1+A+B+C+D=0.$$

The inequality $f_4(a,b,c) \geq 0$ holds for all real numbers a, b, c if and only if

$$3(1+A) \geq C^2+CD+D^2.$$

Notice that

$$\frac{4}{S}f_4(a,b,c) = (U+V+C+D)^2 + 3\left(U-V+\frac{C-D}{3}\right)^2$$
$$+\frac{4}{3}(3+3A-C^2-CD-D^2),$$

where

$$S=\sum a^2b^2-\sum a^2bc,$$

$$U=\frac{\sum a^3b-\sum a^2bc}{S},$$

$$V=\frac{\sum ab^3-\sum a^2bc}{S}.$$

For $A=B=0$, $C=-2$ and $D=1$, we get the following inequality

$$a^4+b^4+c^4+ab^3+bc^3+ca^3\geq 2(a^3b+b^3c+c^3a),$$

with equality for $a=b=c$, and also for

$$\frac{a}{\sin\frac{\pi}{9}}=\frac{b}{\sin\frac{7\pi}{9}}=\frac{c}{\sin\frac{13\pi}{9}}$$

(or any cyclic permutation) - *Vasile Cirtoaje*, 1991.

22. CYCLIC INEQUALITIES OF DEGREE THREE AND FOUR

Consider the third degree cyclic homogeneous polynomial

$$f_3(a,b,c)=\sum a^3+Babc+C\sum a^2b+D\sum ab^2,$$

where B,C,D are real constants. The following theorem holds.

Theorem 1 (see [37]). *The cyclic inequality $f_3(a,b,c)\geq 0$ holds for all nonnegative numbers a,b,c if and only if*

$$f_3(1,1,1)\geq 0$$

and

$$f_3(a,1,0)\geq 0$$

for all $a\geq 0$.

Consider now the fourth degree cyclic homogeneous polynomial

$$f_4(a,b,c)=\sum a^4+A\sum a^2b^2+Babc\sum a+C\sum a^3b+D\sum ab^3,$$

where A,B,C,D are real constants.

The following theorem states the necessary and sufficient conditions that $f_4(a,b,c)\geq 0$ for all real numbers a,b,c.

Theorem 2 (see [22]). *The inequality $f_4(a,b,c)\geq 0$ holds for all real numbers a,b,c if and only if $g_4(t)\geq 0$ for all $t\geq 0$, where*

$$g_4(t)=3(2+A-C-D)t^4-Ft^3+3(4-B+C+D)t^2+1+A+B+C+D,$$

$$F = \sqrt{27(C-D)^2 + E^2}, \quad E = 8 - 4A + 2B - C - D.$$

Note that in the special case $f_4(1,1,1) = 0$ (when $1 + A + B + C + D = 0$), Theorem 1 yields Theorem 0 from the preceding section 21.

The following theorem states some strong sufficient conditions that $f_4(a,b,c) \geq 0$ for all real numbers a, b, c.

Theorem 3 (see [23]). *The inequality $f_4(a,b,c) \geq 0$ holds for all real numbers a, b, c if the following two conditions are satisfied*:

(a) $1 + A + B + C + D \geq 0$;

(b) *there exists a real number $t \in (-\sqrt{3}, \sqrt{3})$ such that $f(t) \geq 0$, where*

$$f(t) = 2Gt^3 - (6 + 2A + B + 3C + 3D)t^2 + 2(1 + C + D)Gt + H,$$

$$G = \sqrt{1 + A + B + C + D}, \quad H = 2 + 2A - B - C - D - C^2 - CD - D^2.$$

The following theorem states the necessary and sufficient conditions that $f_4(a,b,c) \geq 0$ for all $a, b, c \geq 0$.

Theorem 4 (see [43]). *Let*

$$E = 8 - 4A + 2B - C - D, \quad F = \sqrt{27(C-D)^2 + E^2},$$

$$g_4(t) = 3(2 + A - C - D)t^4 - Ft^3 + 3(4 - B + C + D)t^2 + 1 + A + B + C + D,$$

$$g_3(t) = \frac{2E}{F}t^3 + 3t^2 - 1.$$

For $F = 0$, the inequality $f_4(a,b,c) \geq 0$ holds for all $a, b, c \geq 0$ if and only if $g_4(t) \geq 0$ for all $t \in [0,1]$.

For $F \neq 0$, the inequality $f_4(a,b,c) \geq 0$ holds for all $a, b, c \geq 0$ if and only if the following two conditions are satisfied:

(a) $g_4(t) \geq 0$ *for all $t \in [0, t_1]$, where $t_1 \in [1/2, 1]$ such that $g_3(t_1) = 0$;*

(b) $f_4(a, 1, 0) \geq 0$ *for all $a \geq 0$.*

The following theorem states some strong sufficient conditions that $f_4(a,b,c) \geq 0$ for all $a, b, c \geq 0$.

Theorem 5 (see [43]). *The inequality $f_4(a,b,c) \geq 0$ holds for all $a, b, c \geq 0$ if*

$$1 + A + B + C + D \geq 0$$

and one of the following two conditions is satisfied:

(a) $3(1 + A) \geq C^2 + CD + D^2$;

(b) $3(1 + A) < C^2 + CD + D^2$, *and there exists $t \geq 0$ such that*

$$(C + 2D)t^2 + 6t + 2C + D \geq 2\sqrt{(t^4 + t^2 + 1)(C^2 + CD + D^2 - 3 - 3A)}.$$

23. VASC'S POWER EXPONENTIAL INEQUALITY

Theorem. *Let* $0 < k \le e$.

(a) *If* $a, b > 0$*, then (Vasile Cîrtoaje, 2006)*

$$a^{ka} + b^{kb} \ge a^{kb} + b^{ka};$$

(b) *If* $a, b \in (0, 1]$*, then (Vasile Cîrtoaje, 2010)*

$$2\sqrt{a^{ka}b^{kb}} \ge a^{kb} + b^{ka}.$$

Chapter 2

Symmetric Polynomial Inequalities in Real Variables

2.1 Application

2.1. Let a, b, c, d be real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 9.$$

Prove that

$$a^3 + b^3 + c^3 + d^3 \leq 27.$$

2.2. If a, b, c are real numbers such that

$$a + b + c = 0,$$

then

$$(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) \leq 0.$$

2.3. Let a, b, c be real numbers such that

$$a + b \geq 0, \quad b + c \geq 0, \quad c + a \geq 0.$$

Prove that

$$9(a + b)(b + c)(c + a) \geq 8(a + b + c)(ab + bc + ca).$$

2.4. Let a,b,c be real numbers such that

$$ab+bc+ca=3.$$

Prove that

$$(3a^2+1)(3b^2+1)(3c^2+1)\ge 64.$$

When does equality hold?

2.5. If a and b are real numbers, then

$$3(1-a+a^2)(1-b+b^2)\ge 2(1-ab+a^2b^2).$$

2.6. If a,b,c are real numbers, then

$$3(1-a+a^2)(1-b+b^2)(1-c+c^2)\ge 1+abc+a^2b^2c^2.$$

2.7. If a,b,c are real numbers, then

$$(a^2+b^2+c^2)^3\ge (a+b+c)(ab+bc+ca)(a^3+b^3+c^3).$$

2.8. If a,b,c are real numbers, then

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2)\ge [ab(a+b)+bc(b+c)+ca(c+a)-2abc]^2.$$

2.9. If a,b,c are real numbers, then

$$(a^2+1)(b^2+1)(c^2+1)\ge 2(ab+bc+ca).$$

2.10. If a,b,c are real numbers, then

$$(a^2+1)(b^2+1)(c^2+1)\ge \frac{5}{16}(a+b+c+1)^2.$$

2.11. If a,b,c are real numbers, then

(a) $$a^6+b^6+c^6-3a^2b^2c^2+2(a^2+bc)(b^2+ca)(c^2+ab)\ge 0;$$

(b) $$a^6+b^6+c^6-3a^2b^2c^2\ge (a^2-2bc)(b^2-2ca)(c^2-2ab).$$

2.12. If a, b, c are real numbers, then

$$\frac{2}{3}(a^6 + b^6 + c^6) + a^3b^3 + b^3c^3 + c^3a^3 + abc(a^3 + b^3 + c^3) \geq 0.$$

2.13. If a, b, c are real numbers, then

$$4(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq (a-b)^2(b-c)^2(c-a)^2.$$

2.14. If a, b, c are real numbers, then

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 3(a^2b + b^2c + c^2a)(ab^2 + bc^2 + ca^2).$$

2.15. If a, b, c are real numbers such that $abc > 0$, then

$$4\left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right) \geq 9(a + b + c).$$

2.16. If a, b, c are real numbers, then

(a) $(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) \leq (a^2 + b^2 + c^2)(ab + bc + ca)^2;$

(b) $(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) \leq (a + b + c)^2(a^2b^2 + b^2c^2 + c^2a^2).$

2.17. If a, b, c are real numbers such that

$$ab + bc + ca \geq 0,$$

then

$$27(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) \leq (a + b + c)^6.$$

2.18. If a, b, c are real numbers such that

$$a^2 + b^2 + c^2 = 2,$$

then

$$(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) + 2 \geq 0.$$

2.19. If a, b, c are real numbers such that

$$a + b + c = 3,$$

then

$$3(a^4 + b^4 + c^4) + a^2 + b^2 + c^2 + 6 \geq 6(a^3 + b^3 + c^3).$$

2.20. If a, b, c are real numbers such that

$$abc = 1,$$

then

$$3(a^2 + b^2 + c^2) + 2(a + b + c) \geq 5(ab + bc + ca).$$

2.21. If a, b, c are real numbers such that

$$abc = 1,$$

then

$$a^2 + b^2 + c^2 + 6 \geq \frac{3}{2}\left(a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

2.22. If a, b, c are real numbers, then

$$(1 + a^2)(1 + b^2)(1 + c^2) + 8abc \geq \frac{1}{4}(1 + a)^2(1 + b)^2(1 + c)^2.$$

2.23. Let a, b, c be real numbers such that

$$a + b + c = 0.$$

Prove that

$$a^{12} + b^{12} + c^{12} \geq \frac{2049}{8}a^4b^4c^4.$$

2.24. If a, b, c are real numbers such that $abc \geq 0$, then

$$a^2 + b^2 + c^2 + 2abc + 4 \geq 2(a + b + c) + ab + bc + ca.$$

2.25. Let a, b, c be real numbers such that

$$a + b + c = 3.$$

(a) If $a, b, c \geq -3$, then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

(b) If $a, b, c \geq -7$, then

$$\frac{1-a}{(1+a)^2} + \frac{1-b}{(1+b)^2} + \frac{1-c}{(1+c)^2} \geq 0.$$

2.26. If a, b, c are real numbers, then

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

2.27. If a, b, c are real numbers, then

$$\left(\frac{a^2+b^2+c^2}{3}\right)^3 \geq a^2b^2c^2 + \frac{1}{16}(a-b)^2(b-c)^2(c-a)^2.$$

2.28. If a, b, c are real numbers, then

$$(a^2+b^2+c^2)^3 \geq \frac{108}{5}a^2b^2c^2 + 2(a-b)^2(b-c)^2(c-a)^2.$$

2.29. If a, b, c are real numbers, then

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2) \geq (a-b)^2(b-c)^2(c-a)^2.$$

2.30. If a, b, c are real numbers, then

$$32(a^2+bc)(b^2+ca)(c^2+ab) + 9(a-b)^2(b-c)^2(c-a)^2 \geq 0.$$

2.31. If a, b, c are real numbers, then

$$a^4(b-c)^2 + b^4(c-a)^2 + c^4(a-b)^2 \geq \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

2.32. If a,b,c are real numbers, then

$$a^2(b-c)^4+b^2(c-a)^4+c^2(a-b)^4 \ge \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

2.33. If a,b,c are real numbers, then

$$a^2(b^2-c^2)^2+b^2(c^2-a^2)^2+c^2(a^2-b^2)^2 \ge \frac{3}{8}(a-b)^2(b-c)^2(c-a)^2.$$

2.34. If a,b,c are real numbers such that

$$ab+bc+ca=3,$$

then

(a) $$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge 3(a+b+c)^2;$$

(b) $$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge \frac{3}{2}(a^2+b^2+c^2).$$

2.35. If a,b,c are real numbers, then

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge 3(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2).$$

2.36. If a,b,c are real numbers, not all of the same sign, then

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge 3(ab+bc+ca)^3.$$

2.37. If a,b,c are real numbers, then

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge \frac{3}{8}(a^2+b^2)(b^2+c^2)(c^2+a^2).$$

2.38. If a,b,c are real numbers, then

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2) \ge (a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2).$$

2.39. If a, b, c are real numbers, then

$$9(1+a^4)(1+b^4)(1+c^4) \geq 8(1+abc+a^2b^2c^2)^2.$$

2.40. If a, b, c are real numbers, then

$$2(1+a^2)(1+b^2)(1+c^2) \geq (1+a)(1+b)(1+c)(1+abc).$$

2.41. If a, b, c are real numbers, then

$$3(a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2) \geq a^3b^3+b^3c^3+c^3a^3.$$

2.42. If a, b, c are nonzero real numbers, then

$$\sum \frac{b^2-bc+c^2}{a^2} + 2\sum \frac{a^2}{bc} \geq \left(\sum a\right)\left(\sum \frac{1}{a}\right).$$

2.43. Let a, b, c be real numbers. Prove that

(a) if $a, b, c \in [0,1]$, then

$$abc-(b+c-a)(c+a-b)(a+b-c) \leq 1;$$

(b) if $a, b, c \in [-1,1]$, then

$$abc-(b+c-a)(c+a-b)(a+b-c) \leq 4.$$

2.44. Let a, b, c be real numbers. Prove that

(a) if $a, b, c \in [0,1]$, then

$$\sum a^2(a-b)(a-c) \leq 1;$$

(b) if $a, b, c \in [-1,1]$, then

$$\sum a^2(a-b)(a-c) \leq 4.$$

2.45. Let a, b, c be real numbers such that

$$ab + bc + ca = abc + 2.$$

Prove that

$$a^2 + b^2 + c^2 - 3 \geq (2 + \sqrt{3})(a + b + c - 3).$$

2.46. Let a, b, c be real numbers such that

$$(a + b)(b + c)(c + a) = 10.$$

Prove that

$$(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) + 12a^2b^2c^2 \geq 30.$$

2.47. Let a, b, c be real numbers such that

$$(a + b)(b + c)(c + a) = 5.$$

Prove that

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) + 12a^2b^2c^2 \geq 15.$$

2.48. Let a, b, c be real numbers such that

$$a + b + c = 1, \quad a^3 + b^3 + c^3 = k.$$

Prove that

(a) if $k = 25$, then $|a| \leq 1$ or $|b| \leq 1$ or $|c| \leq 1$;

(b) if $k = -11$, then $1 < a \leq 2$ or $1 < b \leq 2$ or $1 < c \leq 2$.

2.49. Let a, b, c be real numbers such that

$$a + b + c = a^3 + b^3 + c^3 = 2.$$

Prove that $a, b, c \notin \left[\frac{5}{4}, 2\right]$.

2.50. If a, b, c and k are real numbers, then

$$\sum (a - b)(a - c)(a - kb)(a - kc) \geq 0.$$

2.51. If a,b,c are real numbers, then

$$\sum a^2(a-b)(a-c) \ge \frac{(a-b)^2(b-c)^2(c-a)^2}{a^2+b^2+c^2+ab+bc+ca}.$$

2.52. Let $x_1,x_2,\ldots,x_n$ $(n \ge 3)$ be real numbers such that

$$x_1+x_2+\ldots+x_n = a+b, \qquad x_1^2+x_2^2+\cdots+x_n^2 = a^2+b^2,$$

where a and b are fixed real numbers such that $a \ne 0$, $b \ne 0$, $a \ne b$. Then, there exist $x_1,x_2,\ldots,x_n$ such that

(a) $$x_1x_2\cdots x_n > 0;$$

(b) $$x_1x_2\cdots x_n < 0.$$

2.53. Let $a \ge b \ge c$ be real numbers such that

$$a+b+c = p, \qquad ab+bc+ca = q,$$

where p and q are fixed real numbers satisfying $p^2 \ge 3q$. Prove that the product

$$r = abc$$

is minimal only when $a = b$, and is maximal only when $b = c$.

2.54. Let a,b,c be real numbers. Prove that

(a) for fixed

$$a+b+c = p, \qquad abc = r,$$

the sum

$$q = ab+bc+ca$$

is maximal only when two of a,b,c are equal;

(b) for fixed

$$ab+bc+ca = q, \qquad abc = r \ne 0,$$

the product

$$p_1 = abc(a+b+c)$$

is maximal only when two of a,b,c are equal.

2.55. Let a,b,c be real numbers such that $a+b+c=3$. Prove that

(a) $$(ab+bc+ca)^2 \ge 9abc;$$

(b) $$(ab+bc+ca)^2+9 \ge 18abc;$$

(c) $$(ab+bc+ca-3)^2 \ge 27(abc-1).$$

2.56. Let a,b,c be real numbers such that

$$ab+bc+ca+abc=4.$$

Prove that

(a) if $abc>0$, then

$$2(a+b+c)+ab+bc+ca \le \frac{9}{abc};$$

(b) if $abc<0$, then

$$2(a+b+c)+ab+bc+ca \ge \frac{9}{abc}.$$

2.57. If a,b,c are real numbers such that

$$a+b+c+abc=4,$$

then

$$a^2+b^2+c^2+3 \ge 2(ab+bc+ca).$$

2.58. If a,b,c are real numbers such that

$$ab+bc+ca=3abc,$$

then

$$4(a^2+b^2+c^2)+9 \ge 7(ab+bc+ca).$$

2.59. Let $a,b,c \le \frac{6}{5}$ be real numbers such that $a^2+b^2+c^2=4$. If

$$k=\frac{16(2+15\sqrt{2})}{125} \approx 2.97,$$

then

$$ab+bc+ca+k \ge abc.$$

2.60. Let $f_4(a,b,c)$ be a symmetric homogeneous polynomial of degree four. Prove that the inequality $f_4(a,b,c) \geq 0$ holds for all real numbers a,b,c if and only if $f_4(a,1,1) \geq 0$ for all real a.

2.61. If a,b,c are real numbers, then

$$10(a^4+b^4+c^4)+64(a^2b^2+b^2c^2+c^2a^2) \geq 33\sum ab(a^2+b^2).$$

2.62. If a,b,c are real numbers such that

$$a+b+c=3,$$

then

$$3(a^4+b^4+c^4)+33 \geq 14(a^2+b^2+c^2).$$

2.63. If a,b,c are real numbers such that

$$a^2+b^2+c^2=3,$$

then

$$a^4+b^4+c^4+3(ab+bc+ca) \leq 12.$$

2.64. Let α,β,γ be real numbers such that

$$1+\alpha+\beta=2\gamma.$$

The inequality

$$\sum a^4+\alpha\sum a^2b^2+\beta abc\sum a \geq \gamma\sum ab(a^2+b^2)$$

holds for any real numbers a,b,c if and only if

$$1+\alpha \geq \gamma^2.$$

2.65. If a,b,c are real numbers such that

$$a^2+b^2+c^2=2,$$

then

$$ab(a^2-ab+b^2-c^2)+bc(b^2-bc+c^2-a^2)+ca(c^2-ca+a^2-b^2) \leq 1.$$

2.66. If a,b,c are real numbers, then

$$(a+b)^4+(b+c)^4+(c+a)^4 \ge \frac{4}{7}(a^4+b^4+c^4).$$

2.67. Let a,b,c be real numbers. If

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc,$$

then

$$(3-p)r+\frac{p^2+q^2-pq}{3} \ge q.$$

2.68. If a,b,c are real numbers, then

$$\frac{ab(a+b)+bc(b+c)+ca(c+a)}{(a^2+1)(b^2+1)(c^2+1)} \le \frac{3}{4}.$$

2.69. If a,b,c are real numbers such that $abc>0$, then

$$\left(a+\frac{1}{a}-1\right)\left(b+\frac{1}{b}-1\right)\left(c+\frac{1}{c}-1\right)+2 \ge \frac{1}{3}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

2.70. If a,b,c are real numbers, then

$$\left(a^2+\frac{1}{2}\right)\left(b^2+\frac{1}{2}\right)\left(c^2+\frac{1}{2}\right) \ge \left(a+b-\frac{1}{2}\right)\left(b+c-\frac{1}{2}\right)\left(c+a-\frac{1}{2}\right).$$

2.71. If a,b,c are real numbers such that

$$a+b+c=3,$$

then

$$\frac{a(a-1)}{8a^2+9}+\frac{b(b-1)}{8b^2+9}+\frac{c(c-1)}{8c^2+9} \ge 0.$$

2.72. If a,b,c are real numbers such that

$$a+b+c=3,$$

then

$$\frac{(a-11)(a-1)}{2a^2+1}+\frac{(b-11)(b-1)}{2b^2+1}+\frac{(c-11)(c-1)}{2c^2+1} \ge 0.$$

2.73. If a, b, c are real numbers, then

$$(a^2+2)(b^2+2)(c^2+2) \geq 9(ab+bc+ca).$$

2.74. If a, b, c are real numbers such that

$$ab+bc+ca=3,$$

then

$$4(a^4+b^4+c^4)+11abc(a+b+c) \geq 45.$$

2.75. Any sixth degree symmetric homogeneous polynomial $f_6(a,b,c)$ can be written in the form

$$f_6(a,b,c)=Ar^2+B(p,q)r+C(p,q),$$

where A is called *the highest coefficient* of f_6, and

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc.$$

In the case $A \leq 0$, prove that the inequality $f_6(a,b,c) \geq 0$ holds for all real numbers a, b, c if and only if $f_6(a,1,1) \geq 0$ for all real a.

2.76. If a, b, c are real numbers such that

$$ab+bc+ca=-1,$$

then

(a) $$5(a^2+b^2)(b^2+c^2)(c^2+a^2) \geq 8;$$

(b) $$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \geq 1.$$

2.77. If a, b, c are real numbers, then

(a) $$\sum a^2(a-b)(a-c)(a+2b)(a+2c)+(a-b)^2(b-c)^2(c-a)^2 \geq 0;$$

(b) $$\sum a^2(a-b)(a-c)(a-4b)(a-4c)+7(a-b)^2(b-c)^2(c-a)^2 \geq 0.$$

2.78. If a, b, c are real numbers, then

$$(a^2+2bc)(b^2+2ca)(c^2+2ab)+(a-b)^2(b-c)^2(c-a)^2 \geq 0.$$

2.79. If a,b,c are real numbers, then

$$(2a^2+5ab+2b^2)(2b^2+5bc+2c^2)(2c^2+5ca+2a^2)+(a-b)^2(b-c)^2(c-a)^2\ge 0.$$

2.80. If a,b,c are real numbers, then

$$\left(a^2+\frac{2}{3}ab+b^2\right)\left(b^2+\frac{2}{3}bc+c^2\right)\left(c^2+\frac{2}{3}ca+a^2\right)\ge \frac{64}{27}(a^2+bc)(b^2+ca)(c^2+ab).$$

2.81. If a,b,c are real numbers, then

$$\sum a^2(a-b)(a-c)\ge \frac{2(a-b)^2(b-c)^2(c-a)^2}{a^2+b^2+c^2}.$$

2.82. If a,b,c are real numbers, then

$$\sum (a-b)(a-c)(a-2b)(a-2c)\ge \frac{8(a-b)^2(b-c)^2(c-a)^2}{a^2+b^2+c^2}.$$

2.83. If a,b,c are real numbers, no two of which are zero, then

$$\frac{a^2+3bc}{b^2+c^2}+\frac{b^2+3ca}{c^2+a^2}+\frac{c^2+3ab}{a^2+b^2}\ge 0.$$

2.84. If a,b,c are real numbers, no two of which are zero, then

$$\frac{a^2+6bc}{b^2-bc+c^2}+\frac{b^2+6ca}{c^2-ca+a^2}+\frac{c^2+6ab}{a^2-ab+b^2}\ge 0.$$

2.85. If a,b,c are real numbers such that

$$ab+bc+ca\ge 0,$$

then

$$\frac{4a^2+23bc}{b^2+c^2}+\frac{4b^2+23ca}{c^2+a^2}+\frac{4c^2+23ab}{a^2+b^2}\ge 0.$$

2.86. If a, b, c are real numbers such that

$$ab + bc + ca = 3,$$

then

$$20(a^6 + b^6 + c^6) + 43abc(a^3 + b^3 + c^3) \geq 189.$$

2.87. If a, b, c are real numbers such that

$$ab + bc + ca \geq 0,$$

then

(a) $$(a^2 + b^2 + c^2)(ab + bc + ca)^2 \geq abc(4a^3 + 4b^3 + 4c^3 + 15abc);$$

(b) $$4(a + b + c)^6 \geq 81abc(5a^3 + 5b^3 + 5c^3 + 21abc).$$

2.88. If a, b, c are real numbers, then

$$4\sum(a^2 + bc)(a - b)(a - c)(a - 3b)(a - 3c) \geq 7(a - b)^2(b - c)^2(c - a)^2.$$

2.89. Let a, b, c be real numbers such that

$$ab + bc + ca \geq 0.$$

For any real k, prove that

$$\sum 4bc(a - b)(a - c)(a - kb)(a - kc) + (a - b)^2(b - c)^2(c - a)^2 \geq 0.$$

2.90. If a, b, c are real numbers, then

$$\left[(a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2)\right]^2 \geq 4(ab + bc + ca)(a^2b^2 + b^2c^2 + c^2a^2).$$

2.91. If a, b, c are real numbers such that

$$a + b + c = 3,$$

then

$$\frac{(a - 1)(a - 25)}{a^2 + 23} + \frac{(b - 1)(b - 25)}{b^2 + 23} + \frac{(c - 1)(c - 25)}{c^2 + 23} \geq 0.$$

2.92. If a, b, c are real numbers such that $abc \neq 0$, then

$$\left(\frac{b+c}{a}\right)^2+\left(\frac{c+a}{b}\right)^2+\left(\frac{a+b}{c}\right)^2 > 2.$$

2.93. If a, b, c are real numbers, then

(a) $$(a^2+1)(b^2+1)(c^2+1) \geq \frac{8}{3\sqrt{3}}\,|(a-b)(b-c)(c-a)|;$$

(b) $$(a^2-a+1)(b^2-b+1)(c^2-c+1) \geq |(a-b)(b-c)(c-a)|.$$

2.94. If a, b, c are real numbers such that

$$a+b+c=3,$$

then

$$(1-a+a^2)(1-b+b^2)(1-c+c^2) \geq 1.$$

2.95. If a, b, c are real numbers such that

$$a+b+c=0,$$

then

$$\frac{a(a-4)}{a^2+2}+\frac{b(b-4)}{b^2+2}+\frac{c(c-4)}{c^2+2} \geq 0.$$

2.96. If a, b, c are real numbers such that

$$a, b, c \leq 1+\sqrt{2}, \qquad a+b+c \geq 0,$$

then

$$2abc+a^2+b^2+c^2+1 \geq 2(ab+bc+ca).$$

2.97. If a, b, c are real numbers such that $a+b+c=2$ and $ab+bc+ca>0$, then

$$(a^2+bc)(b^2+ca)(c^2+ab)+abc \leq 1.$$

2.98. If a, b, c are real numbers such that

$$a^2 + b^2 + c^2 = 3, \qquad a \geq \frac{4}{3},$$

then

$$3(abc + 1) \geq 2(ab + bc + ca).$$

2.99. If a, b, c are real numbers such that $a \geq \dfrac{8}{7}$ and $a^2 + b^2 + c^2 = 3$, then

$$\frac{3 - a - b - c}{1 - abc} \geq \frac{49}{100}.$$

2.100. If $a, b, c \in [-1, 1]$, then

$$a^3 + b^3 + c^3 + abc \leq \frac{15}{16}(a + b + c) + \frac{19}{16}.$$

2.101. If a, b, c are real numbers, then

$$(a^3 + b^3 + c^3)^2 \geq (a^4 + b^4 + c^4)(ab + bc + ca).$$

2.102. Let $a_1, a_2, \ldots, a_n$ be real numbers such that

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n.$$

Prove that:

(a) for $n = 3$,

$$\frac{a_1 + a_2 + a_3}{3} + \min_{i \neq j}(a_i - a_j)^2 \leq \frac{5}{3};$$

(b) for $n = 5$,

$$\frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} + \min_{i \neq j}(a_i - a_j)^2 \leq 1.$$

2.103. Let $a_1, a_2, \ldots, a_7$ be real numbers such that

$$a_1^2 + a_2^2 + \cdots + a_7^2 = n.$$

Prove that:

$$(a) \qquad \sqrt{\frac{|a_1 + a_2 + \cdots + a_7|}{7}} \;+\; \min_{i \neq j}(a_i - a_j)^2 \;\leq\; 1;$$

$$(b) \qquad \sqrt{\frac{|a_1 + a_2 + \cdots + a_7|}{7}} + 8\min_{i \neq j}(a_i - a_j)^2 \leq \frac{19}{8}.$$

2.104. Let f be a differentiable convex function on a closed interval $\mathbb{I} = [a, b]$. If $a_1, a_2, \ldots, a_n \in \mathbb{I}$, then Jensen's difference

$$D = f(a_1) + f(a_2) + \cdots + f(a_n) - nf\left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)$$

is maximal when all $a_i \in \{a, b\}$.

2.105. If a, b, c are real numbers, then

$$2(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \geq (abc - 1)^2.$$

2.106. If a, b, c are real numbers, then

$$(1 + \sqrt{2})(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1) \geq a^2b^2c^2 + 1.$$

2.107. If a, b, c, d are real numbers, then

$$(1 - a + a^2)(1 - b + b^2)(1 - c + c^2)(1 - d + d^2) \geq \left(\frac{1 + abcd}{2}\right)^2.$$

2.108. If a, b, c, d are real numbers, then

$$3(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1) \geq a^2b^2c^2d^2 - abcd + 1.$$

2.109. If a, b, c, d are real numbers, then

$$(a^2 - a + 2)(b^2 - b + 2)(c^2 - c + 2)(d^2 - d + 2) \geq (a + b + c + d)^2.$$

2.110. If a, b, c, d are real numbers such that

$$a + b + c + d \geq a^2 + b^2 + c^2 + d^2,$$

then

$$4abcd + 3(a^2 + b^2 + c^2 + d^2) + 24 \geq 10(a + b + c + d).$$

2.111. Let a, b, c, d be real numbers such that $abcd > 0$. Prove that

$$\left(a + \frac{1}{a}\right)\left(b + \frac{1}{b}\right)\left(c + \frac{1}{c}\right)\left(d + \frac{1}{d}\right) \geq (a + b + c + d)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right).$$

2.112. Let a, b, c, d be real numbers such that

$$a + b + c + d = 4, \qquad a^2 + b^2 + c^2 + d^2 = 7.$$

Prove that

$$a^3 + b^3 + c^3 + d^3 \leq 16.$$

2.113. Let a, b, c, d be real numbers such that

$$a + b + c + d = 0.$$

Prove that

$$12(a^4 + b^4 + c^4 + d^4) \leq 7(a^2 + b^2 + c^2 + d^2)^2.$$

2.114. Let a, b, c, d be real numbers such that

$$a + b + c + d = 0.$$

Prove that

$$(a^2 + b^2 + c^2 + d^2)^3 \geq 3(a^3 + b^3 + c^3 + d^3)^2.$$

2.115. If a, b, c, d are real numbers such that

$$a + b + c + d = 0,$$

then

$$a^4 + b^4 + c^4 + d^4 + 28abcd \geq 0.$$

2.116. If a, b, c, d are real numbers such that

$$abcd = 1.$$

Prove that

$$(1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) \geq (a + b + c + d)^2.$$

2.117. Let a, b, c, d be real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

Prove that

$$(abc)^3 + (bcd)^3 + (cda)^3 + (dab)^3 \leq 4.$$

2.118. Let a, b, c, d be real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 1.$$

Prove that

$$(1 - a)^4 + (1 - b)^4 + (1 - c)^4 + (1 - d)^4 \geq a^4 + b^4 + c^4 + d^4.$$

2.119. If $a, b, c, d \geq \frac{-1}{2}$ such that

$$a + b + c + d = 4,$$

then

$$\frac{1 - a}{1 - a + a^2} + \frac{1 - b}{1 - b + b^2} + \frac{1 - c}{1 - c + c^2} + \frac{1 - d}{1 - d + d^2} \geq 0.$$

2.120. If a, b, c, d are real numbers such that $a \geq b \geq c \geq d$ and

$$a^2 + b^2 + c^2 + d^2 = 4,$$

then

$$a^2c^2 + b^2d^2 \leq 2.$$

2.121. Let a, b, c, d be real numbers such that

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If $a \geq b \geq c \geq d$, then

$$1 - abcd \leq (a - d)^2.$$

2.122. If $a, b, c, d, e \geq -3$ such that

$$a + b + c + d + e = 5,$$

then

$$\frac{1-a}{1+a+a^2} + \frac{1-b}{1+b+b^2} + \frac{1-c}{1+c+c^2} + \frac{1-d}{1+d+d^2} + \frac{1-e}{1+e+e^2} \geq 0.$$

2.123. Let a, b, c, d, e be real numbers such that

$$a + b + c + d + e = 0.$$

Prove that

$$30(a^4 + b^4 + c^4 + d^4 + e^4) \geq 7(a^2 + b^2 + c^2 + d^2 + e^2)^2.$$

2.124. If a, b, c, d, e are real numbers such that $a + b + c + d + e = 5$, then

$$(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1) \geq 1.$$

2.125. If a, b, c, d, e are real numbers, then

$$4(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1) \geq (abcde - 1)^2.$$

2.126. If $a_1, a_2, \ldots, a_5$ are real numbers such that

$$a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 = 0,$$

then

$$\sum_{i<j} a_i a_j \leq 0.$$

2.127. If $a_1, a_2, \ldots, a_{13}$ are real numbers such that

$$a_1 + a_2 + \cdots + a_{13} = \frac{13}{2},$$

then

$$\frac{8a_1+7}{a_1^2-a_1+1} + \frac{8a_2+7}{a_2^2-a_2+1} + \cdots + \frac{8a_{13}+7}{a_{13}^2-a_{13}+1} \le \frac{572}{3}.$$

2.128. Let $a_1, a_2, \ldots, a_n \ge -1$ such that

$$a_1 + a_2 + \cdots + a_n = 0.$$

Prove that

$$(n-2)(a_1^2 + a_2^2 + \cdots + a_n^2) \ge a_1^3 + a_2^3 + \cdots + a_n^3.$$

2.129. Let $a_1, a_2, \ldots, a_n \ge -1$ such that

$$a_1 + a_2 + \cdots + a_n = 0.$$

Prove that

$$(n-2)(a_1^2 + a_2^2 + \cdots + a_n^2) + (n-1(a_1^3 + a_2^3 + \cdots + a_n^3) \ge 0.$$

2.130. Let $a_1, a_2, \ldots, a_n \ge n-1-\sqrt{n^2-n+1}$ be nonzero real numbers such that

$$a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \ge n.$$

2.131. Let $a_1, a_2, \ldots, a_n \le \dfrac{n}{n-2}$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n = n.$$

If k is a positive integer, $k \ge 2$, then

$$a_1^k + a_2^k + \cdots + a_n^k \ge n.$$

2.132. If $a_1, a_2, \ldots, a_n \ge \dfrac{-n}{n-2}$, $n \ge 3$, then

$$\frac{1}{a_1^2}+\frac{1}{a_2^2}+\cdots+\frac{1}{a_n^2} \ge \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}.$$

2.133. If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are real numbers such that

$$a_1, a_2, \ldots, a_n \ge \frac{-(3n-2)}{n-2}, \quad a_1+a_2+\cdots+a_n = n,$$

then

$$\frac{1-a_1}{(1+a_1)^2}+\frac{1-a_2}{(1+a_2)^2}+\cdots+\frac{1-a_n}{(1+a_n)^2} \ge 0.$$

2.134. Let $a_1, a_2, \ldots, a_n$ be real numbers.

(a) If $k \ge n$, then

$$\frac{(a_1+a_2+\cdots+a_n+k-n)^2}{(a_1^2+k-1)(a_2^2+k-1)\cdots(a_n^2+k-1)} \le \frac{1}{k^{n-2}};$$

(b) If $k \ge \dfrac{n}{2}$, then

$$\frac{a_1+a_2+\cdots+a_n+k-n}{(a_1^2+2k-1)(a_2^2+2k-1)\cdots(a_n^2+2k-1)} \le \frac{1}{2(2k)^{n-1}};$$

(c)
$$\frac{(a_1+a_2+\cdots+a_n)^2}{(a_1^2+n-1)(a_2^2+n-1)\cdots(a_n^2+n-1)} \le \frac{1}{n^{n-2}};$$

(d)
$$\frac{a_1+a_2+\cdots+a_n}{(a_1^2+2n-1)(a_2^2+2n-1)\cdots(a_n^2+2n-1)} \le \frac{1}{2(2n)^{n-1}}.$$

2.135. Let $a_1, a_2, \ldots, a_n$ be real numbers.

(a) If $k \ge \dfrac{n}{4}$, then

$$\frac{(a_1+a_2+\cdots+a_n+2k-n)^2}{(a_1^2-a_1+k)(a_2^2-a_2+k)\cdots(a_n^2-a_n+k)} \le \frac{4}{k^{n-2}};$$

(b)
$$\frac{(a_1+a_2+\cdots+a_n)^2}{(a_1^2-a_1+\frac{n}{2})(a_2^2-a_2+\frac{n}{2})\cdots(a_n^2-a_n+\frac{n}{2})} \le \frac{2^n}{n^{n-2}}.$$

2.136. Let $a_1, a_2, \ldots, a_n$ be real numbers.

(a) If $k \ge n$, then

$$\frac{(a_1+a_2+\cdots+a_n)^2+n(k-n)}{(a_1^2+k-1)(a_2^2+k-1)\cdots(a_n^2+k-1)} \le \frac{n}{k^{n-1}};$$

(b)

$$\frac{(a_1+a_2+\cdots+a_n)^2+n^2}{(a_1^2+2n-1)(a_2^2+2n-1)\cdots(a_n^2+2n-1)} \le \frac{n}{(2n)^{n-1}}.$$

2.2 Solutions

P 2.1. *Let a, b, c, d be real numbers such that*

$$a^2 + b^2 + c^2 + d^2 = 9.$$

Prove that

$$a^3 + b^3 + c^3 + d^3 \le 27.$$

Solution. From $a^2 + b^2 + c^2 + d^2 = 9$, we get

$$a^2 \le 9, \quad a \le 3, \quad a^2(a-3) \le 0, \quad a^3 \le 3a^2.$$

Similarly,

$$b^3 \le 3b^2, \quad c^3 \le 3c^2, \quad d^3 \le 3d^2.$$

Therefore, we have

$$a^3 + b^3 + c^3 + d^3 \le 3(a^2 + b^2 + c^2 + d^2) = 27.$$

The equality holds for $a = 3$ and $b = c = d = 0$ (or any cyclic permutation thereof). □

P 2.2. *If a, b, c are real numbers such that*

$$a + b + c = 0,$$

then

$$(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) \le 0.$$

First Solution. Among a, b, c there are two with the same sign; assume that $bc \ge 0$. We need to show that

$$(2b^2 + ca)(2c^2 + ab) \le 0.$$

This is equivalent to

$$[2b^2 - c(b+c)][2c^2 - (b+c)b] \le 0,$$

$$(b-c)^2(2b+c)(b+2c) \ge 0.$$

Since

$$(2b+c)(b+2c) = 2(b^2+c^2) + 5bc \ge 0,$$

the conclusion follows. The equality holds for $\frac{-a}{2} = b = c$ (or any cyclic permutation).

Second Solution. We have

$$2a^2 + bc = (a-b)(a-c) + a(a+b+c) = (a-b)(a-c),$$

$$2b^2 + ca = (b-c)(b-a) + b(a+b+c) = (b-c)(b-a),$$

$$2c^2 + ab = (c-a)(c-b) + c(a+b+c) = (c-a)(c-b).$$

Therefore,

$$(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) = -(a-b)^2(b-c)^2(c-a)^2 \le 0.$$

□

P 2.3. *Let a, b, c be real numbers such that*

$$a + b \ge 0, \quad b + c \ge 0, \quad c + a \ge 0.$$

Prove that

$$9(a+b)(b+c)(c+a) \ge 8(a+b+c)(ab+bc+ca).$$

(Nguyen Van Huyen, 2014)

Solution. Write the desired inequality in the form

$$a(b-c)^2 + b(c-a)^2 + c(a-b)^2 \ge 0.$$

For $a, b, c \ge 0$, the inequality is clearly true. Otherwise, without loss of generality, assume that $a \le b \le c$, $a < 0$. From $a + b \ge 0$, it follows that

$$a < 0 < b \le c, \quad a + b \ge 0.$$

Replacing a by $-a$, we need to show that

$$0 < a \le b \le c$$

involves

$$-a(c-b)^2 + b(c+a)^2 + c(a+b)^2 \ge 0.$$

This is true since $c(a+b)^2 > 0$ and

$$-a(c-b)^2 + b(c+a)^2 \ge -b(c-b)^2 + b(c+a)^2 = b(a+b)(a-b+2c) > 0.$$

The equality holds for $a = b = c \ge 0$.

□

P 2.4. *Let a, b, c be real numbers such that*

$$ab + bc + ca = 3.$$

Prove that

$$(3a^2+1)(3b^2+1)(3c^2+1) \ge 64.$$

When does equality hold?

Solution. Using the substitution

$$a = \frac{x}{\sqrt{3}}, \quad b = \frac{y}{\sqrt{3}}, \quad c = \frac{z}{\sqrt{3}},$$

we need to show that

$$(x^2+1)(y^2+1)(z^2+1) \ge 64$$

for all real x, y, z such that $xy + yz + zx = 9$.

First Solution. Applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(x^2+1)(y^2+1)(z^2+1) &= (x^2+1)[(y+z)^2+(yz-1)^2] \\ &\ge [x(y+z)+(yz-1)]^2 = 64.\end{aligned}$$

The equality holds for $xy + yz + zx = 9$ and $\dfrac{y+z}{x} = yz - 1$; that is, for

$$y + z = (yz-1)x = \frac{(yz-1)(9-yz)}{y+z},$$

$$(y+z)^2 + (yz-1)(yz-9) = 0,$$

$$(y-z)^2 + (yz-3)^2 = 0,$$

$$y = z = \pm\sqrt{3}.$$

In addition, from $xy + yz + zx = 9$, we get

$$x = y = z = \pm\sqrt{3}.$$

Therefore, the original inequality becomes an equality for

$$a = b = c = \pm 1.$$

Second Solution. We have

$$(x^2+1)(y^2+1)(z^2+1) - 64 = x^2y^2z^2 + \sum x^2y^2 + \sum x^2 - 63$$

$$= x^2y^2z^2 + \left(\sum xy\right)^2 - 2xyz\sum x + \left(\sum x\right)^2 - 2\sum xy - 63$$

$$= x^2y^2z^2 - 2xyz\sum x + \left(\sum x\right)^2 = \left(xyz - \sum x\right)^2 \ge 0.$$

Third Solution. We have

$$(3a^2+1)(3b^2+1)(3c^2+1) - 64 = 27a^2b^2c^2 + 9\sum a^2b^2 + 3\sum a^2 - 63$$

$$= 27a^2b^2c^2 + 9\left(\sum ab\right)^2 - 18abc\sum a + 3\left(\sum a\right)^2 - 6\sum ab - 63$$

$$= 27a^2b^2c^2 - 18abc\sum a + 3\left(\sum a\right)^2 = 3\left(3abc - \sum a\right)^2 \ge 0.$$

□

P 2.5. *If a and b are real numbers, then*

$$3(1-a+a^2)(1-b+b^2) \ge 2(1-ab+a^2b^2).$$

(Titu Andreescu, 2006)

Solution. We write the inequality as follows:

$$(3-3b+b^2)a^2 - (3-5b+3b^2)a + 1 - 3b + 3b^2 \ge 0.$$

Since

$$3-3b+b^2 > \frac{9}{4} - 3b + b^2 = \frac{(3-2b)^2}{4} \ge 0,$$

$$1-3b+3b^2 > \frac{3}{4} - 3b + 3b^2 = \frac{3(1-2b)^2}{4} \ge 0,$$

$$3-5b+3b^2 > \frac{25}{12} - 5b + 3b^2 = \frac{(5-6b)^2}{12} \ge 0,$$

it suffices to consider the case $a > 0$. By the AM-GM, we have

$$(3-3b+b^2)a^2 + 1 - 3b + 3b^2 \ge 2a\sqrt{(3-3b+b^2)(1-3b+3b^2)}.$$

Thus, we only need to show that

$$2\sqrt{(3-3b+b^2)(1-3b+3b^2)} \ge 3-5b+3b^2.$$

This is true if

$$4(3-3b+b^2)(1-3b+3b^2) \ge (3-5b+3b^2)^2,$$

which is equivalent to

$$3(b^2-3b+1)^2 \ge 0.$$

The equality occurs for $a > 0$, $(3-3b+b^2)a^2 = 1-3b+3b^2$ and $b^2-3b+1 = 0$. Since

$$a^2 = \frac{1-3b+3b^2}{3-3b+b^2} = \frac{(-b^2)+3b^2}{3-3b+(3b-1)} = b^2,$$

the equality occurs for

$$a = b = \frac{3 \pm \sqrt{5}}{2}.$$

□

P 2.6. *If a, b, c are real numbers, then*

$$3(1-a+a^2)(1-b+b^2)(1-c+c^2) \ge 1+abc+a^2b^2c^2.$$

(Vasile Cîrtoaje and Mircea Lascu, 1989)

First Solution. From the identity

$$2(1-a+a^2)(1-b+b^2) = 1+a^2b^2+(a-b)^2+(1-a)^2(1-b)^2,$$

it follows that

$$2(1-a+a^2)(1-b+b^2) \ge 1+a^2b^2.$$

Thus, it is enough to prove that

$$3(1+a^2b^2)(1-c+c^2) \ge 2(1+abc+a^2b^2c^2).$$

This inequality is equivalent to

$$(3+a^2b^2)c^2-(3+2ab+3a^2b^2)c+1+3a^2b^2 \ge 0,$$

$$[2(3+a^2b^2)c-3-2ab-3a^2b^2]^2+3(1-ab)^4 \ge 0.$$

The equality holds for $a=b=c=1$.

Second Solution. Write the required inequality as

$$3(1-a+a^2)(1-b+b^2)(1-c+c^2)-abc \ge 1+a^2b^2c^2.$$

Replacing a, b, c with $|a|, |b|, |c|$, respectively, the left side of this inequality remains unchanged or decreases, while the right side remains unchanged. Therefore, it suffices to prove the inequality only for $a, b, c \ge 0$. For $a=b=c$, the inequality is true since

$$3(1-a+a^2)^3-(1+a^3+a^6) = (1-a)^4(2-a+2a^2) \ge 0.$$

Multiplying the inequalities

$$\sqrt[3]{3}(1-a+a^2) \ge \sqrt[3]{1+a^3+a^6},$$

$$\sqrt[3]{3}(1-b+b^2) \ge \sqrt[3]{1+b^3+b^6},$$

$$\sqrt[3]{3}(1-c+c^2) \ge \sqrt[3]{1+c^3+c^6},$$

we get

$$3(1-a+a^2)(1-b+b^2)(1-c+c^2) \ge \sqrt[3]{(1+a^3+a^6)(1+b^3+b^6)(1+c^3+c^6)}.$$

Therefore, it suffices to prove that

$$\sqrt[3]{(1+a^3+a^6)(1+b^3+b^6)(1+c^3+c^6)} \ge 1+abc+a^2b^2c^2,$$

which follows immediately from Hölder's inequality.

□

P 2.7. *If a, b, c are real numbers, then*

$$(a^2+b^2+c^2)^3 \ge (a+b+c)(ab+bc+ca)(a^3+b^3+c^3).$$

(Vasile Cîrtoaje, 2007)

Solution. Substituting a, b, c by $|a|, |b|, |c|$, respectively, the left side of the inequality remains unchanged, while the right side either remains unchanged or increases. Therefore, it suffices to prove the inequality only for $a, b, c \ge 0$. Let

$$p = a+b+c, \quad q = ab+bc+ca.$$

Since

$$q^2 - 3abcp = \frac{a^2(b-c)^2 + b^2(c-a)^2 + c^2(a-b)^2}{2} \ge 0,$$

we have

$$(a+b+c)(a^3+b^3+c^3) = p(p^3-3pq+3abc) \le p^4 - 3p^2q + q^2.$$

Thus, it suffices to show that

$$(p^2-2q)^3 \ge q(p^4-3p^2q+q^2),$$

which is equivalent to the obvious inequality

$$(p^2-3q)^2(p^2-q) \ge 0.$$

The equality holds for $a = b = c$.

□

P 2.8. *If a, b, c are real numbers, then*

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2) \ge [ab(a+b)+bc(b+c)+ca(c+a)-2abc]^2.$$

(Vo Quoc Ba Can, 2009)

Solution. Since

$$(a^2+b^2)(a^2+c^2) = (a^2+bc)^2 + (ab-ac)^2$$

and

$$2(b^2+c^2) = (b+c)^2 + (b-c)^2,$$

the required inequality follows by applying the Cauchy-Schwarz inequality as follows

$$\begin{aligned} 2(a^2+b^2)(b^2+c^2)(c^2+a^2) &\ge [(a^2+bc)(b+c)+(ab-ac)(b-c)]^2 \\ &= [ab(a+b)+bc(b+c)+ca(c+a)-2abc]^2. \end{aligned}$$

The equality holds when two of a, b, c are equal.

□

P 2.9. *If a, b, c are real numbers, then*

$$(a^2+1)(b^2+1)(c^2+1) \ge 2(ab+bc+ca).$$

First Solution. Substituting a, b, c by $|a|, |b|, |c|$, respectively, the left side of this inequality remains unchanged, while the right side remains unchanged or increases. Therefore, it suffices to prove the inequality only for $a, b, c \ge 0$. Without loss of generality, assume that $a \ge b \ge c \ge 0$. Since

$$2(ab+bc+ca) \le 3a(b+c) \le \frac{3(a^2+1)(b+c)}{2},$$

it suffices to prove that

$$2(b^2+1)(c^2+1) \ge 3(b+c),$$

which is equivalent to

$$2(b+c)^2 - 3(b+c) + 2(bc-1)^2 \ge 0.$$

Case 1: $4bc \le 1$. We have

$$2(b+c)^2 - 3(b+c) + 2(bc-1)^2 = 2\left(b+c-\frac{3}{4}\right)^2 + \frac{(1-4bc)(7-4bc)}{8} \ge 0.$$

Case 2: $4bc \ge 1$. We get the required inequality by summing

$$\frac{9(b+c)^2}{8} - 3(b+c) + 2 \ge 0,$$

and

$$\frac{7(b+c)^2}{8} + 2b^2c^2 - 4bc \ge 0.$$

We have

$$\frac{9(b+c)^2}{8} - 3(b+c) + 2 = \frac{[3(b+c)-4]^2}{8} \ge 0$$

and

$$\frac{7(b+c)^2}{8} + 2b^2c^2 - 4bc \ge \frac{7bc}{2} + 2b^2c^2 - 4bc = \frac{bc(4bc-1)}{2} \ge 0.$$

For $a \ge b \ge c \ge 0$, the equality holds only if

$$2(ab+bc+ca) = 3a(b+c) = \frac{3(a^2+1)(b+c)}{2};$$

that is, only if either $b = c = 0$ or $a = 1$ and $b + c = 2bc$. If $b = c = 0$, then the original inequality becomes $a^2 + 1 \ge 0$, which is strict. If $a = 1$ and $b + c = 2bc$, then from

$$(a^2+1)(b^2+1)(c^2+1) = 2(ab+bc+ca)$$

we get

$$\frac{b+c}{2} = bc = \frac{5 \pm \sqrt{5}}{10} < 1.$$

This is not possible because from $2bc = b + c \geq 2\sqrt{bc}$ we get $bc \geq 1$. Therefore, the original inequality is strict (without equality).

Second Solution. Write the inequality as

$$(b^2+1)(c^2+1)\left[a - \frac{b+c}{(b^2+1)(c^2+1)}\right]^2 + A \geq 0,$$

where

$$A = (b^2+1)(c^2+1) - 2bc - \frac{(b+c)^2}{(b^2+1)(c^2+1)}.$$

We need to show that $A \geq 0$. By virtue of the Cauchy-Schwarz inequality,

$$(b^2+1)(c^2+1) \geq (b+c)^2.$$

Then,

$$A \geq (b^2+1)(c^2+1) - 2bc - 1 = b^2c^2 + (b-c)^2 \geq 0.$$

The equality holds only if $b^2c^2 + (b-c)^2 = 0$; that is, only if $b = c = 0$. If $b = c = 0$, then the original inequality becomes $a^2 + 1 \geq 0$, which is strict.

□

P 2.10. *If a, b, c are real numbers, then*

$$(a^2+1)(b^2+1)(c^2+1) \geq \frac{5}{16}(a+b+c+1)^2.$$

(Vasile Cîrtoaje, 2006)

First Solution. Since the equality holds for for

$$a = b = c = \frac{1}{2},$$

we replace a, b, c respectively by $a/2, b/2, c/2$. Thus, the inequality becomes

$$(a^2+4)(b^2+4)(c^2+4) \geq 5(a+b+c+2)^2,$$

with equality for $a = b = c = 1$. To prove this, we apply the Cauchy-Schwarz inequality in the form

$$(a^2+4)\left[1 + \left(\frac{b+c+2}{2}\right)^2\right] \geq (a+b+c+2)^2.$$

Therefore, it suffices to prove that

$$(b^2+4)(c^2+4) \geq 5\left[1+\left(\frac{b+c+2}{2}\right)^2\right].$$

This inequality is equivalent to

$$11(b+c)^2-20(b+c)+4b^2c^2-32bc+24 \geq 0.$$

Since

$$4b^2c^2-8bc+4=4(bc-1)^2 \geq 0,$$

it suffices to show that

$$11(b+c)^2-20(b+c)-24bc+20 \geq 0.$$

Indeed,

$$\begin{aligned}11(b+c)^2-20(b+c)-24bc+20 &\geq 11(b+c)^2-20(b+c)-6(b+c)^2+20\\ &=5(b+c-2)^2 \geq 0.\end{aligned}$$

Second Solution. Among a^2, b^2, c^2 there are two either less than or equal to $\frac{1}{4}$, or greater than or equal to $\frac{1}{4}$. Let b^2 and c^2 be these numbers; that is,

$$(4b^2-1)(4c^2-1) \geq 0.$$

Then, we have

$$\frac{16}{5}(b^2+1)(c^2+1)=5\left(\frac{4b^2-1}{5}+1\right)\left(\frac{4c^2-1}{5}+1\right)$$

$$\geq 5\left(\frac{4b^2-1}{5}+\frac{4c^2-1}{5}+1\right)=4b^2+4c^2+3.$$

Therefore, it suffices to prove that

$$(a^2+1)(4b^2+4c^2+3) \geq (a+b+c+1)^2.$$

Writing this inequality as

$$\left(a^2+\frac{1}{4}+\frac{1}{4}+\frac{1}{2}\right)(1+4b^2+4c^2+2) \geq (a+b+c+1)^2,$$

we recognize the Cauchy-Schwarz inequality.

□

P 2.11. *If a,b,c are real numbers, then*

(a) $$a^6+b^6+c^6-3a^2b^2c^2+2(a^2+bc)(b^2+ca)(c^2+ab)\ge 0;$$

(b) $$a^6+b^6+c^6-3a^2b^2c^2\ge (a^2-2bc)(b^2-2ca)(c^2-2ab).$$

Solution. (a) Since

$$(a^2+bc)(b^2+ca)(c^2+ab)=2a^2b^2c^2+\sum a^3b^3+abc\sum a^3,$$

we can write the desired inequality as follows

$$\sum a^6+2\sum a^3b^3+2abc\sum a^3+a^2b^2c^2\ge 0,$$

$$(\sum a^3)^2+2abc\sum a^3+a^2b^2c^2\ge 0,$$

$$(\sum a^3+abc)^2\ge 0.$$

The equality holds for

$$a^3+b^3+c^3+abc=0.$$

(b) Since

$$(a^2-2bc)(b^2-2ca)(c^2-2ab)=-7a^2b^2c^2-2\sum a^3b^3+4abc\sum a^3,$$

we can write the desired inequality as follows

$$\sum a^6+2\sum a^3b^3-4abc\sum a^3+4a^2b^2c^2\ge 0,$$

$$(\sum a^3)^2-4abc\sum a^3+4a^2b^2c^2\ge 0,$$

$$(\sum a^3-2abc)^2\ge 0.$$

The equality holds for

$$a^3+b^3+c^3-2abc=0.$$

□

P 2.12. *If a,b,c are real numbers, then*

$$\frac{2}{3}(a^6+b^6+c^6)+a^3b^3+b^3c^3+c^3a^3+abc(a^3+b^3+c^3)\ge 0.$$

Solution. Write the inequality as follows

$$\frac{4}{3}(a^6+b^6+c^6)+2(a^3b^3+b^3c^3+c^3a^3)+2abc(a^3+b^3+c^3)\ge 0,$$

$$\frac{1}{3}(a^6+b^6+c^6)+(a^3+b^3+c^3)^2+2abc(a^3+b^3+c^3)\ge 0.$$

By virtue of the AM-GM inequality, we have

$$\frac{1}{3}(a^6+b^6+c^6)\ge a^2b^2c^2.$$

Therefore, it suffices to show that

$$a^2b^2c^2+(a^3+b^3+c^3)^2+2abc(a^3+b^3+c^3)\ge 0,$$

which is equivalent to

$$(abc+a^3+b^3+c^3)^2\ge 0.$$

The equality holds for $-a=b=c$ (or any cyclic permutation).

□

P 2.13. *If a,b,c are real numbers, then*

$$4(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)\ge (a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2009)

Solution. Using the identity

$$4xy=(x+y)^2-(x-y)^2,$$

we have

$$\begin{aligned}4(a^2+ab+b^2)(a^2+ac+c^2)=&[(2a^2+ab+ac+2bc)+(b-c)^2]^2\\&-(a+b+c)^2(b-c)^2\\=&(2a^2+ab+ac+2bc)^2+3a^2(b-c)^2.\end{aligned}$$

From this result and

$$4(b^2+bc+c^2)=(b-c)^2+3(b+c)^2,$$

we get

$$16\prod(a^2+ab+b^2)=\left[(2a^2+ab+ac+2bc)^2+3a^2(b-c)^2\right]\left[(b-c)^2+3(b+c)^2\right].$$

Next, the Cauchy-Schwarz inequality gives

$$16\prod(a^2+ab+b^2)\ge\left[(2a^2+ab+ac+2bc)(b-c)+3a(b-c)(b+c)\right]^2$$
$$=4(b-c)^2(a-b)^2(a-c)^2.$$

The equality holds for

$$ab(a+b)+bc(b+c)+ca(c+a)=0.$$

Remark. The inequality is a consequence of the identity

$$4\prod(a^2+ab+b^2)=3[ab(a+b)+bc(b+c)+ca(c+a)]^2+(a-b)^2(b-c)^2(c-a)^2.$$

□

P 2.14. *If a,b,c are real numbers, then*

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)\ge 3(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2).$$

(Gabriel Dospinescu, 2009)

Solution *(by Vo Quoc Ba Can).* As we have shown in the proof of the preceding P 2.13,

$$16\prod(a^2+ab+b^2)=\left[(2a^2+ab+ac+2bc)^2+3a^2(b-c)^2\right]\left[3(b+c)^2+(b-c)^2\right].$$

Thus, by the Cauchy-Schwarz inequality, we get

$$16\prod(a^2+ab+b^2)\ge 3[(b+c)(2a^2+ab+ac+2bc)+a(b-c)^2]^2$$
$$=12[(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)]^2.$$

To prove the desired inequality, it suffices to show that

$$[(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)]^2\ge 4(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2).$$

Indeed, this is equivalent to

$$[(a^2b+b^2c+c^2a)-(ab^2+bc^2+ca^2)]^2\ge 0,$$

$$(a-b)^2(b-c)^2(c-a)^2\ge 0.$$

The equality holds when two of a,b,c are equal.

Remark. The inequality is a consequence of the identity

$$\prod(a^2+ab+b^2)=3(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)+(a-b)^2(b-c)^2(c-a)^2.$$

□

P 2.15. *If a, b, c are real numbers such that $abc > 0$, then*

$$4\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right) \ge 9(a+b+c).$$

Solution. It suffices to show that

$$4(a^2+1)(b^2+1)(c^2+1) \ge 9abc(a+b+c)$$

for any real a, b, c.

First Solution. It is easy to check that the equality occurs for $a = b = c = \sqrt{2}$. Therefore, using the notation

$$a = x\sqrt{2}, \quad b = y\sqrt{2}, \quad c = z\sqrt{2},$$

the inequality can be written as

$$(2x^2+1)(2y^2+1)2z^2+1) \ge 9xyz(x+y+z),$$

with equality for $x = y = z = 1$. Since

$$(xy+yz+zx)^2 - 3xyz(x+y+z) = \frac{1}{2}\sum x^2(y-z)^2 \ge 0,$$

it suffices to prove the stronger inequality

$$(2x^2+1)(2y^2+1)(2z^2+1) \ge 3(xy+yz+zx)^2.$$

Let

$$A = (y^2-1)(z^2-1), \quad B = (z^2-1)(x^2-1), \quad C = (x^2-1)(y^2-1).$$

From

$$ABC = (x^2-1)^2(y^2-1)^2(z^2-1)^2 \ge 0,$$

it follows that at least one of A, B, C is nonnegative. Due to symmetry, assume that

$$A = (y^2-1)(z^2-1) \ge 0.$$

Applying the Cauchy-Schwarz inequality, we have

$$(xy+yz+zx)^2 \le (x^2+1+x^2)(y^2+y^2z^2+z^2).$$

Therefore, it suffices to show that

$$(2y^2+1)(2z^2+1) \ge 3(y^2+y^2z^2+z^2),$$

which reduces to the obvious inequality

$$(y^2-1)(z^2-1) \ge 0.$$

Second Solution. Since

$$4(b^2+1)(c^2+1)-3[(b+c)^2+b^2c^2]=(b-c)^2+(bc-2)^2\geq 0,$$

it suffices to show that

$$(a^2+1)[(b+c)^2+b^2c^2]\geq 3abc(a+b+c).$$

Applying the Cauchy-Schwarz inequality, we get

$$(a^2+1)[(b+c)^2+b^2c^2]\geq [a(b+c)+bc]^2\geq 3abc(a+b+c).$$

□

P 2.16. *If a,b,c are real numbers, then*

(a) $$(a^2+2bc)(b^2+2ca)(c^2+2ab)\leq (a^2+b^2+c^2)(ab+bc+ca)^2;$$

(b) $$(2a^2+bc)(2b^2+ca)(2c^2+ab)\leq (a+b+c)^2(a^2b^2+b^2c^2+c^2a^2).$$

(Vasile Cîrtoaje, 2005)

Solution. (a) Let $q=ab+bc+ca$. Since

$$a^2+2bc=q+(a-b)(a-c),$$

$$b^2+2ca=q+(b-c)(b-a),$$

$$c^2+2ab=q+(c-a)(c-b),$$

we can rewrite the required inequality as follows

$$[q+(a-b)(a-c)][q+(b-c)(b-a)][q+(c-a)(c-b]\leq q^2(a^2+b^2+c^2),$$

$$q^3+q^2\sum(a-b)(a-c)-(a-b)^2(b-c)^2(c-a)^2\leq q^2(a^2+b^2+c^2).$$

Since

$$\sum(a-b)(a-c)=a^2+b^2+c^2-q,$$

the inequality reduces to the obvious form

$$(a-b)^2(b-c)^2(c-a)^2\geq 0.$$

The equality holds for $a=b$, or $b=c$, or $c=a$.

(b) For $a=0$, the required inequality reduces to

$$b^2c^2(b-c)^2\geq 0.$$

Otherwise, for $abc\neq 0$, the inequality follows from the inequality in (a) by substituting a, b, c with $1/a$, $1/b$, $1/c$, respectively. The equality occurs for $a=b$, or $b=c$, or $c=a$.

□

P 2.17. *If a, b, c are real numbers such that*

$$ab + bc + ca \geq 0,$$

then

$$27(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) \leq (a + b + c)^6.$$

Solution. In virtue of the AM-GM inequality, we have

$$\begin{aligned}(a + b + c)^6 &= [a^2 + b^2 + c^2 + (ab + bc + ca) + (ab + bc + ca)]^3 \\ &\geq 27(a^2 + b^2 + c^2)(ab + bc + ca)^2.\end{aligned}$$

Thus, the required inequality follows immediately from the inequality (a) in P 2.16. The equality holds for $a = b = c$.

□

P 2.18. *If a, b, c are real numbers such that*

$$a^2 + b^2 + c^2 = 2,$$

then

$$(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) + 2 \geq 0.$$

(Vasile Cîrtoaje, 2011)

Solution *(by Vo Quoc Ba Can)*. If a, b, c have the same sign, then the inequality is trivial. Otherwise, since the inequality is symmetric and does not change by substituting $-a, -b, -c$ for a, b, c, respectively, it suffices to consider the case where $a \leq 0$ and $b, c \geq 0$. Replacing now $-a$ with a, we need to prove the inequality

$$(a^2 + 2bc)(b^2 - 2ac)(c^2 - 2ab) + 2 \geq 0 \tag{*}$$

for all $a, b, c \geq 0$ satisfying

$$a^2 + b^2 + c^2 = 2.$$

If $b^2 - 2ac$ and $c^2 - 2ab$ have the same sign, then the inequality is also trivial. Due to symmetry in b and c, we may assume that

$$b^2 - 2ac \geq 0 \geq c^2 - 2ab.$$

On the other hand, it is easy to check that (*) becomes an equality for $a = b = 1$ and $c = 0$, when

$$a^2 + 2bc = b^2 - 2ac = ab - \frac{c^2}{2}.$$

Then, we rewrite the inequality (*) in the form

$$(a^2 + 2bc)(b^2 - 2ac)\left(ab - \frac{c^2}{2}\right) \leq 1.$$

Using the AM-GM inequality, we have

$$27(a^2+2bc)(b^2-2ac)\left(ab-\frac{c^2}{2}\right)\le\left[(a^2+2bc)+(b^2-2ac)+\left(ab-\frac{c^2}{2}\right)\right]^3.$$

Thus, it suffices to prove that

$$(a^2+2bc)+(b^2-2ac)+\left(ab-\frac{c^2}{2}\right)\le 3.$$

This inequality can be written in the homogeneous form

$$2(a^2+2bc)+2(b^2-2ac)+(2ab-c^2)\le 3(a^2+b^2+c^2),$$

which is equivalent to

$$(a-b+2c)^2\ge 0.$$

The original inequality is an equality for $a=-1$, $b=1$ and $c=0$ (or any permutation).

Remark. In the same manner, we can prove the following generalization.

- *Let a,b,c be real numbers such that $a^2+b^2+c^2=2$. If $0<k\le 2$, then*

$$(a^2+kbc)(b^2+kca)(c^2+kab)+k\ge 0.$$

□

P 2.19. *If a,b,c are real numbers such that*

$$a+b+c=3,$$

then

$$3(a^4+b^4+c^4)+a^2+b^2+c^2+6\ge 6(a^3+b^3+c^3).$$

(Vasile Cîrtoaje, 2006)

Solution. Write the inequality as $F(a,b,c)\ge 0$, where

$$F(a,b,c)=3(a^4+b^4+c^4)+(a^2+b^2+c^2)-6(a^3+b^3+c^3)+6.$$

Due to symmetry, we may assume that $a\le b\le c$. To prove the required inequality, we use the mixing variables method. More precisely, we show that

$$F(a,b,c)\ge F(a,x,x)\ge 0,$$

where $x=(b+c)/2$, $x\ge 1$. We have

$$F(a,b,c)-F(a,x,x)=3(b^4+b^4-2x^4)+(b^2+c^2-2x^2)-6(b^3+c^3-2x^3)$$

$$= 3[(b^2+c^2)^2-4x^4]+6(x^4-b^2c^2)+(b^2+c^2-2x^2)-6(b^3+c^3-2x^3)$$

$$= (b^2+c^2-2x^2)[3(b^2+c^2+2x^2)+1]+6(x^2-bc)(x^2+bc)-12x(b^2+c^2-bc-x^2).$$

Since

$$b^2+c^2-2x^2=\frac{1}{2}(b-c)^2,$$

$$x^2-bc=\frac{1}{4}(b-c)^2,$$

$$b^2+c^2-bc-x^2=\frac{3}{4}(b-c)^2,$$

we get

$$F(a,b,c)-F(a,x,x)=\frac{1}{2}(b-c)^2[3(b^2+c^2+2x^2)+1+3(x^2+bc)-18x]$$

$$=\frac{1}{2}(b-c)^2[3(x^2-bc)+18x(x-1)+1]\geq 0.$$

Also,

$$F(a,x,x)=F(3-2x,x,x)=6(x-1)^2(3x-4)^2\geq 0.$$

This completes the proof. The equality holds for $a=b=c=1$, and for $a=1/3$ and $b=c=4/3$ (or any cyclic permutation).

□

P 2.20. *If a,b,c are real numbers such that*

$$abc=1,$$

then

$$3(a^2+b^2+c^2)+2(a+b+c)\geq 5(ab+bc+ca).$$

Solution. Without loss of generality, assume that $a\geq b\geq c$. From $abc=1$, it follows that either $a,b,c>0$ or $a>0$ and $b,c<0$.

Case 1: $a,b,c>0$. Let

$$p=a+b+c, \quad q=ab+bc+ca.$$

The AM-GM inequality

$$a+b+c\geq 3\sqrt[3]{abc}$$

gives $p\geq 3$, while Schur's inequality

$$p^3+9abc\geq 4pq$$

gives

$$q \le \frac{p^3+9}{4p}.$$

Write the required inequality as

$$3(p^2-2q)+2p \ge 5q,$$

$$3p^2+2p \ge 11q.$$

This is true since

$$3p^2+2p-11q \ge 3p^2+2p-\frac{11(p^3+9)}{4p} = \frac{(p-3)(p^2+11p+33)}{4p} \ge 0.$$

Case 2: $a>0$ and $b,c<0$. Substituting $-b$ for b and $-c$ for c, we need to prove that

$$3(a^2+b^2+c^2)+2a+5a(b+c) \ge 2(b+c)+5bc$$

for $a,b,c>0$ satisfying $abc=1$. It suffices to show that

$$3(b^2+c^2)-5bc \ge (2-5a)(b+c).$$

Since

$$\frac{3(b^2+c^2)-5bc}{b+c} \ge \frac{b+c}{4} \ge \frac{\sqrt{bc}}{2} = \frac{1}{2\sqrt{a}},$$

we only need to prove that

$$\frac{1}{2\sqrt{a}} \ge 2-5a.$$

Indeed, by the AM-GM inequality, we get

$$5a+\frac{1}{2\sqrt{a}} = 5a+\frac{1}{4\sqrt{a}}+\frac{1}{4\sqrt{a}} \ge 3\sqrt[3]{5a\cdot\frac{1}{4\sqrt{a}}\cdot\frac{1}{4\sqrt{a}}} > 2.$$

This completes the proof. The equality holds for $a=b=c=1$.

□

P 2.21. *If a,b,c are real numbers such that*

$$abc=1,$$

then

$$a^2+b^2+c^2+6 \ge \frac{3}{2}\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

Solution. Substituting a, b, c by $|a|, |b|, |c|$, respectively, the left side of the inequality remains unchanged, while the right side either remains unchanged or increases. Therefore, it suffices to prove the inequality only for $a, b, c > 0$.

First Solution. Write the inequality in the form

$$3(6x^2 - 3x + 4) \geq 7(ab + bc + ca),$$

where

$$x = \frac{a+b+c}{3}.$$

By virtue of the AM-GM inequality, we have $x \geq 1$. The third degree Schur's inequality states that

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

which is equivalent to

$$ab + bc + ca \leq \frac{3(3x^3+1)}{4x}.$$

Therefore, it suffices to show that

$$3(6x^2 - 3x + 4) \geq \frac{21(3x^3+1)}{4x}.$$

This inequality reduces to

$$(x-1)(3x^2 - 9x + 7) \geq 0,$$

which is true because

$$3x^2 - 9x + 7 = 3(x - \frac{3}{2})^2 + \frac{1}{4} > 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. Use the mixing variables technique. Let

$$F(a,b,c) = a^2 + b^2 + c^2 + 6 - \frac{3}{2}\left(a + b + c + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

Assume that

$$a = \min\{a, b, c\}$$

and show that

$$F(a,b,c) \geq F(a,x,x) \geq 0,$$

where

$$x = \sqrt{bc}, \quad x \geq 1.$$

We have

$$\begin{aligned}F(a,b,c)-F(a,x,x) &= (b-c)^2-\frac{3}{2}\left(b+c-2\sqrt{bc}+\frac{1}{b}+\frac{1}{c}-\frac{2}{\sqrt{bc}}\right)\\ &= \frac{1}{2}(\sqrt{b}-\sqrt{c})^2\left[2(\sqrt{b}+\sqrt{c})^2-3-\frac{3}{bc}\right]\\ &\ge \frac{1}{2}(\sqrt{b}-\sqrt{c})^2\left(8\sqrt{bc}-3-\frac{3}{bc}\right)\\ &\ge \frac{1}{2}(\sqrt{b}-\sqrt{c})^2(8-3-3)\ge 0\end{aligned}$$

and

$$\begin{aligned}F(a,x,x) &= F(\frac{1}{x^2},x,x)\\ &= \frac{x^6-6x^5+12x^4-6x^3-3x^2+2}{2x^4}\\ &= \frac{(x-1)^2(x^4-4x^3+3x^2+4x+2}{2x^4}\\ &= \frac{(x-1)^2[(x^2-2x-1)^2+x^2+1]}{2x^4}\ge 0.\end{aligned}$$

□

P 2.22. *If a,b,c are real numbers, then*

$$(1+a^2)(1+b^2)(1+c^2)+8abc\ge\frac{1}{4}(1+a)^2(1+b)^2(1+c)^2.$$

Solution. It is easy to check that

$$(1+a^2)(1+b^2)(1+c^2)+8abc=(1+abc)^2+(a+bc)^2+(b+ca)^2+(c+ab)^2.$$

Thus, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(1+a^2)(1+b^2)(1+c^2)+8abc &\ge \frac{[(1+abc)+(a+bc)+(b+ca)+(c+ab)]^2}{4}\\ &= \frac{1}{4}(1+a)^2(1+b)^2(1+c)^2.\end{aligned}$$

The equality holds for $b=c=1$ (or any cyclic permutation), and also for $a=b=c=-1$.

□

P 2.23. *Let a, b, c be real numbers such that*

$$a+b+c=0.$$

Prove that

$$a^{12}+b^{12}+c^{12} \geq \frac{2049}{8}a^4b^4c^4.$$

Solution. Consider only the nontrivial case $abc \neq 0$, and rewrite the inequality as follows

$$a^{12}+b^{12}+(a+b)^{12} \geq \frac{2049}{8}a^4b^4(a+b)^4,$$

$$(a^6+b^6)^2-2a^6b^6+(a^2+b^2+2ab)^6 \geq \frac{2049}{8}a^4b^4(a^2+b^2+2ab)^2.$$

Let us denote

$$d=\frac{a^2+b^2}{ab}, \quad |d| \geq 2.$$

Since

$$a^6+b^6=(a^2+b^2)^3-3a^2b^2(a^2+b^2),$$

the inequality can be restated as

$$(d^3-3d)^2-2+(d+2)^6 \geq \frac{2049}{8}(d+2)^2,$$

which is equivalent to

$$(d-2)(2d+5)^2(4d^3+12d^2+87d+154) \geq 0.$$

Since this inequality is obvious for $d \geq 2$, we only need to show that

$$4d^3+12d^2+87d+154 \leq 0$$

for $d \leq -2$. Indeed,

$$4d^3+12d^2+87d+154 < 4d^3+12d^2+85d+154$$
$$=(d+2)[(2d+1)^2+76] \leq 0.$$

The equality holds for $a=b=-c/2$ (or any cyclic permutation).

□

P 2.24. *If a,b,c are real numbers such that $abc \geq 0$, then*

$$a^2+b^2+c^2+2abc+4 \geq 2(a+b+c)+ab+bc+ca.$$

(Vasile Cîrtoaje, 2012)

Solution. Let us denote

$$x = a(b-1)(c-1), \quad y = b(c-1)(a-1), \quad z = c(a-1)(b-1).$$

Since

$$xyz = abc(a-1)^2(b-1)^2(c-1)^2 \ge 0,$$

at least one of x, y, z is nonnegative; let

$$a(1-b)(1-c) \ge 0.$$

Since

$$abc \ge a(b+c-1),$$

it suffices to show that

$$a^2 + b^2 + c^2 + 2a(b+c-1) + 4 \ge 2(a+b+c) + ab + bc + ca,$$

which is equivalent to

$$a^2 - (4-b-c)a + b^2 + c^2 - bc - 2(b+c) + 4 \ge 0,$$

$$\left(a - 2 + \frac{b+c}{2}\right)^2 + \frac{3}{4}(b-c)^2 \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = 2$ (or any cyclic permutation).

□

P 2.25. *Let a, b, c be real numbers such that*

$$a + b + c = 3.$$

(a) If $a, b, c \ge -3$, then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \ge \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

(b) If $a, b, c \ge -7$, then

$$\frac{1-a}{(1+a)^2} + \frac{1-b}{(1+b)^2} + \frac{1-c}{(1+c)^2} \ge 0.$$

(Vasile Cîrtoaje, 2012)

Solution. Assume that

$$a = \min\{a, b, c\},$$

and denote

$$t = \frac{b+c}{2},$$

$$E(a,b,c) = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c},$$

$$F(a,b,c) = \frac{1-a}{(1+a)^2} + \frac{1-b}{(1+b)^2} + \frac{1-c}{(1+c)^2}.$$

(a) From $a, b, c \geq -3$ and $a + b + c = 3$, it follows that

$$-3 \leq a \leq \frac{a+b+c}{3} = 1.$$

We will show that

$$E(a,b,c) \geq E(a,t,t) \geq 0.$$

We have

$$\begin{aligned}
E(a,b,c) - E(a,t,t) &= \frac{1}{b^2} + \frac{1}{c^2} - \frac{2}{t^2} - \left(\frac{1}{b} + \frac{1}{c} - \frac{2}{t}\right) \\
&= \frac{(b-c)^2(b^2+c^2+4bc)}{b^2c^2(b+c)^2} - \frac{(b-c)^2}{bc(b+c)} \\
&= \frac{(b-c)^2[(b+c)^2 - bc(b+c-2)]}{b^2c^2(b+c)^2}.
\end{aligned}$$

Since

$$\begin{aligned}
(b+c)^2 - bc(b+c-2) &= (b+c)^2 - bc(1-a) \\
&\geq (b+c)^2 - \frac{(b+c)^2(1-a)}{4} \\
&= \frac{(b+c)^2(3+a)}{4} \geq 0,
\end{aligned}$$

we have

$$E(a,b,c) - E(a,t,t) \geq 0.$$

Also,

$$E(a,t,t) = \frac{1-a}{a^2} + \frac{2(1-t)}{t^2} = \frac{3(a-1)^2(a+3)}{a^2(3-a)^2} \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = -3$ and $b = c = 3$ (or any cyclic permutation).

(b) From

$$t \geq \frac{a+b+c}{3} = 1, \quad t = \frac{3-a}{2} \leq 5,$$

it follows that

$$t \in [1,5].$$

We will show that

$$F(a,b,c) \ge F(a,t,t) \ge 0.$$

Write the left inequality as follows:

$$\left[\frac{1-b}{(1+b)^2} - \frac{1-t}{(1+t)^2}\right] + \left[\frac{1-c}{(1+c)^2} - \frac{1-t}{(1+t)^2}\right] \ge 0,$$

$$(b-c)\left[\frac{(b-1)t-b-3}{(1+b)^2} - \frac{(c-1)t-c-3}{(1+c)^2}\right] \ge 0,$$

$$(b-c)^2[(3+b+c-bc)t+3(b+c)+bc] \ge 0,$$

$$(b-c)^2[2t^2+9t+5-bc(t-1)] \ge 0.$$

The last inequality is true since

$$2t^2+9t+5-bc(t-1) \ge 2t^2+9t+5-t^2(t-1) = (5-t)(1+t)^2 \ge 0.$$

Also, we have

$$\begin{aligned} F(a,t,t) &= \frac{1-a}{(1+a)^2} + \frac{2(1-t)}{(1+t)^2} \\ &= \frac{t-1}{2(2-t)^2} + \frac{2(1-t)}{(1+t)^2} \\ &= \frac{3(1-t)^2(5-t)}{2(2-t)^2(1+t)^2} \ge 0. \end{aligned}$$

The proof is completed. The equality occurs for $a=b=c=1$, and also for $a=-7$ and $b=c=5$ (or any cyclic permutation).

□

P 2.26. *If a,b,c are real numbers, then*

$$a^6+b^6+c^6-3a^2b^2c^2 \ge \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

(Sungyoon Kim, 2006)

Solution. Applying the Cauchy-Schwarz inequality, we have

$$\begin{aligned} a^6+b^6+c^6-3a^2b^2c^2 &= \frac{1}{2}(a^2+b^2+c^2)[(b^2-c^2)^2+(c^2-a^2)^2+(a^2-b^2)^2] \\ &\ge \frac{1}{2}[a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)]^2 \\ &= \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2. \end{aligned}$$

Thus, the proof is completed. The equality holds for $a=b=c$, and also for $a=0$ and $b+c=0$ (or any cyclic permutation).

□

P 2.27. *If a,b,c are real numbers, then*

$$\left(\frac{a^2+b^2+c^2}{3}\right)^3 \ge a^2b^2c^2+\frac{1}{16}(a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2011)

Solution *(by Vo Quoc Ba Can).* Without loss of generality, assume that b and c have the same sign; that is

$$bc \ge 0.$$

Let

$$x=\sqrt{\frac{b^2+c^2}{2}}.$$

Since

$$\begin{aligned}\left(\frac{a^2+b^2+c^2}{3}\right)^3-a^2b^2c^2 &= \left(\frac{a^2+2x^2}{3}\right)^3-a^2x^4+a^2(x^4-b^2c^2)\\ &=\frac{(a^2-x^2)^2(a^2+8x^2)}{27}+a^2(x^4-b^2c^2)\\ &=\frac{(2a^2-b^2-c^2)^2(a^2+4b^2+4c^2)}{108}+\frac{a^2(b^2-c^2)^2}{4},\end{aligned}$$

the desired inequality can be rewritten as

$$(2a^2-b^2-c^2)^2(a^2+4b^2+4c^2)\ge\frac{27}{4}(b-c)^2[(a-b)^2(a-c)^2-4a^2(b+c)^2].$$

According to the inequalities

$$x^2-y^2\le 2x(x+y)$$

and

$$2xy\le\frac{1}{2}(x+y)^2,$$

we have

$$\begin{aligned}(a-b)^2(a-c)^2-4a^2(b+c)^2 &\le 2(a-b)(a-c)[(a-b)(a-c)+2a(b+c)]\\ &=2(a^2-b^2)(a^2-c^2)\le\frac{1}{2}(2a^2-b^2-c^2)^2.\end{aligned}$$

Therefore, it suffices to show that

$$8(a^2+4b^2+4c^2)\ge 27(b-c)^2,$$

which is equivalent to the obvious inequality

$$8a^2+5b^2+5c^2+54bc\ge 0.$$

The equality holds for $a=b=c$, and for $-a=b=c$ (or any cyclic permutation).

□

P 2.28. *If a, b, c are real numbers, then*

$$(a^2+b^2+c^2)^3 \geq \frac{108}{5}a^2b^2c^2 + 2(a-b)^2(b-c)^2(c-a)^2.$$

(Vo Quoc Ba Can and Vasile Cîrtoaje, 2011)

Solution. Write the inequality as $f(a,b,c) \geq 0$, where

$$f(a,b,c) = (a^2+b^2+c^2)^3 - \frac{108}{5}a^2b^2c^2 - 2(a-b)^2(b-c)^2(c-a)^2.$$

Without loss of generality, assume that b and c have the same sign. Since $f(-a,-b,-c) = f(a,b,c)$, we may consider $b \geq 0$, $c \geq 0$. In addition, for $a > 0$, we have

$$f(a,b,c) - f(-a,b,c) = 8a(b+c)(a^2+bc)(b-c)^2 \geq 0.$$

Therefore, it suffices to prove the desired inequality for $a \leq 0$, $b \geq 0$, $c \geq 0$. For $b = c = 0$, the inequality is trivial. Otherwise, due to homogeneity, we may assume that $b + c = 1$. Denoting $x = bc$, we can write the desired inequality as follows:

$$(a^2+1-2x)^3 \geq \frac{108}{5}a^2x^2 + 2(1-2x)(a^2-a+x)2,$$

$$\frac{2}{5}(4a-5)^2x^2 + 2(a+1)(a^3-9a^2+5a-3)x + (a+1)^2(a^4-2a^3+4a^2-2a+1) \geq 0.$$

This inequality holds if

$$\frac{2}{5}(4a-5)^2(a^4-2a^3+4a^2-2a+1) \geq (a^3-9a^2+5a-3)^2.$$

Since

$$10(a^4-2a^3+4a^2-2a+1) = (a+1)^2 + (3a^2-4a+3)^2 \geq (3a^2-4a+3)^2,$$

it suffices to prove that

$$(4a-5)^2(3a^2-4a+3)^2 \geq 25(a^3-9a^2+5a-3)^2.$$

This is true for $a \leq 0$ if

$$(5-4a)(3a^2-4a+3) \geq 5(-a^3+9a^2-5a+3),$$

which reduces to

$$a(a+1)^2 \leq 0.$$

Thus, the proof is completed. The equality holds for $a = 0$ and $b + c = 0$ (or any cyclic permutation).

□

P 2.29. *If a, b, c are real numbers, then*

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2) \ge (a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2011)

First Solution. Since

$$2(a^2+b^2) = (a-b)^2 + (a+b)^2$$

and

$$(b^2+c^2)(c^2+a^2) = (ab+c^2)^2 + (bc-ac)^2,$$

by virtue of the Cauchy-Schwarz inequality, we have

$$\begin{aligned} 2(a^2+b^2)(b^2+c^2)(c^2+a^2) &\ge [(a-b)(ab+c^2)+(a+b)(bc-ac)]^2 \\ &= (a^2b+b^2c+c^2a-ab^2-bc^2-ca^2)^2 \\ &= (a-b)^2(b-c)^2(c-a)^2. \end{aligned}$$

This completes the proof. The equality holds for $(a-b)(bc-ac) = (a+b)(ab+c^2)$, which is equivalent to

$$(a+b+c)(ab+bc+ca) = 5abc.$$

Second Solution. Making the substitution

$$x = \sum ab^2 = ab^2+bc^2+ca^2, \quad y = \sum a^2b = a^2b+b^2c+c^2a,$$

we have

$$\sum a^2b^4 = (\sum ab^2)^2 - 2abc\sum a^2b = x^2 - 2abcy,$$

$$\sum a^4b^2 = (\sum a^2b)^2 - 2abc\sum ab^2 = y^2 - 2abcx,$$

hence

$$\begin{aligned} \prod(a^2+b^2) &= \sum a^2b^4 + \sum a^4b^2 + 2a^2b^2c^2 \\ &= x^2+y^2-2abc(x+y)+2a^2b^2c^2. \end{aligned}$$

Then, the desired inequality is equivalent to

$$2[x^2+y^2-2abc(x+y)+2a^2b^2c^2] \ge (x-y)^2,$$

$$(x+y)^2 - 4abc(x+y) + 4a^2b^2c^2 \ge 0,$$

$$(x+y-2abc)^2 \ge 0.$$

□

P 2.30. *If a,b,c are real numbers, then*

$$32(a^2+bc)(b^2+ca)(c^2+ab)+9(a-b)^2(b-c)^2(c-a)^2 \ge 0.$$

(Vasile Cîrtoaje, 2011)

Solution *(by Vo Quoc Ba Can).* For $a,b,c \ge 0$, the inequality is trivial. Otherwise, since the inequality is symmetric and does not change by substituting $-a,-b,-c$ for a,b,c, we may assume that $a \le 0$ and $b,c \ge 0$. Substituting $-a$ for a, we need to prove that

$$32(a^2+bc)(b^2-ac)(c^2-ab)+9(a+b)^2(a+c)^2(b-c)^2 \ge 0$$

for all $a,b,c \ge 0$. By the AM-GM inequality, we have

$$(a+b)^2(a+c)^2 = [a(b+c)+(a^2+bc)]^2 \ge 4a(b+c)(a^2+bc).$$

Thus, it suffices to prove that

$$32(a^2+bc)(b^2-ac)(c^2-ab)+36a(b+c)(a^2+bc)(b-c)^2 \ge 0,$$

which is true if

$$8(b^2-ac)(c^2-ab)+9a(b+c)(b-c)^2 \ge 0.$$

Since

$$\begin{aligned}(b^2-ac)(c^2-ab) &= bc(bc+a^2)-a(b^3+c^3)\\ &\ge 2abc\sqrt{bc}-a(b^3+c^3) = -a(b\sqrt{b}-c\sqrt{c})^2,\end{aligned}$$

it is enough to show that

$$9(b+c)(b-c)^2-8(b\sqrt{b}-c\sqrt{c})^2 \ge 0.$$

Using the notation

$$\sqrt{b}=x, \quad \sqrt{c}=y,$$

this inequality can be rewritten as

$$(x-y)^2[9(x^2+y^2)(x+y)^2-8(x^2+xy+y^2)^2] \ge 0,$$

It is true because, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}9(x^2+y^2)(x+y)^2 &= 9[(x-y)^2+2xy][(x-y)^2+4xy]\\ \ge 9[(x-y)^2+2\sqrt{2}xy]^2 &\ge 9\left[\frac{2\sqrt{2}}{3}(x-y)^2+2\sqrt{2}xy\right]^2\\ &= 8(x^2+xy+y^2)^2 \ge 0.\end{aligned}$$

The equality occurs when two of a,b,c are zero, and when $-a=b=c$ (or any cyclic permutation).

□

P 2.31. *If a, b, c are real numbers, then*

$$a^4(b-c)^2 + b^4(c-a)^2 + c^4(a-b)^2 \geq \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2011)

Solution. Since

$$\begin{aligned} b^4(c-a)^2 + c^4(a-b)^2 &\geq \frac{1}{2}[b^2(c-a) + c^2(a-b)]^2 \\ &= \frac{1}{2}(b-c)^2(bc-ca-ab)^2, \end{aligned}$$

it suffices to prove that

$$2a^4 + (ab-bc+ca)^2 \geq (a-b)^2(a-c)^2,$$

which is equivalent to

$$a^2(a^2-2bc+2ca+2ab) \geq 0.$$

Therefore, the desired inequality is true if

$$a^2-2bc+2ca+2ab \geq 0.$$

Indeed, from

$$\sum(a^2-2bc+2ca+2ab) = (a+b+c)^2 \geq 0,$$

due to symmetry, we may assume that $a^2-2bc+2ca+2ab \geq 0$. Thus, the proof is completed. The equality occurs when $a=b=c$, when two of a, b, c are equal to zero, and when $a=0$ and $b+c=0$ (or any cyclic permutation).

□

P 2.32. *If a, b, c are real numbers, then*

$$a^2(b-c)^4 + b^2(c-a)^4 + c^2(a-b)^4 \geq \frac{1}{2}(a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2011)

Solution. Let us denote

$$x = \sum ab^2 = ab^2+bc^2+ca^2, \quad y = \sum a^2b = a^2b+b^2c+c^2a.$$

Since

$$\sum a^2b^4 = (\sum ab^2)^2 - 2abc\sum a^2b = x^2-2abcy,$$

$$\sum a^4b^2 = (\sum a^2b)^2 - 2abc\sum ab^2 = y^2-2abcx,$$

$$\sum a^2b^2(a^2+b^2) = x^2+y^2-2abc(x+y),$$

we have

$$\begin{aligned}\sum a^2(b-c)^4 &= \sum a^2(b^4-4b^3c+6b^2c^2-4bc^3+c^4)\\ &= \sum a^2b^2(a^2+b^2)-4abc(\sum ab^2+\sum a^2b)+18a^2b^2c^2\\ &= x^2+y^2-6abc(x+y)+18a^2b^2c^2.\end{aligned}$$

Therefore, we can write the desired inequality as

$$x^2+y^2-6abc(x+y)+18a^2b^2c^2 \ge \frac{1}{2}(x-y)^2,$$

which is equivalent to the obvious inequality

$$(x+y-6abc)^2 \ge 0.$$

The equality holds for

$$a(b-c)^2+b(c-a)^2+c(a-b)^2 = 0.$$

□

P 2.33. *If a,b,c are real numbers, then*

$$a^2(b^2-c^2)^2+b^2(c^2-a^2)^2+c^2(a^2-b^2)^2 \ge \frac{3}{8}(a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2011)

Solution. We see that the inequality remains unchanged and the product

$$(a+b)(b+c)(c+a)$$

changes its sign by replacing a,b,c with $-a,-b,-c$, respectively. Thus, without loss of generality, we may assume that

$$(a+b)(b+c)(c+a) \ge 0.$$

According to this condition, at least one of a,b,c is nonnegative. So, we may consider $a \ge 0$, and hence

$$a(a+b)(b+c)(c+a) \ge 0.$$

Since

$$b^2(c^2-a^2)^2+c^2(a^2-b^2)^2 \ge \frac{1}{2}\left[b(c^2-a^2)+c(a^2-b^2)\right]^2 = \frac{1}{2}(b-c)^2(a^2+bc)^2,$$

it suffices to show that

$$2a^2(b+c)^2+(a^2+bc)^2 \ge \frac{3}{4}(a-b)^2(a-c)^2,$$

which is equivalent to

$$(a+b)(a+c)[a^2+5a(b+c)+bc] \ge 0,$$

$$(a+b)(a+c)[(a+b)(a+c)+4a(b+c)] \ge 0,$$

$$(a+b)^2(a+c)^2+4a(a+b)(b+c)(c+a) \ge 0.$$

Since the last inequality is clearly true, the proof is completed. The equality holds for $a=b=c$, for $-a=b=c$ (or any cyclic permutation), and for $b=c=0$ (or any cyclic permutation).

□

P 2.34. *If a,b,c are real numbers such that*

$$ab+bc+ca=3,$$

then

(a) $$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge 3(a+b+c)^2;$$

(b) $$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge \frac{3}{2}(a^2+b^2+c^2).$$

(Vasile Cîrtoaje, 1995)

Solution. Let

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc.$$

We have

$$\prod(b^2+bc+c^2)=\prod[(b+c)^2-bc]$$
$$=\prod(b+c)^2-\sum bc(a+b)^2(a+c)^2+abc\sum a(b+c)^2-a^2b^2c^2.$$

Since

$$\prod(b+c)^2=(pq-r)^2=r^2-2pqr+p^2q^2,$$

$$\sum bc(a+b)^2(a+c)^2=\sum bc(a^2+q)^2=r\sum a^3+2pqr+q^2$$
$$=r(3r+p^3-3pq)+2pqr+q^2=3r^2+(p^3-pq)r+q^3$$

and

$$abc\sum a(b+c)^2=r(3r+pq)=3r^2+pqr,$$

we get

$$\prod(b^2+bc+c^2)=(p^2-q)q^2-p^3r.$$

(a) Write the inequality as follows

$$3\prod(b^2+bc+c^2)\ge(a+b+c)^2(ab+bc+ca)^2,$$

$$(2p^2-3q)q^2-3p^3r\ge 0,$$

$$q^2(p^2-3q)+p^2(q^2-3pr)\ge 0,$$

$$q^2\sum(b-c)^2+p^2\sum a^2(b-c)^2\ge 0.$$

Clearly, the last inequality holds for all real a,b,c. The equality holds when $a=b=c=\pm 1$.

(b) Write the inequality in the homogeneous forms

$$2\prod(b^2+bc+c^2)\ge(a^2+b^2+c^2)(ab+bc+ca)^2,$$

$$2(p^2-q)q^2-2p^3r-(p^2-2q)q^2\ge 0,$$

$$p^2(q^2-2pr)\ge 0,$$

$$(a+b+c)^2(a^2b^2+b^2c^2+c^2a^2)\ge 0.$$

The equality holds when $a+b+c=0$ and $ab+bc+ca=3$.

□

P 2.35. *If a,b,c are real numbers, then*

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)\ge 3(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2).$$

(Vasile Cîrtoaje, 2011)

Solution. As we have shown in the proof of the preceding P 2.34,

$$\prod(b^2+bc+c^2)=(p^2-q)q^2-p^3r,$$

where

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc.$$

Thus, we can write the desired inequality as

$$(p^2-q)q^2-p^3r\ge 3q(q^2-2pr),$$

$$q^2(p^2-4q)+p(6q-p^2)r\ge 0.$$

Consider further two cases: $6q-p^2\ge 0$ and $6q-p^2\le 0$.

Case 1: $6q - p^2 \geq 0$. By Schur's inequality of degree four, we have

$$pr \geq \frac{1}{6}(p^2 - q)(4q - p^2).$$

Therefore, it suffices to show that

$$q^2(p^2 - 4q) + \frac{1}{6}(6q - p^2)(p^2 - q)(4q - p^2) \geq 0,$$

which is equivalent to the obvious inequality

$$(p^2 - 4q)^2(p^2 - 3q) \geq 0.$$

Case 2: $6q - p^2 \leq 0$. Since

$$pr \leq \frac{1}{3}q^2,$$

it suffices to show that

$$q^2(p^2 - 4q) + \frac{1}{3}(6q - p^2)q^2 \geq 0,$$

which is equivalent to the obvious inequality

$$q^2(p^2 - 3q) \geq 0.$$

This completes the proof. The inequality is an equality for $a = b = c$, for $a = 0$ and $b = c$ (or any cyclic permutation), and for $b = c = 0$ (or any cyclic permutation).

□

P 2.36. *If a, b, c are real numbers, not all of the same sign, then*

$$(a^2 + ab + b^2)(b^2 + bc + c^2)(c^2 + ca + a^2) \geq 3(ab + bc + ca)^3.$$

(Vasile Cîrtoaje, 2011)

Solution. Since the inequality is symmetric and does not change by substituting $-a, -b, -c$ for a, b, c, we may assume that $a \leq 0$ and $b, c \geq 0$. Substituting $-a$ for a, we need to prove that

$$(a^2 - ab + b^2)(b^2 + bc + c^2)(c^2 - ca + a^2) \geq 3(bc - ab - ac)^3$$

for $a, b, c \geq 0$. Since the left hand side of this inequality is nonnegative, consider further the nontrivial case

$$bc - ab - ac > 0.$$

Since

$$b^2 + bc + c^2 - 3(bc - ab - ac) = (b - c)^2 + 3a(b + c) \geq 0,$$

it suffices to show that

$$(a^2-ab+b^2)(a^2-ac+c^2)\ge(bc-ab-ac)^2.$$

First Solution. From $bc-ab-ac>0$, it follows that $a=\min\{a,b,c\}$. Since

$$a^2-ab+b^2\ge(b-a)^2,\quad a^2-ac+c^2\ge(c-a)^2,$$

it suffices to show that

$$(b-a)^2(c-a)^2\ge(bc-ab-ac)^2.$$

This is true if $(b-a)(c-a)\ge bc-ab-ac$; indeed,

$$(b-a)(c-a)-(bc-ab-ac)=a^2\ge 0.$$

The original inequality is an equality when two of a,b,c are zero, and when $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Since

$$4(a^2-ab+b^2)=(a+b)^2+3(a-b)^2,$$

$$4(a^2-ac+c^2)=(a+c)^2+3(a-c)^2,$$

we can apply the Cauchy-Schwarz inequality as follows

$$16(a^2-ab+b^2)(b^2+bc+c^2)\ge[(a+b)(a+c)+3(a-b)(a-c)]^2.$$

Thus, we only need to show that

$$(a+b)(a+c)+3(a-b)(a-c)\ge 4(bc-ab-ac),$$

which is equivalent to the obvious inequality $a(2a+b+c)\ge 0$.

□

P 2.37. *If a,b,c are real numbers, then*

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)\ge\frac{3}{8}(a^2+b^2)(b^2+c^2)(c^2+a^2).$$

(Vasile Cîrtoaje, 2011)

Solution. If a,b,c have the same sign, then the inequality follows from

$$a^2+ab+b^2\ge a^2+b^2,\quad b^2+bc+c^2\ge b^2+c^2,\quad c^2+ca+a^2\ge c^2+a^2.$$

Consider now that a, b, c have not the same sign. Since the inequality is symmetric and does not change by substituting $-a, -b, -c$ for a, b, c, we may assume that $a \le 0$ and $b, c \ge 0$. Substituting $-a$ for a, we need to prove that

$$(a^2 - ab + b^2)(a^2 - ac + c^2)(b^2 + bc + c^2) \ge \frac{3}{8}(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)$$

for $a, b, c \ge 0$. Write this inequality in the form

$$[(a^2 + b^2) + (a - b)^2][(a^2 + c^2) + (a - c)^2][2(b^2 + c^2) + 2bc] \ge$$

$$\ge 3(a^2 + b^2)(b^2 + c^2)(c^2 + a^2).$$

It suffices to show that

$$2(b^2 + c^2)[(a - b)^2(a^2 + c^2) + (a - c)^2(a^2 + b^2)] + 2bc(a^2 + b^2)(a^2 + c^2) \ge$$

$$\ge (a^2 + b^2)(b^2 + c^2)(c^2 + a^2),$$

which is equivalent to

$$2(b^2 + c^2)[(a - b)^2(a^2 + c^2) + (a - c)^2(a^2 + b^2)] \ge (b - c)^2(a^2 + b^2)(a^2 + c^2).$$

For the nontrivial case where

$$a^2 + b^2 \ne 0, \quad b^2 + c^2 \ne 0, \quad c^2 + a^2 \ne 0,$$

we rewrite the inequality in the form

$$\frac{(a - b)^2}{a^2 + b^2} + \frac{(a - c)^2)}{a^2 + c^2} \ge \frac{(b - c)^2}{2(b^2 + c^2)}.$$

Consider further two cases.

Case 1: $2a^2 \le b^2 + c^2$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a - b)^2}{a^2 + b^2} + \frac{(a - c)^2)}{a^2 + c^2} \ge \frac{[(b - a) + (a - c)]^2}{(a^2 + b^2) + (a^2 + c^2)} = \frac{(b - c)^2}{2a^2 + b^2 + c^2}.$$

Thus, it suffices to show that

$$\frac{1}{2a^2 + b^2 + c^2} \ge \frac{1}{2(b^2 + c^2)},$$

which reduces to $b^2 + c^2 \ge 2a^2$.

Case 2: $2a^2 \ge b^2 + c^2$. By the Cauchy-Schwarz inequality, we have

$$\frac{(a - b)^2}{a^2 + b^2} + \frac{(a - c)^2}{a^2 + c^2} \ge \frac{[c(b - a) + b(a - c)]^2}{c^2(a^2 + b^2) + b^2(a^2 + c^2)} = \frac{a^2(b - c)^2}{a^2(b^2 + c^2) + 2b^2c^2}.$$

Therefore, it suffices to prove that

$$\frac{a^2}{a^2(b^2+c^2)+2b^2c^2} \geq \frac{1}{2(b^2+c^2)}.$$

This reduces to

$$a^2(b^2+c^2) \geq 2b^2c^2,$$

which is true because

$$2a^2(b^2+c^2)-4b^2c^2 \geq (b^2+c^2)^2-4b^2c^2 = (b^2-c^2)^2 \geq 0.$$

Thus, the proof is completed. The equality holds when two of a,b,c are zero, and when $-a=b=c$ (or any cyclic permutation).

□

P 2.38. *If a,b,c are real numbers, then*

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2) \geq (a^2-ab+b^2)(b^2-bc+c^2)(c^2-ca+a^2).$$

(Vasile Cîrtoaje, 2014)

Solution. If a,b,c have the same sign, then the inequality follows from

$$a^2+b^2 \geq a^2-ab+b^2, \quad b^2+c^2 \geq b^2-bc+c^2, \quad c^2+a^2 \geq c^2-ca+a^2.$$

Consider now that a,b,c have not the same sign. Since the inequality is symmetric and does not change by substituting $-a,-b,-c$ for a,b,c, we may assume that $a \leq 0$ and $b,c \geq 0$. Substituting $-a$ for a, we need to prove that

$$2(a^2+b^2)(b^2+c^2)(c^2+a^2) \geq (b^2-bc+c^2)(c^2+ca+a^2)(a^2+ab+b^2)$$

for $a,b,c \geq 0$. Using the notation

$$A=b^2+c^2, \quad B=c^2+a^2, \quad C=a^2+b^2,$$

we can write the inequality as follows:

$$2ABC \geq (A-bc)(B+ca)(C+ab),$$

$$ABC+a^2b^2c^2 \geq ab(AB-c^2C)+ac(AC-b^2B)-bc(BC-a^2A),$$

$$ABC+a^2b^2c^2 \geq ab(c^4+a^2b^2)+ac(b^4+a^2c^2)-bc(a^4+b^2c^2).$$

It suffices to show that

$$ABC+a^2b^2c^2 \geq ab(c^4+a^2b^2)+ac(b^4+a^2c^2)+bc(a^4+b^2c^2).$$

Moreover, since

$$2ab \le a^2 + b^2, \quad 2ac \le a^2 + c^2, \quad 2bc \le b^2 + c^2,$$

it is enough to prove that

$$2ABC + 2a^2b^2c^2 \ge (a^2+b^2)(c^4+a^2b^2) + (a^2+c^2)(b^4+a^2c^2) + (b^2+c^2)(a^4+b^2c^2).$$

Indeed, this inequality reduces to the obvious inequality

$$6a^2b^2c^2 \ge 0.$$

The equality holds when two of a, b, c are zero.

□

P 2.39. *If a, b, c are real numbers, then*

$$9(1+a^4)(1+b^4)(1+c^4) \ge 8(1+abc+a^2b^2c^2)^2.$$

(Vasile Cîrtoaje, 2004)

Solution. Substituting a, b, c by $|a|, |b|, |c|$, respectively, the left side of the inequality remains unchanged, while the right side either remains unchanged or increases. Therefore, it suffices to prove the inequality only for $a, b, c \ge 0$. If $a = b = c$, then the inequality reduces to

$$9(1+a^4)^3 \ge 8(1+a^3+a^6)^2,$$

$$9(a^2 + \frac{1}{a^2})^3 \ge 8(a^3 + \frac{1}{a^3} + 1)^2.$$

Setting

$$x = a + \frac{1}{a},$$

this inequality can be written as follows

$$9(x^2-2)^3 \ge 8(x^3-3x+1)^2,$$

$$x^6 - 6x^4 - 16x^3 + 36x^2 + 48x - 80 \ge 0,$$

$$(x-2)^2[x(x^3-8) + 4(x^3-5) + 6x^2] \ge 0.$$

Since $x \ge 2$, the last inequality is clearly true. Multiplying now the inequalities

$$9(1+a^4)^3 \ge 8(1+a^3+a^6)^2,$$

$$9(1+b^4)^3 \ge 8(1+b^3+b^6)^2,$$

$$9(1+c^4)^3 \ge 8(1+c^3+c^6)^2,$$

we get

$$[9(1+a^4)(1+b^4)(1+c^4)]^3 \ge 8^3(1+a^3+a^6)^2(1+b^3+b^6)^2(1+c^3+c^6)^2.$$

From this inequality and Hölder's inequality

$$(1+a^3+a^6)(1+b^3+b^6)(1+c^3+c^6) \ge (1+abc+a^2b^2c^2)^3,$$

the conclusion follows. The equality holds for $a=b=c=1$.

□

P 2.40. *If a,b,c are real numbers, then*

$$2(1+a^2)(1+b^2)(1+c^2) \ge (1+a)(1+b)(1+c)(1+abc).$$

(Vasile Cîrtoaje, 2001)

Solution. Substituting a,b,c by $|a|,|b|,|c|$, respectively, the left side of the inequality remains unchanged, while the right side either remains unchanged or increases. Therefore, it suffices to prove the inequality only for $a,b,c \ge 0$.

First Solution. For $a=b=c$, the inequality reduces to

$$2(1+a^2)^3 \ge (1+a)^3(1+a^3).$$

This is true since

$$2(1+a^2)^3-(1+a)^3(1+a^3) = (1-a)^4(1+a+a^2) \ge 0.$$

Multiplying the inequalities

$$2(1+a^2)^3 \ge (1+a)^3(1+a^3),$$

$$2(1+b^2)^3 \ge (1+b)^3(1+b^3),$$

$$2(1+c^2)^3 \ge (1+c)^3(1+c^3),$$

we get

$$8(1+a^2)^3(1+b^2)^3(1+c^2)^3 \ge (1+a)^3(1+b)^3(1+c)^3(1+a^3)(1+b^3)(1+c^3).$$

Using this result, we still have to show that

$$(1+a^3)(1+b^3)(1+c^3) \ge (1+abc)^3,$$

which is just Hölder's inequality. We can also prove this inequality by adding the inequalities

$$a^3b^3+b^3c^3+c^3a^3 \ge 3a^2b^2c^2,$$

$$a^3+b^3+c^3 \ge 3abc.$$

The equality holds for $a = b = c = 1$.

Second Solution. We use the substitution

$$a = \frac{1-x}{1+x}, \quad b = \frac{1-y}{1+y}, \quad c = \frac{1-z}{1+z},$$

where $x, y, z \in (-1,\ 1]$. Since

$$\frac{1+a^2}{1+a} = \frac{1+x^2}{1+x}, \quad \frac{1+b^2}{1+b} = \frac{1+y^2}{1+y}, \quad \frac{1+c^2}{1+c} = \frac{1+z^2}{1+z}$$

and

$$1 + abc = \frac{2(1+xy+yz+zx)}{(1+x)(1+y)(1+z)},$$

the required inequality becomes

$$(1+x^2)(1+y^2)(1+z^2) \ge 1 + xy + yz + zx,$$

$$x^2y^2 + y^2z^2 + z^2x^2 + x^2 + y^2 + z^2 \ge xy + yz + zx,$$

$$x^2y^2 + y^2z^2 + z^2x^2 + \frac{1}{2}(x-y)^2 + \frac{1}{2}(y-z)^2 + \frac{1}{2}(z-x)^2 \ge 0.$$

□

P 2.41. *If a, b, c are real numbers, then*

$$3(a^2 - ab + b^2)(b^2 - bc + c^2)(c^2 - ca + a^2) \ge a^3b^3 + b^3c^3 + c^3a^3.$$

(Titu Andreescu, 2006)

Solution. Substituting a, b, c by $|a|, |b|, |c|$, respectively, the left side of the inequality remains unchanged or decreases, while the right side remains unchanged or increases. Therefore, it suffices to prove the inequality for $a, b, c \ge 0$. If $a = 0$, then the inequality reduces to $b^2c^2(b-c)^2 \ge 0$. Consider further then $a, b, c > 0$. We first show that

$$3(a^2 - ab + b^2)^3 \ge a^6 + a^3b^3 + b^6.$$

Indeed, setting

$$x = \frac{a}{b} + \frac{b}{a}, \quad x \ge 2,$$

we can write this inequality as

$$3(x-1)^3 \ge x^3 - 3x + 1,$$

$$(x-2)^2(2x-1) \ge 0.$$

Using this result, we have

$$27(a^2-ab+b^2)^3(b^2-bc+c^2)^3(c^2-ca+a^2)^3 \geq$$
$$\geq (a^6+a^3b^3+b^6)(b^6+b^3c^3+c^6)(c^6+c^3a^3+a^6).$$

Therefore, it suffices to show that

$$(a^6+a^3b^3+b^6)(b^6+b^3c^3+c^6)(c^6+c^3a^3+a^6) \geq (a^3b^3+b^3c^3+c^3a^3)^3.$$

Writing this inequality in the form

$$(a^3b^3+b^6+a^6)(b^6+b^3c^3+c^6)(a^6+c^6+c^3a^3) \geq (a^3b^3+b^3c^3+c^3a^3)^3,$$

we see that it is just Hölder's inequality. The equality holds when $a=b=c$, when $a=0$ and $b=c$ (or any cyclic permutation), and when two of a,b,c are 0.

□

P 2.42. *If a,b,c are nonzero real numbers, then*

$$\sum \frac{b^2-bc+c^2}{a^2}+2\sum \frac{a^2}{bc} \geq \left(\sum a\right)\left(\sum \frac{1}{a}\right).$$

(Vasile Cîrtoaje, 2010)

Solution. We have

$$\sum \frac{b^2-bc+c^2}{a^2}+2\sum \frac{a^2}{bc} = \sum\left(\frac{b^2-bc+c^2}{a^2}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)$$
$$= \sum \frac{(b^2-bc+c^2)(ab+bc+ca)}{a^2bc}$$
$$= \frac{ab+bc+ca}{a^2b^2c^2}\sum bc(b^2-bc+c^2).$$

Then, we can write the inequality as

$$(ab+bc+ca)\left[\sum bc(b^2-bc+c^2)-abc\sum a\right] \geq 0.$$

Since

$$\sum bc(b^2-bc+c^2)-abc\sum a = \left(\sum bc\right)\left(\sum a^2\right)-\sum b^2c^2-2abc\sum a$$
$$= \left(\sum bc\right)\left(\sum a^2\right)-\left(\sum bc\right)^2$$
$$= \left(\sum bc\right)\left(\sum a^2-\sum bc\right),$$

the inequality is equivalent to

$$(ab+bc+ca)^2(a^2+b^2+c^2-ab-bc-ca) \geq 0,$$

which is true. The equality holds for $a=b=c$, and also for $ab+bc+ca=0$.

□

P 2.43. *Let a,b,c be real numbers. Prove that*

(a) if $a,b,c \in [0,1]$, then

$$abc-(b+c-a)(c+a-b)(a+b-c) \le 1;$$

(b) if $a,b,c \in [-1,1]$, then

$$abc-(b+c-a)(c+a-b)(a+b-c) \le 4.$$

(Vasile Cîrtoaje, 2011)

Solution. We will show that if $a,b,c \in [m,M]$, where $M \ge 0$, then

$$abc-(b+c-a)(c+a-b)(a+b-c) \le M(M-m)^2.$$

Without loss of generality, assume that

$$M \ge a \ge b \ge c \ge m.$$

We have two cases to consider.

Case 1: $a \le 0$. The required inequality is true, since

$$abc-(b+c-a)(c+a-b)(a+b-c) \le 0 \le M(M-m)^2.$$

Indeed, substituting $-a,-b,-c$ for a,b,c, respectively, the left inequality can be restated as

$$abc \ge (b+c-a)(c+a-b)(a+b-c),$$

where $a,b,c \ge 0$. This is just the well-known Schur's inequality of degree three.

Case 2: $a > 0$. Since $(M-m)^2 \ge (a-c)^2$ and $M \ge a$, we have

$$M(M-m)^2 \ge a(a-c)^2.$$

Therefore, it suffices to show that

$$abc-(b+c-a)(c+a-b)(a+b-c) \le a(a-c)^2,$$

which is equivalent to

$$(b-c)[a^2+(b-2c)a-b^2+c^2] \ge 0.$$

This is true since $b-c \ge 0$ and

$$\begin{aligned} a^2+(b-2c)a-b^2+c^2 &= (a-b)(a+2b-2c)+(b-c)^2 \\ &\ge 2(a-b)(b-c)+(b-c)^2 \ge 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for $a=M$ and $b=c=m$ (or any cyclic permutation).

□

P 2.44. *Let a, b, c be real numbers. Prove that*

(a) if $a, b, c \in [0, 1]$, then

$$\sum a^2(a-b)(a-c) \le 1;$$

(b) if $a, b, c \in [-1, 1]$, then

$$\sum a^2(a-b)(a-c) \le 4.$$

(Vasile Cîrtoaje, 2011)

Solution. We will show that if $a, b, c \in [m, M]$, then

$$\sum a^2(a-b)(a-c) \le (M-m)^2 \cdot \max\{m^2, M^2\}.$$

Without loss of generality, assume that

$$M \ge a \ge b \ge c \ge m.$$

Since

$$b^2(b-c)(b-a) \le 0, \quad (a-c)^2 \le (M-m)^2, \quad \max\{a^2, c^2\} \le \max\{m^2, M^2\},$$

it suffices to show that

$$a^2(a-b)(a-c) + c^2(c-a)(c-b) \le (a-c)^2 \cdot \max\{a^2, c^2\}.$$

This is equivalent to

$$(a-c)^2\left[a^2 + c^2 + ac - ab - bc - \max\{a^2, c^2\}\right] \le 0,$$

that is true if

$$a^2 + c^2 + ac - ab - bc - \max\{a^2, c^2\} \le 0.$$

Case 1: $a^2 \ge c^2$. From

$$a^2 - c^2 = (a-c)(a+c) \ge 0,$$

it follows that $a + c \ge 0$. Then,

$$a^2 + c^2 + ac - ab - bc - \max\{a^2, c^2\} = (a+c)(c-b) \le 0.$$

Case 2: $a^2 \le c^2$. From

$$a^2 - c^2 = (a-c)(a+c) \le 0,$$

it follows that $a + c \le 0$. Then,

$$a^2 + c^2 + ac - ab - bc - \max\{a^2, c^2\} = (a+c)(a-b) \le 0.$$

Thus, the proof is completed. For $M^2 \ge m^2$, the equality holds when $a = M$ and $b = c = m$ (or any cyclic permutation). For $M^2 \le m^2$, the equality holds when when $a = m$ and $b = c = M$ (or any cyclic permutation).

□

P 2.45. *Let a, b, c be real numbers such that*

$$ab + bc + ca = abc + 2.$$

Prove that

$$a^2 + b^2 + c^2 - 3 \geq (2 + \sqrt{3})(a + b + c - 3).$$

(Vasile Cîrtoaje, 2011)

Solution. Substituting $a+1$, $b+1$, $c+1$ for a, b, c, respectively, we need to prove that

$$a + b + c = abc$$

implies

$$a^2 + b^2 + c^2 \geq \sqrt{3}(a + b + c).$$

This inequality is true if

$$(a^2 + b^2 + c^2)^2 \geq 3(a + b + c)^2,$$

which is equivalent to the homogeneous inequality

$$(a^2 + b^2 + c^2)^2 \geq 3abc(a + b + c).$$

Since

$$(ab + bc + ca)^2 - 3abc(a + b + c) = \frac{1}{2}\sum a^2(b - c)^2 \geq 0,$$

it suffices to prove that

$$(a^2 + b^2 + c^2)^2 \geq (ab + bc + ca)^2,$$

which is equivalent to

$$(a^2 + b^2 + c^2 - ab - bc - ca)(a^2 + b^2 + c^2 + ab + bc + ca) \geq 0.$$

This inequality is true since

$$2(a^2 + b^2 + c^2 - ab - bc - ca) = (a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0,$$

$$2(a^2 + b^2 + c^2 + ab + bc + ca) = (a + b)^2 + (b + c)^2 + (c + a)^2 \geq 0.$$

The equality holds for $a = b = c = 1$, and for $a = b = c = 1 + \sqrt{3}$.

□

P 2.46. *Let a, b, c be real numbers such that*

$$(a + b)(b + c)(c + a) = 10.$$

Prove that

$$(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) + 12a^2b^2c^2 \geq 30.$$

(Vasile Cîrtoaje, 2011)

Solution. Since

$$2(b^2+c^2)=(b+c)^2+(b-c)^2$$

and

$$(a^2+b^2)(a^2+c^2)=(a^2+bc)^2+a^2(b-c)^2,$$

by virtue of the Cauchy-Schwarz inequality, we have

$$\begin{aligned}2(a^2+b^2)(b^2+c^2)(c^2+a^2) &\ge [(b+c)(a^2+bc)+a(b-c)^2]^2\\ &=[(a+b)(b+c)(c+a)-4abc]^2\\ &=4(5-2abc)^2.\end{aligned}$$

Thus, it suffices to show that

$$(5-2abc)^2+6a^2b^2c^2\ge 15,$$

which is equivalent to

$$(abc-1)^2=0.$$

Notice that the homogeneous inequality

$$10(a^2+b^2)(b^2+c^2)(c^2+a^2)+120a^2b^2c^2\ge 3(a+b)^2(b+c)^2(c+a)^2$$

becomes an equality for $\frac{a}{k}=b=c$ (or any cyclic permutation), where

$$k+\frac{1}{k}=3.$$

□

P 2.47. *Let a,b,c be real numbers such that*

$$(a+b)(b+c)(c+a)=5.$$

Prove that

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)+12a^2b^2c^2\ge 15.$$

(Vasile Cîrtoaje, 2011)

Solution. Since

$$b^2+bc+c^2=\frac{3}{4}(b+c)^2+\frac{1}{4}(b-c)^2$$

and

$$(a^2+ab+b^2)(a^2+ac+c^2)=\frac{1}{4}(2a^2+ab+ac+2bc)^2+\frac{3}{4}a^2(b-c)^2,$$

by the Cauchy-Schwarz inequality, we have

$$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge$$

$$\ge \frac{3}{16}[(b+c)(2a^2+ab+ac+2bc)+a(b-c)^2]^2$$

$$= \frac{3}{4}[(a+b)(b+c)(c+a)-2abc]^2 = \frac{3}{4}(5-2abc)^2.$$

Thus, it suffices to show that

$$\frac{3}{4}(5-2abc)^2+12a^2b^2c^2 \ge 15,$$

which is equivalent to

$$(2abc-1)^2=0.$$

The homogeneous inequality

$$5(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2)+60a^2b^2c^2 \ge 3(a+b)^2(b+c)^2(c+a)^2$$

becomes an equality for $\frac{a}{k}=b=c$ (or any cyclic permutation), where

$$k+\frac{1}{k}=3.$$

□

P 2.48. *Let a,b,c be real numbers such that*

$$a+b+c=1, \qquad a^3+b^3+c^3=k.$$

Prove that

(a) if $k=25$, then $|a|\le 1$ or $|b|\le 1$ or $|c|\le 1$;

(b) if $k=-11$, then $1<a\le 2$ or $1<b\le 2$ or $1<c\le 2$.

(Vasile Cîrtoaje, 2011)

Solution. Without loss of generality, assume that $a\le b\le c$. If $b=1$, then $a+c=0$, and hence

$$k=a^3+b^3+c^3=1+a^3+c^3=1,$$

which is false in (a) and (b). From $(b-a)(b-c)\le 0$, we get

$$b^2-(a+c)b+ac\le 0,$$

which is equivalent to

$$2b^2-b+ac\le 0.$$

(a) It suffices to show that $|b| \le 1$. We have

$$25-b^3=a^3+c^3=(a+c)^3-3ac(a+c)=(1-b)^3-3ac(1-b),$$

which yields

$$ac=\frac{8+b-b^2}{b-1}.$$

Thus, from

$$2b^2-b+ac=\frac{2(b+1)(4-3b+b^2)}{b-1}\le 0,$$

we get $|b| \le 1$. The equality $|b|=1$ holds for $a=b=-1$ and $c=3$.

(b) It suffices to show that $1<b\le 2$. We have

$$-11-b^3=a^3+c^3=(a+c)^3-3ac(a+c)=(1-b)^3-3ac(1-b),$$

which yields

$$ac=\frac{b^2-b+4}{1-b}.$$

Thus, the inequality $2b^2-b+ac\le 0$ is equivalent to

$$\frac{(b-2)(b^2+1)}{1-b}\ge 0,$$

which involves $1<b\le 2$. The equality $b=2$ holds for $a=-3$ and $b=c=2$.

□

P 2.49. *Let a,b,c be real numbers such that*

$$a+b+c=a^3+b^3+c^3=2.$$

Prove that $a,b,c\notin\left[\frac{5}{4},2\right]$.

(Vasile Cîrtoaje, 2011)

Solution. If $a=2$, then we get a contradiction because

$$b+c=2-a=0,\quad b^3+c^3=2-a^3=-6,\quad b^3+c^3=(b+c)(b^2-bc+c^2)=0.$$

From

$$2=a^3+b^3+c^3=a^3+(b+c)^2-3bc(b+c)=a^3+(2-a)^2-3bc(2-a),$$

we obtain

$$bc=\frac{2(1-a)^2}{2-a}.$$

Thus, the inequality $(b+c)^2 \geq 4bc$ involves

$$(2-a)^2 \geq \frac{8(1-a)^2}{2-a},$$

$$\frac{a(4-2a-a^2)}{2-a} \geq 0,$$

$$a \in (-\infty, -1-\sqrt{5}] \cup [0, -1+\sqrt{5}] \cup (2, \infty).$$

Since

$$-1+\sqrt{5} < \frac{5}{4},$$

it follows that $a \notin \left[\frac{5}{4}, 2\right]$. Similarly, we have

$$b, c \notin \left[\frac{5}{4}, 2\right].$$

□

P 2.50. *If a, b, c and k are real numbers, then*

$$\sum (a-b)(a-c)(a-kb)(a-kc) \geq 0.$$

(Vasile Cîrtoaje, 2005)

Solution. For $a = b = c$, the equality holds. Otherwise, using the substitution

$$m = k+2, \quad u = (1-k)a, \quad b = a+x, \quad c = a+y,$$

the inequality can be written as

$$Au^2 + Bu + C \geq 0,$$

where

$$A = x^2 - xy + y^2,$$

$$B = (x+y)(2A - mxy),$$

$$C = (x+y)^2(A - mxy) + m^2x^2y^2.$$

The quadratic $Au^2 + Bu + C$ has the discriminant

$$D = B^2 - 4AC = -3m^2x^2y^2(x-y)^2.$$

Since $A > 0$ and $D \leq 0$, the conclusion follows. The equality holds for $a = b = c$, and for $a/k = b = c$ (or any cyclic permutation).

Remark 1. The inequality is equivalent to

$$\sum a^4 + k(k+2)\sum a^2b^2 + (1-k^2)abc\sum a \ge (k+1)\sum ab(a^2+b^2).$$

For $k=0$, we get Schur's inequality of degree four

$$a^4+b^4+c^4+abc(a+b+c) \ge \sum ab(a^2+b^2).$$

For $k=1$, we get the inequality

$$a^4+b^4+c^4+3(a^2b^2+b^2c^2+c^2a^2) \ge 2\sum ab(a^2+b^2),$$

with equality for $a=b=c$.
For $k=2$, we get the inequality

$$a^4+b^4+c^4+8(a^2b^2+b^2c^2+c^2a^2) \ge 3(ab+bc+ca)(a^2+b^2+c^2),$$

which can be rewritten as

$$9(a^4+b^4+c^4)+126(a^2b^2+b^2c^2+c^2a^2) \ge 5(a+b+c)^4,$$

with equality for $a=b=c$, and for $a/2=b=c$ (or any cyclic permutation).

Remark 2. The inequality in P 2.50 is equivalent to

$$\sum (a-b)^2(a+b-c-kc)^2 \ge 0.$$

□

P 2.51. *If a,b,c are real numbers, then*

$$\sum a^2(a-b)(a-c) \ge \frac{(a-b)^2(b-c)^2(c-a)^2}{a^2+b^2+c^2+ab+bc+ca}.$$

Solution *(by Michael Rozenberg).* Since

$$\sum a^2(a-b)(a-c) = \frac{1}{2}\sum (b-c)^2(b+c-a)^2,$$

we can write the inequality in the form

$$\left[\sum (b+c)^2\right]\left[\sum (b-c)^2(b+c-a)^2\right] \ge 4(a-b)^2(b-c)^2(c-a)^2.$$

Using now the Cauchy-Schwarz inequality, is suffices to show that

$$\left[\sum (b+c)(b-c)(b+c-a)\right]^2 \ge 4(a-b)^2(b-c)^2(c-a)^2,$$

which is an identity. The equality holds for $a=b=c$, for $a=0$ and $b=c$ (or any cyclic permutation), and for $a=0$ and $b+c=0$ (or any cyclic permutation).

□

P 2.52. *Let $x_1, x_2, \ldots, x_n$ $(n \geq 3)$ be real numbers such that*

$$x_1 + x_2 + \ldots + x_n = a + b, \qquad x_1^2 + x_2^2 + \cdots + x_n^2 = a^2 + b^2,$$

where a and b are fixed real numbers such that $a \neq 0$, $b \neq 0$, $a \neq b$. Then, there exist $x_1, x_2, \ldots, x_n$ such that

(a) $$x_1 x_2 \cdots x_n > 0;$$

(b) $$x_1 x_2 \cdots x_n < 0.$$

Solution. For

$$x_1 = x_2 = y, \qquad x_3 = \cdots = x_n = z,$$

from

$$2y + (n-2)z = a + b, \qquad 2y^2 + (n-2)z^2 = a^2 + b^2,$$

we get the real solution

$$y = \frac{2(a+b) + \sqrt{2(n-2)d}}{2n} > 0, \qquad z = \frac{(n-2)(a+b) - \sqrt{2(n-2)d}}{n(n-2)} < 0,$$

where

$$d = (n-1)(a^2 + b^2) - 2ab > 0.$$

The product

$$x_1 x_2 \cdots x_n = y^2 z^{n-2}$$

is positive for even n, and negative for odd n.

Also, for

$$x_1 = u, \qquad x_2 = \cdots = x_n = v,$$

from

$$u + (n-1)v = a + b, \qquad u^2 + (n-1)v^2 = a^2 + b^2,$$

we get the real solution

$$u = \frac{a + b + \sqrt{(n-1)d}}{2n} > 0, \qquad v = \frac{(n-1)(a+b) - \sqrt{(n-1)d}}{n(n-1)} < 0.$$

The product

$$x_1 x_2 \cdots x_n = u v^{n-1}$$

is negative for even n, and positive for odd n.

□

P 2.53. *Let $a \geq b \geq c$ be real numbers such that*

$$a + b + c = p, \quad ab + bc + ca = q,$$

where p and q are fixed real numbers satisfying $p^2 \geq 3q$. Prove that the product

$$r = abc$$

is minimal only when $a = b$, and is maximal only when $b = c$.

Solution. For $p^2 = 3q$, which is equivalent to

$$(a-b)^2+(b-c)^2+(c-a)^2=0,$$

we get $a = b = c$. Consider further that $p^2 > 3q$.

First Solution. From

$$(b-a)(b-c)\le 0,$$

which is equivalent to

$$b^2+bc-b(a+c)\le 0, \quad b^2+q-2b(a+c)\le 0, \quad b^2+q-2b(p-b)\le 0,$$

we get

$$3b^2-2pb+q\le 0,$$

hence

$$b\in[b_1,\ b_2],$$

where

$$b_1=\frac{p-\sqrt{p^2-3q}}{3}, \quad b_2=\frac{p+\sqrt{p^2-3q}}{3}, \quad b_1<b_2.$$

We have $b = b_1$ and $b = b_2$ when $(b-a)(b-c) = 0$; that is, when $b = c$ and $b = a$, respectively. On the other hand, from

$$abc=b[q-b(a+c)]=bq-b^2(p-b)=b^3-pb^2+qb,$$

we get

$$r(b)=b^3-pb^2+qb.$$

Since

$$r'(b)=3b^2-2pb+q=(b-a)(b-c)\le 0,$$

$r(b)$ is strictly decreasing on $[b_1, b_2]$, therefore $r(b)$ is minimal only for $b = b_2$, when $b = a$, and is maximal only for $b = b_1$, when $b = c$.

Second Solution. We will show that $a \in [a_1,\ a_2]$, where

$$a_1=\frac{p+\sqrt{p^2-3q}}{3}, \quad a_2=\frac{p+2\sqrt{p^2-3q}}{3}, \quad a_1<a_2.$$

From

$$\begin{aligned}0\le(b-c)^2&=(b+c)^2-4bc=(b+c)^2+4a(b+c)-4q\\&=(p-a)^2+4a(p-a)-4q=-3a^2+2pa+p^2-4q,\end{aligned}$$

we get $a \le a_2$, with equality for $b = c$. Similarly, from

$$0\le(a-b)(a-c)=a^2-2a(b+c)+q=a^2-2a(p-a)+q=3a^2-2pa+q,$$

we get $a \geq a_1$, with equality for $a = b$. On the other hand, from

$$abc = a[q - a(b+c)] = aq - a^2(p-a) = a^3 - pa^2 + qa,$$

we get

$$r(a) = a^3 - pa^2 + qa.$$

Since

$$r'(a) = 3a^2 - 2pa + q = (a-b)(a-c) \geq 0,$$

$r(a)$ is strictly increasing on $[a_1, a_2]$, and hence $r(a)$ is minimal only for $a = a_1$, when $a = b$, and is maximal only for $a = a_2$, when $b = c$.

Third Solution. We will show that $c \in [c_1,\ c_2]$, where

$$c_1 = \frac{p - 2\sqrt{p^2 - 3q}}{3}, \quad c_2 = \frac{p - \sqrt{p^2 - 3q}}{3}, \quad c_1 < c_2.$$

From

$$\begin{aligned} 0 \leq (a-b)^2 &= (a+b)^2 - 4ab = (a+b)^2 + 4c(a+b) - 4q \\ &= (p-c)^2 + 4c(p-c) - 4q = -3c^2 + 2pc + p^2 - 4q, \end{aligned}$$

we get $c \geq c_1$, with equality for $a = b$. Similarly, from

$$0 \leq (b-c)(a-c) = c^2 - 2c(a+b) + q = c^2 - 2c(p-c) + q = 3c^2 - 2pc + q \geq 0,$$

we get $c \leq c_2$, with equality for $b = c$. On the other hand, from

$$abc = c[q - c(a+b)] = cq - c^2(p-c) = c^3 - pc^2 + qc,$$

we get

$$r(c) = c^3 - pc^2 + qc.$$

Since

$$r'(c) = 3c^2 - 2pc + q = (b-c)(a-c) \geq 0,$$

$r(c)$ is strictly increasing on $[c_1, c_2]$, and hence $r(c)$ is minimal only for $c = c_1$, when $a = b$, and is maximal only for $c = c_2$, when $b = c$.

Fourth Solution. From

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3 \geq 0,$$

we get $r \in [r_1, r_2]$, where

$$r_1 = \frac{9pq - 2p^3 - 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27},$$

$$r_2 = \frac{9pq - 2p^3 + 2(p^2 - 3q)\sqrt{p^2 - 3q}}{27}.$$

Obviously, r attains its minimum and maximum only when two of a,b,c are equal; that is, when either $a=b$ or $b=c$. For $a=b$, from $a+b+c=p$ and $ab+bc+ca=q$, we get

$$a=b=\frac{p+\sqrt{p^2-3q}}{3}, \quad c=\frac{p-2\sqrt{p^2-3q}}{3},$$

$$r=\frac{(p+\sqrt{p^2-3q})^2(p-2\sqrt{p^2-3q})}{27}=r_1.$$

Similar, for $b=c$, we get

$$b=c=\frac{p-\sqrt{p^2-3q}}{3}, \quad a=\frac{p+2\sqrt{p^2-3q}}{3},$$

$$r=\frac{(p-\sqrt{p^2-3q})^2(p+2\sqrt{p^2-3q})}{27}=r_2.$$

Remark 1. The statement remains valid by replacing "fixed $ab+bc+ca=q$" with "fixed $a^2+b^2+c^2=p_1$". Thus, we can prove (by induction or contradiction method) the following generalization:

- *If $a_1,a_2,\ldots,a_n$ are real numbers such that $a_1\ge a_2\ge\cdots\ge a_n$ and*

$$a_1+a_2+\cdots+a_n=p, \quad a_1^2+a_2^2+\cdots+a_n^2=p_1,$$

where p and p_1 are fixed real numbers satisfying $p^2\le np_1$, then the product

$$r=a_1a_2\cdots a_n$$

is minimal and maximal when $n-1$ of $a_1,a_2,\ldots,a_n$ are equal, more precisely, it is minimal when $a_1=\cdots=a_{n-1}\ge a_n$, and is maximal when $a_1\ge a_2=\cdots=a_n$.

Assume, by the sake of contradiction, that the product r is minimal/maximal when three of $a_1,a_2,\ldots,a_n$ are distinct, for example, when $a_1<a_2<a_3$. According to P 2.53, the product r can be increased/decreased by choosing some suitable numbers b_1,b_2,b_3 such that

$$b_1+b_2+b_3=a_1+a_2+a_3, \quad b_1^2+b_2^2+b_3^2=a_1^2+a_2^2+a_3^2;$$

this is a contradiction.

Remark 2. Another extension is the following (Vasile Cîrtoaje, 2017):

- *If $a,b,c\in[m,M]$ such that $a\ge b\ge c$ and*

$$a+b+c=p, \quad ab+bc+ca=q,$$

where p and q are fixed real numbers satisfying $p^2\ge 3q$, then the product

$$r=abc$$

is minimal when $a = b \geq c$ or $c = m$, and maximal when $a \geq b = c$ or $a = M$.

As we have shown above, if $a, b, c \in \mathbb{R}$, then

$$c \in [c_1, c_2], \quad b \in [b_1, b_2], \quad a \in [a_1, a_2];$$

more precisely, for $p^2 > 3q$ and $u = \sqrt{p^2 - 3q}$, we have:

$$\frac{p-2u}{3} = c_1 < c_2 = \frac{p-u}{3} = b_1 < b_2 = \frac{p+u}{3} = a_1 < a_2 = \frac{p+2u}{3}.$$

In addition, if $a = b$, then $c = c_1$, $b = b_2$, $a = a_1$, and if $b = c$, then $c = c_2$, $b = b_1$, $a = a_2$.
On the other hand, if $a, b, c \in [m, M]$, we have $m \leq c_1$ or $c_1 \leq m \leq c_2$, and $M \geq a_2$ or $a_1 \leq M \leq a_2$. Thus, we have

$$c \in [c_1', c_2], \qquad c_1' = \max\{c_1, m\},$$

and

$$a \in [a_1, a_2'], \qquad a_2' = \min\{a_2, M\}.$$

According to Third Solution, the product $r = abc$ is minimal for $c = c_1'$, when either $c = c_1$ $(a = b \geq c)$ or $c = m$. Similarly, according to Second Solution, the product $r = abc$ is maximal for $a = a_2'$, when either $a = a_2$ $(a \geq b = c)$ or $a = M$.

Remark 3. The result in Remark 2 can be generalized as follows:

- *If $a_1, a_2, \ldots, a_n \in [m, M]$ are real numbers such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and*

$$a_1 + a_2 + \cdots + a_n = p, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = p_1,$$

where p and p_1 are fixed real numbers satisfying $p^2 \leq np_1$, then the product

$$r = a_1 a_2 \cdots a_n$$

is minimal for $a_1 = \cdots = a_{n-1} \geq a_n$ or $a_n = m$, and maximal for $a_1 \geq a_2 = \cdots = a_n$ or $a_1 = M$.

Remark 4. The following result follows from P 2.52 and P 2.53:

- *If $a_1, a_2, \ldots, a_n$ are real numbers such that*

$$a_1 + a_2 + \cdots + a_n = p, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = p_1,$$

where p and p_1 are fixed real numbers satisfying $p^2 \leq np_1$, then the product

$$a_1 a_2 \cdots a_n$$

is minimal and maximal only when the set $(a_1, a_2, \ldots, a_n)$ has at most two distinct elements.

We can prove this statement by the contradiction method. Thus, assume that the product $a_1a_2\cdots a_n$ is minimal (maximal) for a set $(a_1, a_2, \ldots, a_n)$ having three distinct elements (let a_1, a_2, a_3, with $a_1 \neq 0$ and $a_2 \neq 0$, be these elements). If among the numbers $a_3, \ldots, a_n$ there are i numbers ($i \geq 1$) equal to zero (let $a_3, \ldots, a_{i+2}$ be these numbers), according to P 2.52, there exists a set $(b_1, b_2, \ldots, b_{i+2})$ such that

$$b_1 + b_2 + \cdots + b_{i+2} = a_1 + a_2 + \cdots + a_{i+2}, \qquad b_1^2 + b_2^2 + \cdots + b_{i+2}^2 = a_1^2 + a_2^2 + \cdots + a_{i+2}^2,$$

$$b_1b_2\cdots b_{i+2} > 0,$$

and also a set $(c_1, c_2, \ldots, c_{i+2})$ such that

$$c_1 + c_2 + \cdots + c_{i+2} = a_1 + a_2 + \cdots + a_{i+2}, \qquad c_1^2 + c_2^2 + \cdots + c_{i+2}^2 = a_1^2 + a_2^2 + \cdots + a_{i+2}^2,$$

$$c_1c_2\cdots c_{i+2} < 0.$$

Therefore, the set $(a_1, a_2, \ldots, a_n)$ is not minimal (maximal), which is a contradiction. Assume now that all numbers $a_1, a_2, \ldots, a_n$ are nonzero. According to P 2.53, since the numbers a_1, a_2, a_3 are distinct, the product $a_1a_2a_3$ is not minimal nor maximal, therefore the product $a_1a_2\cdots a_n$ is not minimal nor maximal, which is a contradiction.

□

P 2.54. *Let a, b, c be real numbers. Prove that*

(a) for fixed

$$a + b + c = p, \qquad abc = r,$$

the sum

$$q = ab + bc + ca$$

is maximal only when two of a, b, c are equal;

(b) for fixed

$$ab + bc + ca = q, \qquad abc = r \neq 0,$$

the product

$$p_1 = abc(a + b + c)$$

is maximal only when two of a, b, c are equal.

(Vasile Cîrtoaje, 2017)

First Solution. Assume that $a \geq b \geq c$.

(a) If $b = 0$, we have $c \leq 0 \leq a$, $a + c = p$ and $q = ac$. From $q = ac \leq 0$, it follows that q is maximal when one of a and c is zero, therefore when two of a, b, c are equal to zero. Consider now that $b \neq 0$. From

$$q = ac + b(a + c) = \frac{r}{b} + b(p - b),$$

we get

$$q(b) = \frac{r}{b} + pb - b^2,$$

with

$$q'(b) = \frac{-r}{b^2} + p - 2b = \frac{-(b-a)(b-c)}{b}.$$

For $b < 0$ (that implies $c < 0$), we have $q'(b) \le 0$, $q(b)$ is strictly decreasing, $q(b) \le q(c)$, hence $q(b)$ is maximal only for $b = c < 0$.

For $b > 0$ (that implies $a > 0$), we have $q'(b) \ge 0$, $q(b)$ is strictly increasing, $q(b) \le q(a)$, hence $q(b)$ is maximal only for $b = a > 0$.

(b) Similarly, we have

$$p_1(b) = r\left(b + \frac{q - ac}{b}\right) = r\left(b + \frac{q}{b} - \frac{r}{b^2}\right),$$

with

$$p_1'(b) = r\left(1 - \frac{q}{b^2} + \frac{2r}{b^3}\right) = \frac{r(b-a)(b-c)}{b^2}.$$

If $r > 0$, then $p_1'(b) \le 0$ and $p_1(b)$ is strictly decreasing. For $b < 0$ (which involves $c < 0$ and $a > 0$), we have $p_1(b) \le p_1(c)$, hence $p_1(b)$ is maximal only for $b = c < 0$, while for $b > 0$ (which involves $a > 0$ and $c > 0$), we have $p_1(b) \le p_1(c)$, hence $p_1(b)$ is maximal only for $b = c > 0$.

If $r < 0$, then $p_1'(b) \ge 0$ and $p_1(b)$ is strictly increasing. For $b < 0$ (which involves $c < 0$ and $a < 0$), we have $p_1(b) \le p_1(a)$, hence $p_1(b)$ is maximal only for $b = a < 0$, while for $b > 0$ (which involves $a > 0$ and $c < 0$), we have $p_1(b) \le p_1(a)$, hence $p_1(b)$ is maximal only for $b = a > 0$.

Second Solution. From

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3,$$

if follows

$$-27r^2 + 2(9pq - 2p^3)r + p^2q^2 - 4q^3 \ge 0,$$

with equality if and only if two of a, b, c are equal.

(a) Write the inequality $(a-b)^2(b-c)^2(c-a)^2 \ge 0$ as $f(q) \ge 0$, where

$$f(q) = -4q^3 + p^2q^2 + 18prq - 4p^3r - 27r^2.$$

There two possible cases:

$$f(q) = -4(q-q_1)(q-q_2)(q-q_3), \quad q_1 \le q_2 \le q_3,$$

or

$$f(q) = -4(q^2 + bq + c)(q - q_3), \quad b^2 - 4c < 0.$$

In both cases, the inequality $f(q) \geq 0$ involves $q \leq q_3$. Therefore, the maximal value of q is q_3. Since $f(q_3) = 0$, q is maximal when

$$(a-b)^2(b-c)^2(c-a)^2 = 0,$$

therefore when two of a, b, c are equal.

(b) Write the inequality $(a-b)^2(b-c)^2(c-a)^2 \geq 0$ as $f(x) \geq 0$, where $x = rp$ and

$$f(x) = -4x^3 + q^2x^2 + 18qr^2x - 4q^3r^2 - 27r^4.$$

Clearly, we have $f(x) = 0$ if and only if two of a, b, c are equal. There two possible cases:

$$f(x) = -4(x-x_1)(x-x_2)(x-x_3), \quad x_1 \leq x_2 \leq x_3,$$

or

$$f(x) = -4(x^2 + bx + c)(x - x_3), \quad b^2 - 4c < 0.$$

In both cases, the inequality $f(x) \geq 0$ involves $x \leq x_3$. Therefore, the maximal value of x is x_3. Since $f(x_3) = 0$, x is maximal when

$$(a-b)^2(b-c)^2(c-a)^2 = 0,$$

therefore when two of a, b, c are equal.

Remark 1. The inequality in (b) follows immediately from the inequality in (a) by replacing a, b and c with $1/a$, $1/b$ and $1/c$, respectively.

Remark 2. The statement (a) remains valid by replacing "sum $q = ab + bc + ca$ is maximal when two of a, b, c are equal" with "sum $p_1 = a^2 + b^2 + c^2$ is minimal when two of a, b, c are equal". Thus, this statement can be generalized as follows:

- *If $a_1, a_2, \ldots, a_n$ are real numbers such that*

$$a_1 + a_2 + \cdots + a_n = p, \qquad a_1a_2\cdots a_n = r,$$

where p and r are fixed real numbers, then the sum

$$p_1 = a_1^2 + a_2^2 + \cdots + a_n^2$$

is minimal when $n-1$ of $a_1, a_2, \ldots, a_n$ are equal.

□

P 2.55. *Let a, b, c be real numbers such that $a + b + c = 3$. Prove that*

(a) $$(ab + bc + ca)^2 \geq 9abc;$$

(b) $$(ab + bc + ca)^2 + 9 \geq 18abc;$$

(c) $$(ab + bc + ca - 3)^2 \geq 27(abc - 1).$$

(Vasile Cîrtoaje, 2011)

Solution. Let $q = ab + bc + ca$. According to P 2.53, for fixed q, the product abc is maximal when two of a, b, c are equal. Therefore, it suffices to prove the desired inequalities for $b = c$, when $a + 2b = 3$.

The inequality (a) is equivalent to

$$(2ab + b^2)^2 \geq 9ab^2,$$

$$b^2(b-1)^2 \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 3$ and $b = c = 0$ (or any cyclic permutation).

The inequality (b) is equivalent to

$$(2ab + b^2 - 3)^2 + 9 \geq 18ab^2,$$

$$(b-1)^2(b+1)^2 \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 5$ and $b = c = -1$ (or any cyclic permutation).

The inequality (c) is equivalent to

$$(2ab + b^2 - 3)^2 \geq 27(ab^2 - 1),$$

$$(b-1)^2(b+2)^2 \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 7$ and $b = c = -2$ (or any cyclic permutation).

Remark 1. Another solution for the inequality (b) is the following. Using the substitution

$$a = x + 1, \quad b = y + 1, \quad c = z + 1,$$

the inequality becomes

$$(xy + yz + zx)^2 \geq 12(xy + yz + zx) + 18xyz,$$

where x, y, z are real numbers such that

$$x + y + z = 0.$$

Assume that y and z have the same sign, hence $yz \geq 0$. Substituting x by $-y-z$, the inequality can be rewritten in the form

$$(y^2 + yz + z^2)^2 + 12(y^2 + yz + z^2) + 18yz(y+z) \geq 0.$$

Since

$$y^2 + yz + z^2 \geq \frac{3}{4}(y+z)^2 \geq 3yz,$$

it suffices to show that

$$9y^2z^2 + 9(y+z)^2 + 18yz(y+z) \ge 0,$$

which is equivalent to

$$9(yz + y + z)^2 \ge 0.$$

Remark 2. Another solution for the inequality (c) is the following. Assume that $a = \max\{a,b,c\}$, $a \ge 1$. Since

$$3 - ab - bc - ca \ge 3 - a(b+c) - \frac{1}{4}(b+c)^2 = 3 - a(3-a) - \frac{1}{4}(3-a)^2 = \frac{3}{4}(a-1)^2$$

and

$$abc - 1 \le \frac{1}{4}a(b+c)^2 - 1 = \frac{1}{4}a(3-a)^2 - 1 = \frac{1}{4}(a-1)^2(a-4),$$

it suffices to prove that

$$\frac{9}{16}(a-1)^4 \ge \frac{27}{4}(a-1)^2(a-4),$$

which is equivalent to

$$(a-1)^2(a-7)^2 \ge 0.$$

□

P 2.56. *Let a,b,c be real numbers such that*

$$ab + bc + ca + abc = 4.$$

Prove that

(a) if $abc > 0$, then

$$2(a+b+c) + ab + bc + ca \le \frac{9}{abc};$$

(b) if $abc < 0$, then

$$2(a+b+c) + ab + bc + ca \ge \frac{9}{abc}.$$

(Vasile Cîrtoaje, 2018)

Solution. We write both inequalities in the unique form

$$2abc(a+b+c) + abc(ab+bc+ca) \le 9.$$

According to P 2.54-(b), for fixed $ab+bc+ca$ and abc, the product $abc(a+b+c)$ is maximal when two of a,b,c are equal. Therefore, it suffices to prove the inequality for $b=c$; that is, to show that

$$2ab^2(a+2b+2ab+b^2)\le 9$$

for $2ab+b^2+ab^2=4$. Write this hypothesis as

$$(b+2)(b-2+ab)=0.$$

For $b=-2$, the required inequality is equivalent to $8a^2+3\ge 0$, while for $a=\dfrac{2-b}{b}$, it is equivalent to $(b^2-1)^2\ge 0$.

The inequality (a) is an equality for $a=b=c=1$, while the inequality (b) is an equality for $a=-3$ and $b=c=-1$ (or any cyclic permutation).

□

P 2.57. *If a,b,c are real numbers such that*

$$a+b+c+abc=4,$$

then

$$a^2+b^2+c^2+3\ge 2(ab+bc+ca).$$

(Vasile Cîrtoaje, 2011)

First Solution. Write the inequality in the form

$$(a+b+c)^2+3\ge 4(ab+bc+ca).$$

According to P 2.54-(a), for fixed $a+b+c$ and abc, the sum $ab+bc+ca$ is maximal when two of a,b,c are equal. Therefore, considering $b=c$, we need to show that $a+2b+ab^2=4$ involves

$$a^2+2b^2+3\ge 2(2ab+b^2),$$

that is

$$a^2+3\ge 4ab.$$

We have

$$a^2+3-4ab=\left(\frac{4-2b}{b^2+1}\right)^2+3-\frac{4b(4-2b)}{b^2+1}$$

$$=\frac{11b^4-16b^3+18b^2-32b+19}{(b^2+1)^2}=\frac{(b-1)^2(11b^2+6b+19)}{(b^2+1)^2}\ge 0.$$

The equality holds for $a=b=c=1$.

Second Solution. Without loss of generality, assume that $a \geq b \geq c$. The case $a \leq 0$ is not possible, since it involves $a+b+c+abc \leq 0 < 4$. If $a \geq 0 \geq b \geq c$, then

$$a^2+b^2+c^2+3-2(ab+bc+ca) \geq a^2+(b-c)^2+3 > 0.$$

Also, if $a \geq b \geq 0 \geq c$, then

$$a^2+b^2+c^2+3-2(ab+bc+ca) \geq (a-b)^2+c^2+3 > 0.$$

Consider further that $a,b,c \geq 0$ and denote

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

We need to show that

$$p^2+3 \geq 4q$$

for $p+r=4$. By Schur's inequality of degree three, we have

$$p^3+9r \geq 4pq.$$

Therefore, we get

$$p(p^2+3-4q) \geq p^3+3p-(p^3+9r) = 12(p-3).$$

To complete the proof, we need to show that $p \geq 3$. By virtue of the AM-GM inequality, we have

$$p^3 \geq 27r,$$

$$p^3 \geq 27(4-p),$$

$$(p-3)(p^2+3p+36) \geq 0,$$

$$p \geq 3.$$

□

P 2.58. *If a,b,c are real numbers such that*

$$ab+bc+ca = 3abc,$$

then

$$4(a^2+b^2+c^2)+9 \geq 7(ab+bc+ca).$$

(Vasile Cîrtoaje, 2011)

Solution. If one of a, b, c is 0, then the inequality is trivial. Otherwise, write the inequality in the homogeneous form

$$4(a^2+b^2+c^2)+\frac{81a^2b^2c^2}{(ab+bc+ca)^2}\ge 7(ab+bc+ca),$$

or

$$81r^2\ge q^2(15q-4p^2),$$

where

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc.$$

First Solution. For fixed p and q, r^2 is minimal when r is either minimal (when $r\ge 0$) or maximal (when $r\le 0$). According to P 2.53, r is minimal and maximal when two of a, b, c are equal. For $a=b$, the inequality becomes

$$(a-c)^2(4a-c)^2\ge 0.$$

The equality holds for $a=b=c=1$, and for $a=b=2$ and $c=1/2$ (or any cyclic permutation).

Second Solution (by *Vo Quoc Ba Can*). Consider the nontrivial case $15q-4p^2>0$. Substituting a, b, c by $|a|, |b|, |c|$, respectively, the left side of the inequality remains unchanged, while the right side remains unchanged or increases. Therefore, it suffices to prove the inequality only for $a, b, c>0$ and $15q-4p^2>0$. Assume that $a\ge b\ge c>0$. There are two cases to consider.

Case 1: $4b^2\le 3ab+3bc+ca$. Since

$$4q^2(15q-4p^2)\le\left[\frac{q^2}{b}+b(15q-4p^2)\right]^2,$$

it suffices to show that

$$18r\ge\frac{q^2}{b}+b(15q-4p^2),$$

which is equivalent to the obvious inequality

$$(a-b)(b-c)(3ab+3bc+ca-4b^2)\ge 0.$$

Case 2: $4b^2>3ab+3bc+ca$. Since

$$4q^2(15q-4p^2)\le\left[\frac{q^2}{a}+a(15q-4p^2)\right]^2,$$

it suffices to show that

$$18r\ge\frac{q^2}{a}+a(15q-4p^2),$$

which is equivalent to

$$(a-b)(a-c)(4a^2-3ab-bc-3ca)\ge 0.$$

This is true, since

$$4a^2 - 3ab - bc - 3ca = (4b^2 - 3ab - 3bc - ca) + 2(a-b)(2a+2b-c) > 0.$$

□

P 2.59. *Let $a, b, c \le \frac{6}{5}$ be real numbers such that $a^2 + b^2 + c^2 = 4$. If*

$$k = \frac{16(2+15\sqrt{2})}{125} \approx 2.97,$$

then

$$ab + bc + ca + k \ge abc.$$

(Vasile Cîrtoaje, 2018)

Solution. According to Remark 2 from P 2.53, for $a^2+b^2+c^2 = 4$ and $ab+bc+ca = constant$, the product abc is maximal when $a = \frac{6}{5}$ or $a \ge b = c$. Therefore, it is enough to consider these cases.

Case 1: $a = \frac{6}{5}$. We need to prove that if

$$b^2 + c^2 = \frac{64}{25},$$

then

$$bc - 6(b+c) \le 5k.$$

We have

$$bc - 6(b+c) \le \frac{b^2+c^2}{2} + 6\sqrt{2(b^2+c^2)} = \frac{32}{25} + \frac{48\sqrt{2}}{5} = 5k.$$

Case 2: $a \ge b = c$. We need to show that if

$$a^2 + 2b^2 = 4,$$

then

$$k \ge (a-1)b^2 - 2ab.$$

For $0 > a \ge b$, the inequality is true since its right side is negative. Consider next that $a \ge 0$. We get the required inequality by adding the inequalities

$$\frac{32}{125} \ge (a-1)b^2$$

and

$$\frac{48\sqrt{2}}{25} \ge -2ab.$$

The first inequality is true because, for the non-trivial case $1 \le a \le \frac{6}{5}$, we have we have

$$64 - 250(a-1)b^2 = 64 - 125(a-1)(4-a^2)$$
$$= 564 - 500a - 125a^2 + 125a^3 = (6-5a)(94-5a-25a^2) \ge 0.$$

The second inequality is true if

$$\left(\frac{48}{25}\right)^2 \ge 2a^2b^2.$$

Indeed, since $25a^2 \le 36$, we have

$$\left(\frac{48}{25}\right)^2 - 2a^2b^2 = \left(\frac{48}{25}\right)^2 - a^2(4-a^2)$$
$$= \frac{2304 - 2500a^2 + 625a^4}{625} = \frac{(36-25a^2)(64-25a^2)}{625} \ge 0.$$

The equality occurs for $a = \frac{6}{5}$ and $b = c = \frac{-4\sqrt{2}}{5}$ (or any cyclic permutation).

□

P 2.60. *Let $f_4(a,b,c)$ be a symmetric homogeneous polynomial of degree four. Prove that the inequality $f_4(a,b,c) \ge 0$ holds for all real numbers a,b,c if and only if $f_4(a,1,1) \ge 0$ for all real a.*

Solution. Let

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

Any symmetric homogeneous polynomial $f_4(a,b,c)$ can be written as

$$f_4(a,b,c) = Apr + Bp^4 + Cp^2q + Dq^2,$$

where A, B, C, D are real constants. For fixed p and q, the linear function

$$g(r) = Apr + Bp^4 + Cp^2q + Dq^2$$

is minimal when r is either minimal or maximal. By P 2.53, r is minimal and maximal when two of a,b,c are equal. Since $f_4(a,b,c)$ is symmetric, homogeneous and satisfies $f_4(-a,-b,-c) = f_4(a,b,c)$, it follows that the inequality $f_4(a,b,c) \ge 0$ holds for all real numbers a,b,c if and only if $f_4(a,1,1) \ge 0$ and $f_4(a,0,0) \ge 0$ for all real a. Notice that the condition "$f_4(a,0,0) \ge 0$ for all real a" is not necessary because it follows from the condition "$f_4(a,1,1) \ge 0$ for all real a" as follows:

$$f_4(a,0,0) = \lim_{t\to 0} f_4(a,t,t) = \lim_{t\to 0} t^4 f_4(a/t,1,1) \ge 0.$$

Remark. Similarly, we can prove the following statement, which is valid for the extended case where $f_4(a,b,c)$ is only a symmetric polynomial (homogeneous or non-homogeneous).

- *Let $f_4(a,b,c)$ be a symmetric polynomial function of degree $n=4$. The inequality $f_4(a,b,c)\ge 0$ holds for all real numbers a,b,c if and only if $f_4(a,b,b)\ge 0$ for all real numbers a and b.*

Notice that a function $f(a,b,c)$ is symmetric if it is unchanged by any permutation of its variables. A function $f(a,b,c)$ is a polynomial function if it is a polynomial in one variable when the other two variables are fixed. In addition, a polynomial function $f(a,b,c)$ is of degree n if $f(a,a,a)$ is a polynomial of degree n.

□

P 2.61. *If a,b,c are real numbers, then*

$$10(a^4+b^4+c^4)+64(a^2b^2+b^2c^2+c^2a^2)\ge 33\sum ab(a^2+b^2).$$

(Vasile Cîrtoaje, 2008)

Solution. According to P 2.60, it suffices to prove the required inequality for $b=c=1$, when it becomes

$$5a^4-33a^3+64a^2-33a+9\ge 0,$$

$$(a-3)^2(5a^2-3a+1)\ge 0.$$

This is true since

$$5a^2-3a+1=5(a-\frac{3}{10})^2+\frac{11}{20}>0.$$

The equality holds for $a/3=b=c$ (or any cyclic permutation).

□

P 2.62. *If a,b,c are real numbers such that*

$$a+b+c=3,$$

then

$$3(a^4+b^4+c^4)+33\ge 14(a^2+b^2+c^2).$$

(Vasile Cîrtoaje, 2009)

First Solution. Write the inequality as $F(a,b,c) \geq 0$, where

$$F(a,b,c) = 3(a^4+b^4+c^4)+33-14(a^2+b^2+c^2).$$

Due to symmetry, we may assume that $a \leq b \leq c$. Let us denote

$$x = \frac{b+c}{2}, \quad x \geq 1.$$

To prove the desired inequality, we use the mixing variables method. We will show that

$$F(a,b,c) \geq F(a,x,x) \geq 0.$$

We have

$$\begin{aligned} F(a,b,c) - F(a,x,x) &= 3(b^4+b^4-2x^4) - 14(b^2+c^2-2x^2) \\ &= 3[(b^2+c^2)^2-4x^4] + 6(x^4-b^2c^2) - 14(b^2+c^2-2x^2) \\ &= (b^2+c^2-2x^2)[3(b^2+c^2+2x^2)-14] + 6(x^2-bc)(x^2+bc). \end{aligned}$$

Since

$$x^2 - bc = \frac{1}{4}(b-c)^2, \quad b^2+c^2-2x^2 = 2(x^2-bc) = \frac{1}{2}(b-c)^2,$$

we get

$$\begin{aligned} F(a,b,c) - F(a,x,x) &= \frac{1}{2}(b-c)^2[3(b^2+c^2+2x^2)-14+3(x^2+bc)] \\ &= \frac{1}{2}(b-c)^2[3(x^2-bc)+18x^2-14] \geq 0. \end{aligned}$$

Also,

$$F(a,x,x) = F(3-2x,x,x) = 6(x-1)^2(3x-5)^2 \geq 0.$$

This completes the proof. The equality holds for $a=b=c=1$, and for $a=-1/3$ and $b=c=5/3$ (or any cyclic permutation).

Second Solution. Write the inequality in the homogeneous form

$$81(a^4+b^4+c^4) + 11(a+b+c)^4 \geq 42(a^2+b^2+c^2)(a+b+c)^2.$$

According to P 2.60, it suffices to prove this inequality for $b=c=1$, when it becomes

$$25a^4 - 40a^3 + 6a^2 + 8a + 1 \geq 0,$$

$$(a-1)^2(5a+1)^2 \geq 0.$$

□

P 2.63. *If a, b, c are real numbers such that*

$$a^2+b^2+c^2=3,$$

then

$$a^4+b^4+c^4+3(ab+bc+ca)\le 12.$$

Solution. Write the inequality in the homogeneous form

$$3(a^4+b^4+c^4)+3(ab+bc+ca)(a^2+b^2+c^2)\le 4(a^2+b^2+c^2)^2.$$

According to P 2.60, it suffices to prove this inequality for $b=c=1$, when it becomes

$$a^4-6a^3+13a^2-12a+4\ge 0,$$

$$(a-1)^2(a-2)^2\ge 0.$$

The equality holds for $a=b=c=\pm 1$, for $a=\sqrt{2}$ and $b=c=\sqrt{2}/2$ (or any cyclic permutation), and for $a=-\sqrt{2}$ and $b=c=-\sqrt{2}/2$ (or any cyclic permutation).

□

P 2.64. *Let α, β, γ be real numbers such that*

$$1+\alpha+\beta=2\gamma.$$

The inequality

$$\sum a^4+\alpha\sum a^2b^2+\beta abc\sum a\ge \gamma\sum ab(a^2+b^2)$$

holds for any real numbers a, b, c if and only if

$$1+\alpha\ge\gamma^2.$$

(Vasile Cîrtoaje, 2009)

Solution. Let

$$f_4(a,b,c)=\sum a^4+\alpha\sum a^2b^2+\beta abc\sum a-\gamma\sum ab(a^2+b^2.$$

According to P 2.60, the inequality $f_4(a,b,c)\ge 0$ holds for any real numbers a, b, c if and only if $f_4(a,1,1)\ge 0$ for any real a. From

$$f_4(a,1,1)=(a-1)^2[(a-\gamma+1)^2+1+\alpha-\gamma^2],$$

the conclusion follows. The equality holds for $a=b=c$. In addition, if $1+\alpha=\gamma^2$, then the equality holds also for $\dfrac{a}{\gamma-1}=b=c$ (or any cyclic permutation).

Remark. For $\gamma = k+1$ and $\alpha = k(k+2)$ (which involves $1+\alpha = \gamma^2$) , we get

$$\sum a^4 + k(k+2)\sum a^2b^2 + (1-k^2)abc\sum a \ge (k+1)\sum ab(a^2+b^2), \quad k \in \mathbb{R},$$

which is equivalent to the elegant inequality from P 2.50, namely

$$\sum (a-b)(a-c)(a-kb)(a-kc) \ge 0,$$

where the equality holds for $a = b = c$, and also for $a/k = b = c$ (or any cyclic permutation). In addition, for $k = 0$, we get Schur's inequality of degree four

$$\sum a^2(a-b)(a-c),$$

with equality for $a = b = c$, and also for $a = 0$ and $b = c$ (or any cyclic permutation).

□

P 2.65. *If a, b, c are real numbers such that*

$$a^2 + b^2 + c^2 = 2,$$

then

$$ab(a^2 - ab + b^2 - c^2) + bc(b^2 - bc + c^2 - a^2) + ca(c^2 - ca + a^2 - b^2) \le 1.$$

Solution. Write the inequality in the homogeneous form

$$(a^2 + b^2 + c^2)^2 \ge 4\sum ab(a^2 - ab + b^2 - c^2).$$

According to P 2.60, it suffices to prove this inequality for $b = c = 1$, when it can be written as

$$a^2(a-4)^2 \ge 0.$$

The equality holds for

$$a^2 + b^2 + c^2 = 2(ab + bc + ca).$$

□

P 2.66. *If a, b, c are real numbers, then*

$$(a+b)^4 + (b+c)^4 + (c+a)^4 \ge \frac{4}{7}(a^4 + b^4 + c^4).$$

(Vietnam TST, 1996)

Solution. Denote the left side of the inequality by $f_4(a,b,c)$. According to P 2.60, it suffices to prove that $f_4(a,1,1) \ge 0$ for all real a. Indeed,

$$f_4(a,1,1) = \frac{2}{7}(5a^4 + 28a^3 + 42a^2 + 28a + 59) > 0$$

since, for the nontrivial case $a < 0$, we have

$$5a^4 + 28a^3 + 42a^2 + 28a + 59 = (5a^2 - 2a)(a+3)^2 + 9(a + \frac{23}{9})^2 + \frac{2}{9} > 0.$$

The equality holds for $a = b = c = 0$.

□

P 2.67. *Let a,b,c be real numbers. If*

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc,$$

then

$$(3-p)r + \frac{p^2+q^2-pq}{3} \ge q.$$

(Vasile Cîrtoaje, 2011)

First Solution. Write the inequality as

$$(p^2-3q) + (q^2-3pr) \ge pq - 9r,$$

$$\frac{1}{2}\sum(b-c)^2 + \frac{1}{2}\sum a^2(b-c)^2 \ge \sum a(b-c)^2.$$

According to the AM-GM inequality, it suffices to prove that

$$\sqrt{\left[\sum(b-c)^2\right]\left[\sum a^2(b-c)^2\right]} \ge \sum a(b-c)^2.$$

Clearly, this inequality follows immediately from the Cauchy-Schwarz inequality. The equality holds for $a = b = c$, and for $b = c = 1$ (or any cyclic permutation).

Second Solution. Write the inequality as $f_4(a,b,c) \ge 0$, where

$$f_4(a,b,c) = 3(3-p)r + p^2 + q^2 - pq - 3q.$$

is a symmetric polynomial of degree four in a,b,c. According to Remark from the proof of P 2.60, it suffices to prove that $f_4(a,b,b) \ge 0$ for all real numbers a and b. Indeed, we have

$$f_4(a,b,b) = (a-b)^2(b-1)^2 \ge 0.$$

□

P 2.68. *If a, b, c are real numbers, then*

$$\frac{ab(a+b)+bc(b+c)+ca(c+a)}{(a^2+1)(b^2+1)(c^2+1)} \le \frac{3}{4}.$$

(Vasile Cîrtoaje, 2011)

First Solution. We try to get a stronger homogeneous inequality. According to the AM-GM inequality, we have

$$(a^2+1)(b^2+1)(c^2+1) = (a^2b^2c^2+1)+(a^2b^2+b^2c^2+c^2a^2)+(a^2+b^2+c^2)$$

$$\ge 2abc + 2\sqrt{(a^2b^2+b^2c^2+c^2a^2)(a^2+b^2+c^2)}.$$

Therefore, it suffices to prove the homogeneous inequality

$$3abc + 3\sqrt{(a^2b^2+b^2c^2+c^2a^2)(a^2+b^2+c^2)} \ge 2\sum ab(a+b).$$

Using the identity

$$9(a^2+b^2+c^2) = \sum(2a+2b-c)^2$$

together with the Cauchy-Schwarz inequality, we get

$$3\sqrt{(a^2b^2+b^2c^2+c^2a^2)(a^2+b^2+c^2)} =$$

$$= \sqrt{\left[\sum a^2b^2\right]\left[\sum(2a+2b-c)^2\right]} \ge \sum ab(2a+2b-c)$$

$$= 2\sum ab(a+b) - 3abc.$$

The equality holds for $a = b = c = 1$.

Second Solution. Write the inequality as

$$3(abc-1)^2 + f_4(a,b,c) \ge 0,$$

where

$$f_4(a,b,c) = \sum a^2b^2 + \sum a^2 + 2abc - \frac{4}{3}\sum ab(a+b)$$

is a symmetric polynomial of degree four. Clearly, it suffices to prove that $f_4(a,b,c) \ge 0$. According to Remark from P 2.60, it suffices to prove that $f_4(a,b,b) \ge 0$ for all real numbers a and b. Indeed, we have

$$3f_4(a,b,b) = (6b^2-8b+3)a^2 - 2b^2a + b^2(3b^2-8b+6)$$

$$= (6b^2-8b+3)\left(a - \frac{b^2}{6b^2-8b+3}\right)^2 + \frac{18b^2(b-1)^4}{6b^2-8b+3} \ge 0.$$

Remark. The inequality is equivalent to

$$3(abc-1)^2 + \sum(a-1)^2(b-c)^2 + (ab+bc+ca-a-b-c)^2 \ge 0.$$

□

P 2.69. *If a, b, c are real numbers such that $abc > 0$, then*

$$\left(a+\frac{1}{a}-1\right)\left(b+\frac{1}{b}-1\right)\left(c+\frac{1}{c}-1\right)+2 \ge \frac{1}{3}(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2011)

Solution. Let

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

Multiplying by abc, we can rewrite the inequality as

$$r^2 + (4-p-q)r + p^2 + q^2 - \frac{4pq}{3} - p - q + 1 \ge 0.$$

Since the equality holds for $a = b = c = 1$, when $p = q = 3$ and $r = 1$, we write the inequality as

$$\left(r - 1 + \frac{p-q}{2}\right)^2 + f(p,q,r) \ge 0,$$

where

$$\begin{aligned} 12f(p,q,r) &= 24(3-p)r + 9(p^2+q^2) - 10pq - 24q \\ &\ge 24(3-p)r + 8(p^2+q^2-pq) - 24q \end{aligned}$$

Thus, it suffices to prove that $f_4(a,b,c) \ge 0$ for all real a, b, c, where

$$f_4(a,b,c) = 3(3-p)r + p^2 + q^2 - pq - 3q \ge 0.$$

According to Remark from P 2.60, it suffices to prove that $f_4(a,b,b) \ge 0$ for all real numbers a and b. Indeed, we have

$$f_4(a,b,b) = (b-1)^2(a-b)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

Remark. The inequalities in P 2.68 and P 2.69 are particular cases of the following more general statement (*Vasile Cîrtoaje*, 2011).

- *Let a, b, c be real numbers such that $abc > 0$. If*

$$-2 \le k \le 1,$$

then

$$\left(a+\frac{1}{a}+k\right)\left(b+\frac{1}{b}+k\right)\left(c+\frac{1}{c}+k\right)+(1-k)(2+k)^2 \ge$$
$$\ge \frac{1}{3}(2+k)^2(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

□

P 2.70. *If a, b, c are real numbers, then*

$$\left(a^2+\frac{1}{2}\right)\left(b^2+\frac{1}{2}\right)\left(c^2+\frac{1}{2}\right) \ge \left(a+b-\frac{1}{2}\right)\left(b+c-\frac{1}{2}\right)\left(c+a-\frac{1}{2}\right).$$

(Vasile Cîrtoaje, 2011)

Solution. It suffices to prove that $f_4(a,b,c) \ge 0$, where

$$f_4(a,b,c) = \prod\left(a^2+\frac{1}{2}\right) - \prod\left(a+b-\frac{1}{2}\right) - \left(abc+\frac{1}{2}-\frac{a+b+c}{2}\right)^2$$

is a symmetric polynomial of degree four. According to Remark from P 2.60, it suffices to prove that $f_4(a,b,b) \ge 0$ for all real a and b. Indeed, we have

$$2f_4(a,b,b) = [(2b-1)a - b(2-b)]^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

Remark. The inequality is equivalent to

$$(2abc+1-a-b-c)^2 + 2(ab+bc+ca-a-b-c)^2 \ge 0.$$

□

P 2.71. *If a, b, c are real numbers such that*

$$a+b+c=3,$$

then

$$\frac{a(a-1)}{8a^2+9} + \frac{b(b-1)}{8b^2+9} + \frac{c(c-1)}{8c^2+9} \ge 0.$$

(Vasile Cîrtoaje, 2013)

Solution *(by Michael Rozenberg).* Write the inequality as follow

$$\sum \frac{pa(a-1)}{8a^2+9} \ge 0, \quad p > 0,$$

$$\sum \left[\frac{pa(a-1)}{8a^2+9} + 1\right] \ge 3,$$

$$\sum \frac{(p+8)a^2 - pa + 9}{8a^2+9} \ge 3.$$

Choosing $p = 18 + 6\sqrt{17}$, the inequality can be written as

$$\sum \frac{(ka-3)^2}{8a^2+9} \ge 3, \quad k = 3 + \sqrt{17}.$$

Let m be a real constant. According to the Cauchy-Schwarz inequality, we have

$$\sum \frac{(ka-3)^2}{8a^2+9} \ge \frac{\left[\sum (ka-3)(ma+3)\right]^2}{\sum (ma+3)^2(8a^2+9)},$$

with equality for

$$\frac{ka-3}{(8a^2+9)(ma+3)} = \frac{kb-3}{(8b^2+9)(mb+3)} = \frac{kc-3}{(8c^2+9)(mc+3)}. \qquad (*)$$

On the other hand, we can check that the original inequality becomes an equality for $a=b=c$, and also for $a=3/2$ and $b=c=3/4$ (or any cyclic permutation). It is easy to get that the equality conditions (*) are satisfied for $a=3/2$ and $b=c=3/4$ if and only if $m=k$. For this value of m, it suffices to show that

$$\frac{\left[\sum (ka-3)(ka+3)\right]^2}{\sum (ka+3)^2(8a^2+9)} \ge 3,$$

$$\left[\sum (k^2a^2-9)\right]^2 \ge 3\sum (ka+3)^2(8a^2+9),$$

$$[k^2(a^2+b^2+c^2)-27]^2 \ge 3\sum (ka+3)^2(8a^2+9).$$

Write this inequality in the homogeneous form $f_4(a,b,c) \ge 0$, where

$$f_4(a,b,c) = [k^2(a^2+b^2+c^2)-3(a+b+c)^2]^2$$

$$-3\sum (ka+a+b+c)^2[8a^2+(a+b+c)^2].$$

According to P 2.60, it suffices to prove that $f_4(a,1,1) \ge 0$ for all real a. Indeed, this inequality is equivalent to

$$(a-1)^2(a-2)^2 \ge 0.$$

□

P 2.72. *If a,b,c are real numbers such that*

$$a+b+c=3,$$

then

$$\frac{(a-11)(a-1)}{2a^2+1} + \frac{(b-11)(b-1)}{2b^2+1} + \frac{(c-11)(c-1)}{2c^2+1} \ge 0.$$

(Vasile Cîrtoaje, 2013)

Solution. Write the inequality as

$$\sum\left[\frac{(a-11)(a-1)}{2a^2+1}+1\right]\geq 3,$$

$$\sum\frac{(a-2)^2}{2a^2+1}\geq 1.$$

According to the Cauchy-Schwarz inequality, we have

$$\sum\frac{(a-2)^2}{2a^2+1}\geq\frac{\left[\sum(a-2)^2\right]^2}{\sum(a-2)^2(2a^2+1)}.$$

Therefore, it suffices to show that

$$\left[\sum(a-2)^2\right]^2\geq\sum(a-2)^2(2a^2+1),$$

$$(a^2+b^2+c^2)^2\geq 2\sum a^4-8\sum a^3+9\sum a^2-4\sum a+12.$$

Write this inequality in the homogeneous form $f_4(a,b,c)\geq 0$, where

$$\begin{aligned}f_4(a,b,c)&=3(a^2+b^2+c^2)^2-6\sum a^4+8\left(\sum a^3\right)\left(\sum a\right)-3\left(\sum a^2\right)\left(\sum a\right)^2\\&=2\left[\sum a^4+\sum ab(a^2+b^2)-3abc\sum a\right].\end{aligned}$$

According to P 2.60, it suffices to prove that $f_4(a,1,1)\geq 0$ for all real a. Indeed,

$$f_4(a,1,1)=2(a-1)^2(a+2)^2\geq 0.$$

The equality holds for $a=b=c=1$.

□

P 2.73. *If a,b,c are real numbers, then*

$$(a^2+2)(b^2+2)(c^2+2)\geq 9(ab+bc+ca).$$

(Vasile Cîrtoaje, 1994)

Solution. We will prove the sharper inequality $f_4(a,b,c)\geq 0$, where

$$f_4(a,b,c)=(a^2+2)(b^2+2)(c^2+2)-9(ab+bc+ca)-\left(abc-\frac{a+b+c}{3}\right)^2.$$

Since $f_4(a,b,c)$ is a symmetric polynomial of degree four, according to Remark from P 2.60, it suffices to prove that $f_4(a,b,b)\geq 0$ for all real numbers a and b. For fixed b, this inequality is equivalent to $f(a)\geq 0$, where

$$f(a)=7(6b^2+5)a^2+2b(6b^2-83)a+18b^4-13b^2+72.$$

It is true for all real a if and only if

$$7(6b^2+5)(18b^4-13b^2+72) \geq b^2(6b^2-83)^2.$$

Indeed, we have

$$7(6b^2+5)(18b^4-13b^2+72)-b^2(6b^2-83)^2 = 360(b^2-1)^2(2b^2+7) \geq 0.$$

The equality holds for $a=b=c=1$.

Remark. A sharper inequality for all real a,b,c is the following:

$$(a^2+2)(b^2+2)(c^2+2) \geq 3(a+b+c)^2 + \frac{4}{9}\left[(a-b)^2+(b-c)^2+(c-a)^2\right],$$

with equality for $a=b=c=1$.

The proof is similar, and the final inequality of the proof has the form

$$b^2(b^2-1)^2 \geq 0.$$

□

P 2.74. *If a,b,c are real numbers such that*

$$ab+bc+ca=3,$$

then

$$4(a^4+b^4+c^4)+11abc(a+b+c) \geq 45.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality in the homogeneous form

$$4(a^4+b^4+c^4)+11abc(a+b+c) \geq 5(ab+bc+ca)^2.$$

It suffices to prove that there exists a positive number k such that $f_4(a,b,c) \geq 0$, where

$$\begin{aligned} f_4(a,b,c) =& 4(a^4+b^4+c^4)+11abc(a+b+c)-5(ab+bc+ca)^2 \\ & -k(ab+bc+ca)(a^2+b^2+c^2-ab-bc-ca). \end{aligned}$$

According to P 2.60, the inequality $f_4(a,b,c) \geq 0$ holds for all real a,b,c if and only if $f_4(a,1,1) \geq 0$ for all real a. We have

$$\begin{aligned} f_4(a,1,1) &= (a-1)^2(2a+1)(2a+3)-k(2a+1)(a-1)^2 \\ &= (a-1)^2(2a+1)(2a+3-k). \end{aligned}$$

Setting $k=2$, we get

$$f_4(a,1,1) = (a-1)^2(2a+1)^2 \geq 0.$$

The equality holds for $a=b=c=\pm 1$.

□

P 2.75. *Any sixth degree symmetric homogeneous polynomial $f_6(a,b,c)$ can be written in the form*

$$f_6(a,b,c) = Ar^2 + B(p,q)r + C(p,q),$$

where A is called the highest coefficient of f_6, and

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

In the case $A \le 0$, prove that the inequality $f_6(a,b,c) \ge 0$ holds for all real numbers a,b,c if and only if $f_6(a,1,1) \ge 0$ for all real a.

(Vasile Cîrtoaje, 2006)

Solution. For $A \le 0$ and fixed p and q,

$$g(r) = Ar^2 + B(p,q)r + C(p,q)$$

is a concave quadratic function of r. Therefore, $g(r)$ is minimal when r is minimal or maximal. By P 2.53, r is minimal and maximal when two of a,b,c are equal. Since $f_6(a,b,c)$ is symmetric, homogeneous and satisfies $f_6(-a,-b,-c) = f_6(a,b,c)$, it follows that the inequality $f_6(a,b,c) \ge 0$ holds for all real numbers a,b,c if and only if $f_6(a,1,1) \ge 0$ and $f_6(a,0,0) \ge 0$ for all real a. Notice that the condition "$f_6(a,0,0) \ge 0$ for all real a" is not necessary because it follows from the condition "$f_6(a,1,1) \ge 0$ for all real a" as follows:

$$f_6(a,0,0) = \lim_{t\to 0} f_6(a,t,t) = \lim_{t\to 0} t^6 f_6(a/t,1,1) \ge 0.$$

Remark 1. A symmetric homogeneous polynomial of degree six in three variables has the form

$$(A) \qquad f_6(a,b,c) = A_1\sum a^6 + A_2\sum ab(a^4+b^4) + A_3\sum a^2b^2(a^2+b^2)$$

$$+A_4\sum a^3b^3 + A_5abc\sum a^3 + A_6abc\sum ab(a+b) + 3A_7a^2b^2c^2,$$

where $A_1,\ldots,A_7$ are real constants. In order to write this polynomial as a function of p, q and r, we can use the following relations:

$$\sum a^3 = 3r + p^3 - 3pq,$$

$$\sum ab(a+b) = -3r + pq,$$

$$\sum a^3b^3 = 3r^2 - 3pqr + q^3,$$

$$\sum a^2b^2(a^2+b^2) = -3r^2 - 2(p^3 - 2pq)r + p^2q^2 - 2q^3,$$

$$\sum ab(a^4+b^4) = -3r^2 - 2(p^3 - 7pq)r + p^4q - 4p^2q^2 + 2q^3,$$

$$\sum a^6 = 3r^2 + 6(p^3 - 2pq)r + p^6 - 6p^4q + 9p^2q^2 - 2q^3.$$

According to these relations, the highest coefficient A of the polynomial $f_6(a,b,c)$ has the expression

$$(B) \qquad A = 3(A_1 - A_2 - A_3 + A_4 + A_5 - A_6 + A_7).$$

Remark 2. The polynomial

$$P_1(a,b,c) = \sum (A_1a^2 + A_2bc)(B_1a^2 + B_2bc)(C_1a^2 + C_2bc)$$

has the highest coefficient

$$A = 3(A_1 + A_2)(B_1 + B_2)(C_1 + C_2) = P_1(1,1,1).$$

Indeed, since

$$\begin{aligned} P_1(a,b,c) =& A_1B_1C_1 \sum a^6 + A_2B_2C_2 \sum b^3c^3 + \left(\sum A_1B_1C_2\right) abc \sum a^3 \\ &+ 3\left(\sum A_1B_2C_2\right) a^2b^2c^2, \end{aligned}$$

we have

$$\begin{aligned} A &= 3A_1B_1C_1 + 3A_2B_2C_2 + 3\sum A_1B_1C_2 + 3\sum A_1B_2C_2 \\ &= 3(A_1 + A_2)(B_1 + B_2)(C_1 + C_2). \end{aligned}$$

Similarly, we can show that the polynomial

$$P_2(a,b,c) = \sum (A_1a^2 + A_2bc)(B_1b^2 + B_2ca)(C_1c^2 + C_2ab)$$

has the highest coefficient

$$A = 3(A_1 + A_2)(B_1 + B_2)(C_1 + C_2) = P_2(1,1,1),$$

and the polynomial

$$P_3(a,b,c) = \prod (A_1a^2 + A_2bc) = (A_1a^2 + A_2bc)(A_1b^2 + A_2ca)(A_1c^2 + A_2ab)$$

has the highest coefficient

$$A = (A_1 + A_2)^3 = P_3(1,1,1).$$

The polynomial

$$P_4(a,b,c) = \prod (a^2 + mab + b^2)$$

has the highest coefficient

$$A = (m-1)^3.$$

Indeed, since

$$\prod (a^2 + mab + b^2) = \prod (p^2 - 2q - c^2 + mab),$$

$P_4(a,b,c)$ has the same highest coefficient as $R_3(a,b,c) = \prod(-c^2 + mab)$; that is,

$$A = R_3(1,1,1) = (-1+m)^3.$$

As a consequence,

$$P_5(a,b,c) = (a-b)^2(b-c)^2(c-a)^2 = \prod(a^2 - 2ab + b^2)$$

has the highest coefficient

$$A = (-2-1)^2 = -27.$$

Remark 3. We can extend the statement in P 2.75 as follows:

• *Let $f_6(a,b,c)$ be a sixth degree symmetric homogeneous polynomial having the highest coefficient $A \le 0$, and let k_1, k_2 be two fixed real numbers. The inequality $f_6(a,b,c) \ge 0$ holds for all real numbers a,b,c satisfying*

$$k_1(a+b+c)^2 + k_2(ab+bc+ca) \ge 0,$$

if and only if $f_6(a,1,1) \ge 0$ for all real a satisfying $k_1(a+2)^2 + k_2(2a+1) \ge 0$.

Notice that the condition "$f_6(a,0,0) \ge 0$ for all real a satisfying $k_1a^2 \ge 0$" is not necessary because it follows from the condition "$f_6(a,1,1) \ge 0$ for all real a satisfying $k_1(a+2)^2 + k_2(2a+1) \ge 0$". Indeed, for the non-trivial case $k_1 \ge 0$, when the condition "$f_6(a,0,0) \ge 0$ for all real a satisfying $k_1a^2 \ge 0$" becomes "$f_6(a,0,0) \ge 0$ for all real a", we have

$$f_6(a,0,0) = \lim_{t\to 0} f_6(a,t,t) = \lim_{t\to 0} t^6 f_6(a/t,1,1) \ge 0.$$

Remark 4. The statement in P 2.75 and its extension in Remark 3 are also valid in the more general case when $f_6(a,b,c)$ is a symmetric homogeneous function of the form

$$f_6(a,b,c) = Ar^2 + B(p,q)r + C(p,q),$$

where $B(p,q)$ and $C(p,q)$ are rational functions.

□

P 2.76. *If a,b,c are real numbers such that*

$$ab + bc + ca = -1,$$

then

(a) $$5(a^2+b^2)(b^2+c^2)(c^2+a^2) \ge 8;$$

(b) $$(a^2+ab+b^2)(b^2+bc+c^2)(c^2+ca+a^2) \ge 1.$$

(Vasile Cîrtoaje, 2011)

Solution. We use the *highest coefficient method* (see P 2.75). Let

$$p = a + b + c, \quad q = ab + bc + ca.$$

(a) Write the inequality in the homogeneous form $f_6(a, b, c) \geq 0$, where

$$f_6(a, b, c) = 5(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) + 8(ab + bc + ca)^3.$$

From

$$\prod (b^2 + c^2) = \prod (p^2 - 2q - a^2),$$

it follows that $f_6(a, b, c)$ has the highest coefficient $A = -5$. Then, by P 2.75, it suffices to prove that $f_6(a, 1, 1) \geq 0$ for all real a. Indeed, we have

$$f_6(a, 1, 1) = 2(a + 3)^2(5a^2 + 2a + 1) \geq 0.$$

The homogeneous inequality

$$f_6(a, b, c) \geq 0$$

is an equality for $-a/3 = b = c$ (or any cyclic permutation), and for $b = c = 0$ (or any cyclic permutation). The original inequality becomes an equality for $a = -3/\sqrt{5}$ and $b = c = 1/\sqrt{5}$ (or any cyclic permutation), and for $a = 3/\sqrt{5}$ and $b = c = -1/\sqrt{5}$ (or any cyclic permutation).

(b) Write the inequality in the homogeneous form $f_6(a, b, c) \geq 0$, where

$$f_6(a, b, c) = \prod (b^2 + bc + c^2) + (ab + bc + ca)^3.$$

According to Remark 2 from P 2.75, $f_6(a, b, c)$ has the highest coefficient

$$A = (1 - 1)^3 = 0.$$

Then, by P 2.75, it suffices to prove that $f_6(a, 1, 1) \geq 0$ for all real a. Indeed, we have

$$f_6(a, 1, 1) = (a + 2)^2(3a^2 + 2a + 1) \geq 0.$$

The homogeneous inequality $f_6(a, b, c) \geq 0$ is an equality when $a + b + c = 0$, and when $b = c = 0$ (or any cyclic permutation). The original inequality becomes an equality for

$$ab + bc + ca = -1, \quad a + b + c = 0.$$

Remark. As we have shown in the proof of P 2.34,

$$\prod (b^2 + bc + c^2) = (p^2 - q)q^2 - p^3abc.$$

Therefore,

$$f_6(a, b, c) = p^2(q^2 - pabc) = \frac{1}{2}p^2 \sum a^2(b - c)^2 \geq 0.$$

□

P 2.77. *If a, b, c are real numbers, then*

(a) $\sum a^2(a-b)(a-c)(a+2b)(a+2c)+(a-b)^2(b-c)^2(c-a)^2 \ge 0;$

(b) $\sum a^2(a-b)(a-c)(a-4b)(a-4c)+7(a-b)^2(b-c)^2(c-a)^2 \ge 0.$

(Vasile Cîrtoaje, 2008)

Solution. Consider the more general inequality $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c)=f(a,b,c)+m(a-b)^2(b-c)^2(c-a)^2,$$

$$f(a,b,c)=\sum a^2(a-b)(a-c)(a-kb)(a-kc).$$

Since

$$f(a,b,c)=\sum a^2(a^2+2bc-q)[a^2+(k+k^2)bc-kq], \quad q=ab+bc+ca,$$

$f(a,b,c)$ has the same highest coefficient as $P_1(a,b,c)$, where

$$P_1(a,b,c)=\sum a^2(a^2+2bc)[a^2+(k+k^2)bc].$$

According to Remark 2 from P 2.75, $f_6(a,b,c)$ has the highest coefficient

$$A=P_1(1,1,1)-27m=9(k^2+k+1-3m).$$

(a) For $k=-2$ and $m=1$, we get $A=0$. Then, by P 2.75, it suffices to prove the original inequality for $b=c=1$; that is,

$$a^2(a-1)^2(a+2)^2 \ge 0.$$

The equality holds for $a=b=c$, for $a+b+c=0$, and for $a=0$ and $b=c$ (or any cyclic permutation).

(b) For $k=4$ and $m=7$, we get $A=0$. Then, by P 2.75, it suffices to prove the original inequality for $b=c=1$; that is,

$$a^2(a-1)^2(a-4)^2 \ge 0.$$

The equality holds for $a=b=c$, and for $a^2+b^2+c^2=2(ab+bc+ca)$.

Remark. The inequalities in P 2.77 are respectively equivalent to

$$\left(\sum a\right)^2\left[\sum a^4+abc\sum a-\sum ab(a^2+b^2)\right] \ge 0$$

and

$$\left(\sum a^2-\sum ab\right)\left(\sum a^2-2\sum ab\right)^2 \ge 0.$$

□

P 2.78. *If a, b, c are real numbers, then*

$$(a^2+2bc)(b^2+2ca)(c^2+2ab)+(a-b)^2(b-c)^2(c-a)^2 \ge 0.$$

(Vasile Cîrtoaje, 2011)

First Solution *(by Vo Quoc Ba Can).* Without loss of generality, assume that b and c have the same sign. Since $a^2+2bc \ge 0$ and

$$\begin{aligned}(a-b)^2(a-c)^2 &= \frac{1}{4}[(a^2+2bc)+(a^2-2ab-2ac)]^2 \\ &\ge (a^2+2bc)(a^2-2ab-2ac),\end{aligned}$$

it suffices to prove that

$$(b^2+2ca)(c^2+2ab)+(b-c)^2(a^2-2ab-2ac) \ge 0.$$

This inequality is equivalent to

$$(b+c)^2a^2+2bc(b+c)a+b^2c^2 \ge 0,$$

$$[(b+c)a+bc]^2 \ge 0.$$

which is clearly true. The equality holds for $ab+bc+ca=0$.

Second Solution. Denote the left side of the inequality by $f_6(a,b,c)$. According to Remark 2 from P 2.75, $f_6(a,b,c)$ has the highest coefficient

$$A=(1+2)^3-27=0.$$

Then, by P 2.75, it suffices to prove that $f_6(a,1,1) \ge 0$ for all real a. Indeed,

$$f_6(a,1,1)=(a^2+2)(2a+1)^2 \ge 0.$$

Remark 1. The inequality is equivalent to

$$(a^2+b^2+c^2)(ab+bc+ca)^2 \ge 0.$$

Remark 2. The inequality in P 2.78 is a particular case of the following more general statement.

- *If a, b, c are real numbers and*

$$\alpha_k = \begin{cases} \dfrac{9k^2(k^2-k+1)}{4(k+1)^3}, & 1 \le k \le 2 \\ \dfrac{k^2}{4}, & k \ge 2 \end{cases},$$

then

$$(a^2+kbc)(b^2+kca)(c^2+kab)+\alpha_k(a-b)^2(b-c)^2(c-a)^2 \ge 0,$$

with equality for $-ka=b=c$ (or any cyclic permutation), and also for $b=c=0$ (or any cyclic permutation).

□

P 2.79. *If a,b,c are real numbers, then*

$$(2a^2+5ab+2b^2)(2b^2+5bc+2c^2)(2c^2+5ca+2a^2)+(a-b)^2(b-c)^2(c-a)^2 \geq 0.$$

(Vasile Cîrtoaje, 2011)

***Solution*.** Denote the left side of the inequality by $f_6(a,b,c)$. Clearly, $f_6(a,b,c)$ has the same highest coefficient as

$$(5ab-2c^2)(5bc-2a^2)(5ca-2b^2)+(a-b)^2(b-c)^2(c-a)^2.$$

According to Remark 2 from P 2.75, $f_6(a,b,c)$ has the highest coefficient

$$A=(5-2)^3-27=0.$$

Then, by P 2.75, it suffices to prove that $f_6(a,1,1) \geq 0$ for all real a. Indeed,

$$f_6(a,1,1)=9(2a^2+5a+2)^2 \geq 0.$$

The equality holds for $a+b+c=0$, and also for $ab+bc+ca=0$.

Remark 1. The inequality in P 2.79 is equivalent to

$$(a+b+c)^2(ab+bc+ca)^2 \geq 0.$$

Remark 2. The following more general statement holds.

- *Let a,b,c be real numbers. If $k>-2$, then*

$$4\prod(b^2+kbc+c^2) \geq (2-k)(a-b)^2(b-c)^2(c-a)^2.$$

Notice that this inequality is equivalent to

$$(k+2)[(a+b+c)(ab+bc+ca)-(5-2k)abc]^2 \geq 0.$$

□

P 2.80. *If a,b,c are real numbers, then*

$$\left(a^2+\frac{2}{3}ab+b^2\right)\left(b^2+\frac{2}{3}bc+c^2\right)\left(c^2+\frac{2}{3}ca+a^2\right) \geq \frac{64}{27}(a^2+bc)(b^2+ca)(c^2+ab).$$

***Solution*.** Write the inequality as $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c)=P_4(a,b,c)-\frac{64}{27}P_3(a,b,c),$$

$$P_4(a,b,c)=\left(a^2+\frac{2}{3}ab+b^2\right)\left(b^2+\frac{2}{3}bc+c^2\right)\left(c^2+\frac{2}{3}ca+a^2\right),$$

$$P_3(a,b,c) = (a^2+bc)(b^2+ca)(c^2+ab).$$

According to Remark 2 from P 2.75, $f_6(a,b,c)$ has the highest coefficient

$$A = \left(\frac{2}{3}-1\right)^3 - \frac{64}{27}(1+1)^3 < 0.$$

Then, by P 2.75, it suffices to prove that $f_6(a,1,1) \geq 0$ for all real a. Indeed,

$$f_6(a,1,1) = \frac{8}{3}\left(a^2+\frac{2}{3}a+1\right)^2 - \frac{64}{27}(a^2+1)(a+1)^2 = \frac{8}{27}(a-1)^4 \geq 0.$$

The equality holds for $a = b = c$.

□

P 2.81. *If a,b,c are real numbers, then*

$$\sum a^2(a-b)(a-c) \geq \frac{2(a-b)^2(b-c)^2(c-a)^2}{a^2+b^2+c^2}.$$

Solution. Let

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc$$

and

$$f_6(a,b,c) = (a^2+b^2+c^2)\sum a^2(a-b)(a-c) - 2(a-b)^2(b-c)^2(c-a)^2.$$

Clearly, $f_6(a,b,c)$ has the highest coefficient

$$A = -2(-27) = 54.$$

Since $A > 0$, we will use the *highest coefficient cancellation method*. It is easy to check that

$$f_6(1,1,1) = 0, \quad f_6(0,1,1) = 0.$$

Therefore, we define the symmetric homogeneous polynomial of degree three

$$P(a,b,c) = r + Bp^3 + Cpq$$

such that

$$P(1,1,1) = 0, \quad P(0,1,1) = 0;$$

that is,

$$P(a,b,c) = r + \frac{1}{9}p^3 - \frac{4}{9}pq.$$

We will prove the sharper inequality $g_6(a,b,c) \geq 0$, where

$$g_6(a,b,c) = f_6(a,b,c) - 54P^2(a,b,c).$$

Since $g_6(a,b,c)$ has the highest coefficient $A_1 = 0$, it suffices to show that $g_6(a,1,1) \ge 0$ for all real a (see P 2.75). Indeed, we have

$$f_6(a,1,1) = a^2(a^2+2)(a-1)^2, \quad P(a,1,1) = \frac{1}{9}a(a-1)^2,$$

hence

$$g_6(a,1,1) = f_6(a,1,1) - 54P^2(a,1,1) = \frac{1}{3}a^2(a-1)^2(a+2)^2 \ge 0.$$

The equality holds for $a = b = c$, for $a = 0$ and $b = c$ (or any cyclic permutation), and also for $a = 0$ and $b + c = 0$ (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization (*Vasile Cîrtoaje, 2014*).

- *Let a, b, c be real numbers. If $k \in [-1, 2)$, then*

$$\sum a^2(a-b)(a-c) \ge \frac{(2-k)(a-b)^2(b-c)^2(c-a)^2}{a^2+b^2+b^2+k(ab+bc+ca)},$$

with equality for $a = b = c$, and also for $a = 0$ and $b^2 = c^2$ (or any cyclic permutation).

□

P 2.82. *If a, b, c are real numbers, then*

$$\sum (a-b)(a-c)(a-2b)(a-2c) \ge \frac{8(a-b)^2(b-c)^2(c-a)^2}{a^2+b^2+c^2}.$$

Solution. Let

$$\begin{aligned} f_6(a,b,c) = &(a^2+b^2+c^2)\sum (a-b)(a-c)(a-2b)(a-2c) \\ &- 8(a-b)^2(b-c)^2(c-a)^2. \end{aligned}$$

Clearly, $f_6(a,b,c)$ has the highest coefficient

$$A = (-8)(-27) = 216.$$

Since $A > 0$, we will use the *highest coefficient cancellation method*. Since

$$f_6(1,1,1) = 0, \quad f_6(2,1,1) = 0,$$

we define the symmetric homogeneous polynomial of degree three

$$P(a,b,c) = abc + B(a+b+c)^3 + C(a+b+c)(ab+bc+ca)$$

such that

$$P(1,1,1)=0, \quad P(2,1,1)=0.$$

We get $B=1/18$ and $C=-5/18$, hence

$$P(a,b,c)=abc+\frac{1}{18}(a+b+c)^3-\frac{5}{18}(a+b+c)(ab+bc+ca).$$

Consider now the sharper inequality $g_6(a,b,c)\geq 0$, where

$$g_6(a,b,c)=f_6(a,b,c)-216P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A_1=0$. By P 2.75, it suffices to prove that $g_6(a,1,1)\geq 0$ for all real a. We have

$$f_6(a,1,1)=(a^2+2)(a-1)^2(a-2)^2, \quad P(a,1,1)=\frac{1}{18}(a-1)^2(a-2),$$

hence

$$g_6(a,1,1)=f_6(a,1,1)-216P^2(a,1,1)=\frac{1}{3}(a-1)^2(a^2-4)^2\geq 0.$$

The equality holds for $a=b=c$, for $a=0$ and $b+c=0$ (or any cyclic permutation), and also for $a/2=b=c$ (or any cyclic permutation).

Remark. In the same manner, we can prove the following generalization (*Vasile Cîrtoaje, 2014*).

- *Let a,b,c be real numbers. If $k\in\mathbb{R}$, then*

$$\sum(a-b)(a-c)(a-kb)(a-kc)\geq\frac{(k+2)^2(a-b)^2(b-c)^2(c-a)^2}{2(a^2+b^2+c^2)},$$

with equality for $a=b=c$, for $a/k=b=c$ (or any cyclic permutation) if $k\neq 0$, and for $a=0$ and $b+c=0$ (or any cyclic permutation).

□

P 2.83. *If a,b,c are real numbers, no two of which are zero, then*

$$\frac{a^2+3bc}{b^2+c^2}+\frac{b^2+3ca}{c^2+a^2}+\frac{c^2+3ab}{a^2+b^2}\geq 0.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as $f_6(a,b,c)\geq 0$, where

$$f_6(a,b,c)=\sum(a^2+3bc)(a^2+b^2)(a^2+c^2).$$

Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

From

$$f_6(a,b,c) = \sum (a^2 + 3bc)(p^2 - 2q - c^2)(p^2 - 2q - b^2),$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as $f(a,b,c)$, where

$$f(a,b,c) = \sum (a^2 + 3bc)b^2c^2 = 3r^2 + 3\sum b^3c^3 = 12r^2 - 9pqr + 3q^3;$$

that is,

$$A = 12.$$

Since $A > 0$, we will use the *highest coefficient cancellation method*. It is easy to check that

$$f_6(-1,1,1) = 0.$$

So, we define the homogeneous polynomial

$$P(a,b,c) = r + Bp^3 + (B-1)pq,$$

which satisfies the property $P(-1,1,1) = 0$. We will show that there is at least a real value of B such that the following sharper inequality holds

$$f_6(a,b,c) \ge 12P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 12P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A_1 = 0$. By P 2.75, it suffices to prove that $g_6(a,1,1) \ge 0$ for all real a. We have

$$f_6(a,1,1) = (a+1)^2(a^2+1)(a^2-2a+7)$$

and

$$P(a,1,1) = (a+1)[B(a+2)(a+5) - 2(a+1)],$$

hence

$$g_6(a,1,1) = f_6(a,1,1) - 12P^2(a,1,1) = (a+1)^2 g(a),$$

where

$$g(a) = (a^2+1)(a^2-2a+7) - 12[B(a+2)(a+5) - 2(a+1)]^2.$$

Choosing $B = 1/4$, we get

$$4g(a) = a^2(a-1)^2 + 4(4a^2 + a + 4) > 0,$$

hence $g_6(a,1,1) \ge 0$ for all real a. The proof is completed. The equality holds for $-a = b = c$ (or any cyclic permutation).

□

P 2.84. *If a, b, c are real numbers, no two of which are zero, then*

$$\frac{a^2+6bc}{b^2-bc+c^2}+\frac{b^2+6ca}{c^2-ca+a^2}+\frac{c^2+6ab}{a^2-ab+b^2}\ge 0.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as $f_6(a,b,c)\ge 0$, where

$$f_6(a,b,c)=\sum(a^2+6bc)(a^2-ab+b^2)(a^2-ac+c^2).$$

Let

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc.$$

From

$$f_6(a,b,c)=\sum(a^2+6bc)(p^2-2q-c^2-ab)(p^2-2q-b^2-ac),$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as $f(a,b,c)$, where

$$f(a,b,c)=\sum(a^2+6bc)(b^2+ca)(c^2+ab);$$

that is, according to Remark 2 from P 2.75,

$$A=f(1,1,1)=84.$$

Since $A>0$, we use the *highest coefficient cancellation method*. We will show that there are two real numbers B and C such that the following sharper inequality holds

$$f_6(a,b,c)\ge 84P^2(a,b,c),$$

where

$$P(a,b,c)=r+Bp^3+Cpq.$$

Let us denote

$$g_6(a,b,c)=f_6(a,b,c)-84P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient equal to zero. Then, it suffices to prove that $g_6(a,1,1)\ge 0$ for all real a. We have

$$g_6(a,1,1)=f_6(a,1,1)-84P^2(a,1,1),$$

where

$$f_6(a,1,1)=(a^2-a+1)(a^2+a+1)(a^2-2a+8)$$

and

$$P(a,1,1)=a+B(a+2)^3+C(a+2)(2a+1).$$

Let

$$g(a)=g_6(a,1,1).$$

Since $g(-2)=0$, we can have $g(a)\geq 0$ in the vicinity of $a=-2$ only if $g'(-2)=0$, which involves $C=-61/168$. On the other hand, from $g(1)=0$, we get $B=155/1512$. Using these values of B and C, the inequality $g_6(a,1,1)\geq 0$ is equivalent to

$$27216(a^2-a+1)(a^2+a+1)(a^2-2a+8)\geq$$
$$\geq\left[155(a+2)^3-549(a+2)(2a+1)+1512a\right]^2;$$

that is,

$$(a+2)^2(a-1)^2(3191a^2-8734a+49391)\geq 0,$$

which is true for all real a. The proof is completed. The equality holds for $a=b+c=0$ (or any cyclic permutation).

□

P 2.85. *If a,b,c are real numbers such that*

$$ab+bc+ca\geq 0,$$

then

$$\frac{4a^2+23bc}{b^2+c^2}+\frac{4b^2+23ca}{c^2+a^2}+\frac{4c^2+23ab}{a^2+b^2}\geq 0.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as $f_6(a,b,c)\geq 0$, where

$$f_6(a,b,c)=\sum(4a^2+23bc)(a^2+b^2)(a^2+c^2).$$

Let

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc.$$

From

$$f_6(a,b,c)=\sum(4a^2+23bc)(p^2-2q-c^2)(p^2-2q-b^2),$$

it follows that $f_6(a,b,c)$ has the same highest coefficient A as $f(a,b,c)$, where

$$f(a,b,c)=\sum(4a^2+23bc)b^2c^2=12r^2+23\sum b^3c^3=81r^2-69pqr+23q^3;$$

that is,

$$A=81.$$

Since $A>0$, we will use the *highest coefficient cancellation method*. It is easy to check that

$$f(-1,2,2)=0.$$

Therefore, define the homogeneous polynomial

$$P(a,b,c)=r+\frac{4}{27}p^3+Cpq,$$

which satisfies $P(-1,2,2)=0$. We will show that there is at least a real C such that the following sharper inequality holds for $ab+bc+ca\ge 0$:

$$f_6(a,b,c)\ge 81P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c)=f_6(a,b,c)-81P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A_1=0$. Therefore, by Remark 3 from P 2.75, it suffices to prove that $g_6(a,1,1)\ge 0$ for all real a such that $2a+1\ge 0$. We have

$$f_6(a,1,1)=(2a+1)(a^2+1)(2a^3-a^2+14a+39),$$

$$P(a,1,1)=\frac{1}{27}(2a+1)[2a^2+(27C+11)a+54C+32],$$

$$g_6(a,1,1)=f_6(a,1,1)-81P^2(a,1,1).$$

From the condition $g_6(1,1,1)=0$, we get $C=-1/3$. For this value of C, we find

$$P(a,1,1)=\frac{2}{27}(2a+1)(a^2+a+7),$$

then

$$\begin{aligned}g_6(a,1,1)&=\frac{1}{9}(2a+1)(10a^5-29a^4+16a^3+170a^2-322a+155)\\&=\frac{1}{9}(2a+1)(a-1)^2(10a^3-9a^2-12a+155).\end{aligned}$$

We only need to show that

$$10a^3-9a^2-12a+155\ge 0$$

for $a\ge -1/2$. This is clearly true for $-1/2\le a\le 0$. Also, for $a>0$, we have

$$10a^3-9a^2-12a+155=10a(a^2-a+1)+(a-11)^2+34>0.$$

The proof is completed. The equality holds for $-2a=b=c$ (or any cyclic permutation).

□

P 2.86. *If a,b,c are real numbers such that*

$$ab+bc+ca=3,$$

then

$$20(a^6+b^6+c^6)+43abc(a^3+b^3+c^3)\ge 189.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality in the homogeneous form $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = 20(a^6+b^6+c^6)+43abc(a^3+b^3+c^3)-7(ab+bc+ca)^3.$$

Since the highest coefficient of $f_6(a,b,c)$ is positive, namely

$$A = 20\cdot 3 + 43\cdot 3 = 189,$$

we will use the *highest coefficient cancellation method*. From

$$f_6(a,1,1) = (2a+1)(a-1)^2(10a^3+15a^2+44a+33),$$

it follows that

$$f_6(1,1,1) = 0, \quad f_6(-1/2,1,1) = 0.$$

Define the homogeneous function

$$P(a,b,c) = r + Bp^3 + Cpq, \qquad p = a+b+c,\ q = ab+bc+ca,\ r = abc,$$

such that $P(1,1,1) = P(-1/2,1,1) = 0$; that is,

$$P(a,b,c) = r + \frac{4}{27}p^3 - \frac{5}{9}pq,$$

$$P(a,1,1) = \frac{27a+4(a+2)^3-15(a+2)(2a+1)}{27} = \frac{2(a-1)^2(2a+1)}{27}.$$

We will show that the following sharper inequality holds for $ab+bc+ca \geq 0$:

$$f_6(a,b,c) \geq 189P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 189P^2(a,b,c).$$

Since the highest coefficient of $g_6(a,b,c)$ is zero, it suffices to prove that $g_6(a,1,1) \geq 0$ for $2a+1 \geq 0$ (see Remark 3 from P 2.75). We have

$$g_6(a,1,1) = f_6(a,1,1) - 189P^2(a,1,1) = (2a+1)(a-1)^2 g(a),$$

where

$$g(a) = 10a^3+15a^2+44a+33-\frac{28}{27}(a-1)^2(2a+1).$$

Since

$$g(a) \geq 10a^3+15a^2+44a+33-5(a-1)^2(2a+1) = 22(a+1)^2+8a^2+6 > 0,$$

we have $g_6(a,1,1) \geq 0$ for $a \geq -1/2$. Thus, the proof is completed. The equality holds for $a=b=c=1$.

□

P 2.87. *If a,b,c are real numbers such that*

$$ab+bc+ca \ge 0,$$

then

(a) $$(a^2+b^2+c^2)(ab+bc+ca)^2 \ge abc(4a^3+4b^3+4c^3+15abc);$$

(b) $$4(a+b+c)^6 \ge 81abc(5a^3+5b^3+5c^3+21abc).$$

(Vasile Cîrtoaje and Nguyen Van Huyen, 2020)

Solution. Let

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc.$$

We have

$$a^3+b^3+c^3=3r+p^3-3pq.$$

(a) Write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c)=-27r^2-(p^3-3pq)r+p^2q^2-2q^3.$$

Since $f_6(a,b,c)$ has the highest coefficient $A=-27$, it suffices to prove the inequality for $b=c=1$ and $2a+1 \ge 0$ (see Remark 3 from P 2.75). Thus, we need to show that

$$(a^2+2)(2a+1)^2 \ge a(4a^3+15a+8),$$

which is equivalent to

$$(2a+1)(a-1)^2 \ge 0.$$

The equality occurs for $a=b=c$ and for $-2a=b=c$ (or any cyclic permutation).

(b) Write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c)=-2916r^2-405(p^3-3pq)r+4p^6.$$

Since $f_6(a,b,c)$ has the highest coefficient $A=-2916$, it suffices to prove the inequality for $b=c=1$ and $2a+1 \ge 0$ (see Remark 3 from P 2.75). Thus, we need to show that

$$4(a+2)^6 \ge 81a(5a^3+21a+10),$$

which is equivalent to

$$(2a+1)(a-1)^2(2a^3+27a^2-42a+256) \ge 0.$$

This is true since

$$2a^3+27a^2-42a+256=a^2(2a+1)+26a^2-42a+256 > 21a^2-42a+21 \ge 0.$$

The equality occurs for $a=b=c$ and for $-2a=b=c$ (or any cyclic permutation).

□

P 2.88. *If a,b,c are real numbers, then*

$$4\sum(a^2+bc)(a-b)(a-c)(a-3b)(a-3c) \ge 7(a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = 4f(a,b,c) - 7(a-b)^2(b-c)^2(c-a)^2,$$

$$f(a,b,c) = \sum(a^2+bc)(a-b)(a-c)(a-3b)(a-3c).$$

We have

$$f_6(a,1,1) = 4(a^2+1)(a-1)^2(a-3)^2.$$

Let

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

Since

$$(a-b)(a-c) = a^2+2bc-q$$

and

$$(a-3b)(a-3c) = a^2+12bc-3q,$$

$f(a,b,c)$ has the same highest coefficient A_0 as $g(a,b,c)$, where

$$g(a,b,c) = \sum(a^2+bc)(a^2+2bc)(a^2+12bc);$$

that is, according to Remark 2 from P 2.75,

$$A_0 = g(1,1,1) = 3\cdot 2\cdot 3\cdot 13 = 234.$$

Therefore, $f_6(a,b,c)$ has the highest coefficient

$$A = 4A_0 - 7(-27) = 1125.$$

Since the highest coefficient A is positive, we will use the *highest coefficient cancellation method*. There are two cases to consider: $q \ge 0$ and $q < 0$.

Case 1: $q \ge 0$. Since

$$f_6(1,1,1) = f_6(3,1,1) = 0,$$

define the homogeneous function

$$P(a,b,c) = r + Bp^3 + Cpq$$

such that $P(1,1,1) = P(3,1,1) = 0$; that is,

$$P(a,b,c) = r + \frac{2}{45}p^3 - \frac{11}{45}pq,$$

hence

$$P(a,1,1) = \frac{45a + 2(a+2)^3 - 11(a+2)(2a+1)}{45} = \frac{2(a-1)^2(a-3)}{45}.$$

We will show that the following sharper inequality holds for $q \ge 0$:

$$f_6(a,b,c) \ge 1125P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 1125P^2(a,b,c).$$

Since the highest coefficient of $g_6(a,b,c)$ is zero, it suffices to prove that $g_6(a,1,1) \ge 0$ for all real a such that $2a+1 \ge 0$ (see Remark 3 from P 2.75). We have

$$g_6(a,1,1) = f_6(a,1,1) - 1125P^2(a,1,1) = \frac{8(a-1)^2(a-3)^2(a+2)(2a+1)}{9} \ge 0.$$

Case 2: $q < 0$. Define the homogeneous polynomial

$$P(a,b,c) = r + Bp^3 - \left(3B + \frac{1}{9}\right)pq,$$

which satisfies $P(1,1,1) = 0$. We will show that there is a real number B such that the following sharper inequality holds

$$f_6(a,b,c) \ge 1125P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 1125P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient equal to zero. Then, by Remark 3 from P 2.75, it suffices to prove that $g_6(a,1,1) \ge 0$ for $2a+1 < 0$. We have

$$g_6(a,1,1) = f_6(a,1,1) - 1125P^2(a,1,1),$$

where

$$P^2(a,1,1) = \left[a + B(a+2)^3 - \left(3B + \frac{1}{9}\right)(a+2)(2a+1)\right]^2.$$

Let us denote $g(a) = g_6(a,1,1)$. Since $g(-2) = 0$, we can have $g(a) \ge 0$ in the vicinity of $a = -2$ only if $g'(-2) = 0$, which involves $B = 8/135$. Using this value of B, we get

$$P^2(a,1,1) = \frac{4(a-1)^4(4a-7)^2}{25 \cdot 729},$$

$$g_6(a,1,1) = 4(a-1)^2\left[(a^2+1)(a-3)^2 - \frac{5}{81}(a-1)^2(4a-7)^2\right]$$

$$= \frac{4}{81}(a-1)^2(a+2)^2(a^2 - 50a + 121) \ge 0.$$

The proof is completed. The equality holds for $a = b = c$, for $a/3 = b = c$ (or any cyclic permutation), and for $a = 0$ and $b + c = 0$ (or any cyclic permutation).

□

P 2.89. *Let a,b,c be real numbers such that*

$$ab+bc+ca \geq 0.$$

For any real k, prove that

$$\sum 4bc(a-b)(a-c)(a-kb)(a-kc)+(a-b)^2(b-c)^2(c-a)^2 \geq 0.$$

(Vasile Cîrtoaje, 2014)

Solution. Write the inequality as $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c)=4f(a,b,c)+(a-b)^2(b-c)^2(c-a)^2,$$

$$f(a,b,c)=\sum bc(a-b)(a-c)(a-kb)(a-kc).$$

Let

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc.$$

Since

$$(a-b)(a-c)=a^2+2bc-q$$

and

$$(a-kb)(a-kc)=a^2+(k+k^2)bc-kq,$$

$f(a,b,c)$ has the same highest coefficient A_0 as $P_1(a,b,c)$, where

$$P_1(a,b,c)=\sum bc(a^2+2bc)[a^2+(k+k^2)bc];$$

that is, according to Remark 2 from P 2.75,

$$A_0=P_1(1,1,1)=3(1+2)(1+k+k^2)=9(1+k+k^2).$$

Therefore, $f_6(a,b,c)$ has the highest coefficient

$$A=4A_0-27=9(2k+1)^2.$$

We have

$$f_6(a,1,1)=4(a-1)^2(a-k)^2.$$

Consider first that $k=-1/2$, when $A=0$. By P 2.75, it suffices to show that $f_6(a,1,1) \geq 0$ for all real a. Clearly, this is true. Consider further that $k \neq -1/2$, when the highest coefficient A is positive. We will use the *highest coefficient cancellation method*. Since

$$f_6(1,1,1)=f_6(k,1,1)=0,$$

define the homogeneous function

$$P(a,b,c)=r+Cpq+D\frac{q^2}{p}$$

such that $P(1,1,1) = P(k,1,1) = 0$; that is,

$$P(a,b,c) = r + \frac{pq}{3(2k+1)} - \frac{2(k+2)q^2}{3(2k+1)p}.$$

We will show that the following sharper inequality holds for $ab+bc+ca \geq 0$:

$$f_6(a,b,c) \geq 9(2k+1)^2 P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 9(2k+1)^2 P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A_1 = 0$. Then, by Remark 4 from P 2.75, it suffices to prove that $g_6(a,1,1) \geq 0$ for all real a such that $2a+1 \geq 0$. We have

$$\begin{aligned} P(a,1,1) &= a + \frac{(a+2)(2a+1)}{3(2k+1)} - \frac{2(k+2)(2a+1)^2}{3(2k+1)(a+2)} \\ &= \frac{2(a-1)^2(a-k)}{3(2k+1)(a+2)}, \end{aligned}$$

then

$$\begin{aligned} g_6(a,1,1) &= f_6(a,1,1) - 9(2k+1)^2 P^2(a,1,1) \\ &= 4(a-1)^2(a-k)^2 - \frac{4(a-1)^4(a-k)^2}{(a+2)^2} \\ &= \frac{12(a-1)^2(a-k)^2(2a+1)}{(a+2)^2} \geq 0. \end{aligned}$$

The proof is completed. The equality holds for $a = b = c$, for $a/k = b = c$ (or any cyclic permutation) - if $k \neq 0$, and for $b = c = 0$ (or any cyclic permutation).

□

P 2.90. *If a,b,c are real numbers, then*

$$\left[(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)\right]^2 \geq 4(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2).$$

First Solution. Consider the nontrivial case $ab+bc+ca \geq 0$, and write the inequality as $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = [(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2)]^2 - 4(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2).$$

Since

$$(a^2b+b^2c+c^2a)+(ab^2+bc^2+ca^2) = (a+b+c)(ab+bc+ca) - 3abc,$$

f_6 has the highest coefficient

$$A = (-3)^2 = 9.$$

Since $A > 0$, we will use the *highest coefficient cancellation method*. Because

$$f(1,1,1) = f(0,1,1) = 0,$$

define the homogeneous function

$$P(a,b,c) = abc + C(a+b+c)(ab+bc+ca) + D\frac{(ab+bc+ca)^2}{a+b+c}$$

such that $P(1,1,1) = P(0,1,1) = 0$; that is,

$$P(a,b,c) = abc + \frac{(a+b+c)(ab+bc+ca)}{3} - \frac{4(ab+bc+ca)^2}{3(a+b+c)}.$$

We will show that the following sharper inequality holds for $ab+bc+ca \geq 0$:

$$f_6(a,b,c) \geq 9P^2(a,b,c).$$

Let us denote

$$g_6(a,b,c) = f_6(a,b,c) - 9P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A_1 = 0$. Then, by Remark 4 from P 2.75, it suffices to prove that $g_6(a,1,1) \geq 0$ for all real a such that $2a+1 \geq 0$. We have

$$f_6(a,1,1) = 4a^2(a-1)^2,$$

$$P(a,1,1) = a + \frac{(a+2)(2a+1)}{3} - \frac{4(2a+1)^2}{3(a+2)} = \frac{2a(a-1)^2}{3(a+2)},$$

hence

$$f_6(a,1,1) = f_6(a,1,1) - 9P^2(a,1,1) = \frac{12a^2(a-1)^2(2a+1)}{(a+2)^2} \geq 0.$$

The proof is completed. The equality holds for $a = b = c$, for $a = 0$ and $b = c$ (or any cyclic permutation), and for $b = c = 0$ (or any cyclic permutation).

Second Solution (by *Nguyen Van Quy*). Since the inequality remains unchanged by replacing a,b,c with $-a,-b,-c$, we may assume that $a+b+c \geq 0$. In addition, consider the non-trivial case $ab+bc+ca > 0$, when the inequality can be rewritten as

$$(a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2) \geq 2\sqrt{(ab+bc+ca)(a^2b^2 + b^2c^2 + c^2a^2)}.$$

Using the Cauchy-Schwarz inequality, we have

$$(a^2b + b^2c + c^2a) + (ab^2 + bc^2 + ca^2) + (a^3 + b^3 + c^3) = (a^2 + b^2 + c^2)(a+b+c)$$

$$= \sqrt{\left[a^4+b^4+c^4+2(a^2b^2+b^2c^2+c^2a^2)\right]\left[a^2+b^2+c^2+2(ab+bc+ca)\right]}$$
$$\geq \sqrt{(a^4+b^4+c^4)(a^2+b^2+c^2)}+2\sqrt{(a^2b^2+b^2c^2+c^2a^2)(ab+bc+ca)}.$$

Thus, it suffices to show that

$$\sqrt{(a^4+b^4+c^4)(a^2+b^2+c^2)} \geq a^3+b^3+c^3,$$

which follows also from the Cauchy-Schwarz inequality.

□

P 2.91. *If a,b,c are real numbers such that*

$$a+b+c=3,$$

then

$$\frac{(a-1)(a-25)}{a^2+23}+\frac{(b-1)(b-25)}{b^2+23}+\frac{(c-1)(c-25)}{c^2+23} \geq 0.$$

Solution. Denote

$$p=a+b+c, \quad q=ab+bc+ca$$

and write the inequality in the homogeneous form $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c)=\sum(3a-p)(3a-25p)(9b^2+23p^2)(9c^2+23p^2).$$

Since the highest coefficient of f_6 is positive, namely

$$A=3\cdot 9^3,$$

we use the *highest coefficient cancellation method*. Thus, we will prove that there exist two real numbers B and C such that $g_6(a,b,c) \geq 0$, where

$$g_6(a,b,c)=f_6(a,b,c)-A[abc+B(a+b+c)^3+C(a+b+c)(ab+bc+ca)]^2.$$

Since g_6 has the highest coefficient equal to zero, it suffices to show that $g_6(a,1,1) \geq 0$ for all real a (see P 2.75). Notice that

$$f_6(a,1,1)=12(a-1)^2(7a+11)^2[23(a+2)^2+9]$$

and

$$g_6(a,1,1)=f_6(a,1,1)-3\cdot 9^3[a+B(a+2)^3+C(a+2)(2a+1)]^2.$$

Let us denote $g(a)=g_6(a,1,1)$. Since $g(-2)=0$, we can have $g(a) \geq 0$ in the vicinity of $a=-2$ only if $g'(-2)=0$; this involves $C=-13/9$, hence

$$g_6(a,1,1)=12(a-1)^2(7a+11)^2[23(a+2)^2+9]$$
$$-27[9a+9B(a+2)^3-13(a+2)(2a+1)]^2.$$

There are two cases to consider: $5p^2+q\le 0$ and $5p^2+q\ge 0$.

Case 1: $5p^2+q\le 0$. By Remark 3 from the proof of P 2.75, we only need to show that there exist a real number B such that $g_6(a,1,1)\ge 0$ for all real a satisfying $5(a+2)^2+2a+1\le 0$; that is, for $a\in[-3,-7/5]$. From $g_6(-11/7,1,1)=0$, we get $B=28/9$, then

$$\begin{aligned}g_6(a,1,1)=&12(a-1)^2(7a+11)^2[23(a+2)^2+9]\\&-108(7a+11)^2(2a^2+7a+9)^2\\=&-12(a+2)^2(7a+11)^2(13a^2+154a+157)\ge 0.\end{aligned}$$

Case 2: $5p^2+q\ge 0$. By Remark 3 from the proof of P 2.75, we only need to show that there exist a real number B such that $g_6(a,1,1)\ge 0$ for all real a satisfying $5(a+2)^2+2a+1\ge 0$; that is, for $a\in(-\infty,-3]\cup[-7/5,\infty)$. From $g_6(1,1,1)=0$, we get $B=4/9$, then

$$\begin{aligned}g_6(a,1,1)&=12(a-1)^2(7a+11)^2[23(a+2)^2+9]-108(a-1)^4(2a+3)^2\\&=12(a+2)^2(a-1)^2(1091a^2+3650a+3035)\ge 0.\end{aligned}$$

The proof is completed. The equality holds for $a=b=c=1$, and for $a=-11$ and $b=c=7$ (or any cyclic permutation).

□

P 2.92. *If a,b,c are real numbers such that $abc\ne 0$, then*

$$\left(\frac{b+c}{a}\right)^2+\left(\frac{c+a}{b}\right)^2+\left(\frac{a+b}{c}\right)^2>2.$$

(Michael Rozenberg, 2014)

Solution. Assume that

$$a^2=\min\{a^2,b^2,c^2\}.$$

By the Cauchy-Schwarz inequality, we have

$$\left(\frac{c+a}{b}\right)^2+\left(\frac{a+b}{c}\right)^2\ge\frac{[(c+a)+(-a-b)]^2}{b^2+c^2}=\frac{(b-c)^2}{b^2+c^2}.$$

On the other hand,

$$\left(\frac{b+c}{a}\right)^2\ge\frac{(b+c)^2}{b^2+c^2}.$$

Therefore,

$$\left(\frac{b+c}{a}\right)^2+\left(\frac{c+a}{b}\right)^2+\left(\frac{a+b}{c}\right)^2\ge\frac{(b+c)^2}{b^2+c^2}+\frac{(b-c)^2}{b^2+c^2}=2.$$

The equality holds if and only if

$$\left(\frac{c+a}{b}\right)^2+\left(\frac{a+b}{c}\right)^2=\frac{(b-c)^2}{b^2+c^2}$$

and $b+c=0$. Since these relations involves $a=0$, we conclude that the inequality is strict (the equality does not hold).

□

P 2.93. *If a,b,c are real numbers, then*

(a) $$(a^2+1)(b^2+1)(c^2+1)\ge\frac{8}{3\sqrt{3}}\,|(a-b)(b-c)(c-a)|;$$

(b) $$(a^2-a+1)(b^2-b+1)(c^2-c+1)\ge|(a-b)(b-c)(c-a)|.$$

(Kwon Ji Mun, 2011)

Solution. (a) ***First Solution.*** Without loss of generality, assume that $a\le b\le c$, when

$$|(a-b)(b-c)(c-a)|=(a-b)(b-c)(c-a).$$

Denote

$$k=\frac{4}{3\sqrt{3}}$$

and write the inequality as

$$Aa^2+2Ba+C\ge 0,$$

where

$$A=(b^2+1)(c^2+1)+2k(b-c),$$

$$B=-k(b^2-c^2),$$

$$C=(b^2+1)(c^2+1)+2kbc(b-c).$$

Substituting $b=\dfrac{-x}{\sqrt{3}}$ and $c=\dfrac{y}{\sqrt{3}}$, by the Cauchy-Schwarz inequality, we get

$$\begin{aligned}9A&=(x^2+1+2)(1+y^2+2)-8(x+y)\\&\ge(x+y+2)^2-8(x+y)=(x+y-2)^2\ge 0.\end{aligned}$$

We have $A=0$ only for $b=-1/\sqrt{3}$ and $c=1/\sqrt{3}$, when

$$Aa^2+2Ba+C=2Ba+C=64/27.$$

Otherwise, for $A>0$, it suffices to prove that $AC-B^2\ge 0$. Let us denote

$$E=b-c,\quad F=bc+1.$$

Since

$$(b^2+1)(c^2+1) = (b-c)^2 + (bc+1)^2 = E^2 + F^2,$$
$$(b+c)^2 = (b-c)^2 + 4(bc+1) - 4 = E^2 + 4F - 4,$$
$$(b^2-c^2)^2 = (b-c)^2(b+c)^2 = E^2(E^2+4F-4),$$

we have

$$A = E^2 + F^2 + 2kE, \quad B^2 = k^2E^2(E^2+4F-4), \quad C = E^2 + F^2 + 2kE(F-1),$$

and hence

$$AC - B^2 = (E^2+F^2+2kE)(E^2+F^2+2kEF-2kE) - k^2E^2(E^2+4F-4)$$

$$= (E^2+F^2)(E^2+F^2+2kEF) - k^2E^4 = \frac{1}{27}(E+\sqrt{3}F)^2(11E^2 - 2\sqrt{3}EF + 9F^2) \ge 0.$$

The equality holds for

$$b - c + \sqrt{3}(bc+1) = 0, \quad a + \frac{b+c}{1+3bc} = 0$$

(or any permutation).

Second Solution (by *Vo Quoc Ba Can*). Substituting

$$a = x\sqrt{3}, \quad b = y\sqrt{3}, \quad c = z\sqrt{3},$$

the inequality becomes

$$(3x^2+1)(3y^2+1)(3z^2+1) \ge 8|(x-y)(y-z)(z-x)|.$$

It suffices to show that

$$E^2 \ge 64(x-y)^2(y-z)^2(z-x)^2,$$

where

$$E = (3x^2+1)(3y^2+1)(3z^2+1) = 27x^2y^2z^2 + 9\sum x^2y^2 + 3\sum x^2 + 1.$$

It it easy to check that the equality holds for $x = -1$, $y = 0$ and $z = 1$ (or any cyclic permutation), when

$$x + y + z = 0, \quad xy + yz + zx = -1, \quad xyz = 0.$$

From

$$\left(9xyz + \sum x\right)^2 \ge 0$$

and

$$\left(\sum xy + 1\right)^2 \ge 0,$$

we get

$$81x^2y^2z^2 \ge -18xyz\sum x - \sum x^2 - 2\sum xy$$

and

$$1 \ge -\sum x^2y^2 - 2xyx\sum x - 2\sum xy,$$

respectively. Therefore,

$$\begin{aligned}3E \ge &\left(-18xyz\sum x - \sum x^2 - 2\sum xy\right) + 27\sum x^2y^2 + 9\sum x^2\\ &+ 3\left(-\sum x^2y^2 - 2xyx\sum x - 2\sum xy\right)\\ =&24\left(\sum x^2y^2 - xyz\sum x\right) + 8\left(\sum x^2 - \sum xy\right)\\ =&12\sum x^2(y-z)^2 + \frac{4}{3}\sum(2x-y-z)^2.\end{aligned}$$

By the AM-GM inequality, we have

$$3E \ge 8\sqrt{\left[\sum x^2(y-z)^2\right]\left[\sum(2x-y-z)^2\right]}.$$

In addition, by the Cauchy-Schwarz inequality, we get

$$\begin{aligned}E^2 &\ge \frac{64}{9}\left[\sum x(y-z)(2x-y-z)\right]^2\\ &= 64\left(\sum x^2y - \sum xy^2\right)^2\\ &= 64(x-y)^2(y-z)^2(z-x)^2.\end{aligned}$$

(b) Write the inequality as

$$\left[\left(a-\frac{1}{2}\right)^2 + \frac{3}{4}\right]\left[\left(b-\frac{1}{2}\right)^2 + \frac{3}{4}\right]\left[\left(c-\frac{1}{2}\right)^2 + \frac{3}{4}\right] \ge (a-b)(b-c)(c-a).$$

Using the substitution

$$a - \frac{1}{2} = \frac{\sqrt{3}}{2}x, \quad b - \frac{1}{2} = \frac{\sqrt{3}}{2}y, \quad c - \frac{1}{2} = \frac{\sqrt{3}}{2}z,$$

the inequality turns out into the inequality in (a). From the equality conditions in (a), namely

$$y - z + \sqrt{3}(yz+1) = 0, \quad x + \frac{y+z}{1+3yz} = 0,$$

we get the following equality conditions

$$b = \frac{c-1}{c}, \quad a = \frac{1}{1-c}$$

(or any permutation).

□

P 2.94. *If a, b, c are real numbers such that*

$$a+b+c=3,$$

then

$$(1-a+a^2)(1-b+b^2)(1-c+c^2) \geq 1.$$

First Solution. Since

$$2(1-a+a^2)(1-b+b^2)-(a+b-1)^2-1=a^2(b-1)^2+b^2(a-1)^2 \geq 0$$

and

$$2(1-c+c^2) \geq 1+c^2,$$

it is enough to show that

$$[(a+b-1)^2+1](1+c^2) \geq 4.$$

By the Cauchy-Schwartz inequality, we have

$$[(a+b-1)^2+1](1+c^2) \geq [(a+b-1)+c]^2=(a+b+c-1)^2=4.$$

The equality holds for $a=b=c=1$.

Second Solution (by *Marian Tetiva*). Assume that

$$a \leq b \leq c.$$

There are two cases to consider: $a \geq 0$ and $a < 0$.

Case 1: $a \geq 0$. Among the numbers a, b, c always there exist two (let b and c) which are either less than or equal to 1, or greater than or equal to 1. Then,

$$bc(b-1)(c-1) \geq 0$$

and

$$\begin{aligned}(1-b+b^2)(1-c+c^2) &= 1+(b^2-b)+(c^2-c)+(b^2-b)(c^2-c)\\ &\geq 1+(b^2-b)+(c^2-c) \geq 1-(b+c)+\frac{1}{2}(b+c)^2\\ &= 1-(3-a)+\frac{1}{2}(3-a)^2=\frac{1}{2}(5-4a+a^2).\end{aligned}$$

Therefore, it suffices to show that

$$(1-a+a^2)(5-4a+a^2) \geq 2.$$

Indeed,

$$(1-a+a^2)(5-4a+a^2)-2=(a-1)^2(a^2-3a+3) \geq 0.$$

Case 2: $a < 0$. We have

$$c > \frac{3}{2}$$

because

$$c \geq \frac{b+c}{2} > \frac{a+b+c}{2} = \frac{3}{2}.$$

The desired inequality is true since

$$1 - a + a^2 > 1,$$

$$1 - b + b^2 = \left(\frac{1}{2} - b\right)^2 + \frac{3}{4} \geq \frac{3}{4},$$

$$1 - c + c^2 > 1 - c + \frac{3c}{2} = 1 + \frac{c}{2} > 1 + \frac{3}{4} = \frac{7}{4}.$$

□

P 2.95. *If a, b, c are real numbers such that*

$$a + b + c = 0,$$

then

$$\frac{a(a-4)}{a^2+2} + \frac{b(b-4)}{b^2+2} + \frac{c(c-4)}{c^2+2} \geq 0.$$

Solution. Write the inequality as follows

$$\sum \left[\frac{a(a-4)}{a^2+2} + 1\right] \geq 3,$$

$$\sum \frac{(a-1)^2}{a^2+2} \geq \frac{3}{2}.$$

From

$$a^2 = (b+c)^2 \leq 2(b^2 + c^2),$$

we get

$$3a^2 \leq 2(a^2 + b^2 + c^2).$$

Similarly,

$$3b^2 \leq 2(a^2 + b^2 + c^2), \quad 3c^2 \leq 2(a^2 + b^2 + c^2).$$

Therefore, we have

$$\sum \frac{(a-1)^2}{a^2+2} = \sum \frac{3(a-1)^2}{3a^2+6} \geq \sum \frac{3(a-1)^2}{2(a^2+b^2+c^2)+6}$$

$$= \frac{3}{2(a^2+b^2+c^2+3)} \sum (a-1)^2 = \frac{3}{2}.$$

Thus, the proof is completed. The equality holds for $a = b = c = 0$, and also for $a = -2$ and $b = c = 1$ (or any cyclic permutation).

□

P 2.96. *If a, b, c are real numbers such that*

$$a, b, c \le 1 + \sqrt{2}, \quad a + b + c \ge 0,$$

then

$$2abc + a^2 + b^2 + c^2 + 1 \ge 2(ab + bc + ca).$$

(Vasile Cîrtoaje, 2014)

Solution. Assume that

$$a \le b \le c \le 1 + \sqrt{2}.$$

First Solution. There are three cases: $0 \le a \le b \le c$, $a \le 0 \le b \le c$ and $a \le b \le 0 \le c$.

Case 1: $0 \le a \le b \le c$. Among the numbers $1 - a$, $1 - b$ and $1 - c$ there are two which have the same sign; let

$$(1 - b)(1 - c) \ge 0.$$

From

$$\begin{aligned} &2abc + a^2 + b^2 + c^2 + 1 - 2(ab + bc + ca) = \\ &= (a - 1)^2 + (b - c)^2 + 2a + 2abc - 2a(b + c) \\ &= (a - 1)^2 + (b - c)^2 + 2a(1 - b)(1 - c) \ge 0, \end{aligned}$$

the conclusion follows.

Case 2: $a \le 0 \le b \le c$. Denote

$$x = \frac{b + c}{2}, \quad 0 \le x \le 1 + \sqrt{2},$$

and write the inequality as

$$(a + 1)^2 + (b - c)^2 - 2a(1 + b + c - bc) \ge 0.$$

Clearly, it suffices to show that

$$1 + b + c - bc \ge 0.$$

Indeed,

$$1 + b + c - bc \ge 1 + 2x - x^2 = (\sqrt{2} - 1 + x)(\sqrt{2} + 1 - x) \ge 0.$$

Case 3: $a \le b \le 0 \le c$. Write the inequality in the form

$$2abc + (a - b)^2 + c^2 + 1 - 2(a + b)c \ge 0,$$

which is clearly true.

The equality holds for $a=b=c=1$, and also for $a=-1$ and $b=c=1+\sqrt{2}$ (or any cyclic permutation).

Second Solution. According to P 2.53, for fixed $a+b+c$ and $ab+bc+ca$, the product abc is minimal when $a \le b = c$. Therefore, we only need to prove that

$$2ab^2+a^2+1 \ge 4ab$$

for $a \le b \le 1+\sqrt{2}$ and $a+2b \ge 0$. Write the inequality as

$$(a-1)^2+2a(b-1)^2 \ge 0$$

or

$$(a+1)^2-2a(1+2b-b^2) \ge 0.$$

For $a \ge 0$, the inequality is clearly true. For $a \le 0$, it is enough to show that $1+2b-b^2 \ge 0$. This is true because $2b \ge -a \ge 0$ and $b \le 1+\sqrt{2}$.

Remark. Actually, the original inequality holds for the extended conditions

$$a,b,c \le 1+\sqrt{2}, \qquad a+b+c+2\sqrt{2}-1 \ge 0,$$

with equality for for $a=b=c=1$, for $a=-1$ and $b=c=1+\sqrt{2}$ (or any cyclic permutation), and for $a=-1$ and $b=c=1-\sqrt{2}$ (or any cyclic permutation).

□

P 2.97. *If a,b,c are real numbers such that $a+b+c=2$ and $ab+bc+ca>0$, then*

$$(a^2+bc)(b^2+ca)(c^2+ab)+abc \le 1.$$

(Nguyen Van Huyen, 2020)

Solution. Write the inequality in the homogeneous form $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = p^6 - 8abcp^3 - 64P_2(a,b,c),$$

with

$$p=a+b+c, \qquad P_2(a,b,c)=(a^2+bc)(b^2+ca)(c^2+ab).$$

According to Remark 2 from P 2.75, $f_6(a,b,c$ has the highest coefficient

$$A=-64P_2(1,1,1)=-512.$$

Thus, according to P 2.75 and its Remark 3, it is enough to show that $f_6(a,1,1) \ge 0$ for $2a+1 \ge 0$. We have

$$f_6(a,1,1)=(a+2)^6-8a(a+2)^3-64(a^2+1)(a+1)^2$$
$$=a^2(a^4+12a^3-12a^2-16a+16)$$
$$\ge a^2(a^4+12a^3-12a^2-16a+15)=a^2(a-1)^2(a^2+14a+15) \ge 0.$$

The equality occurs for $a=0$ and $b=c=1$ (or any cyclic permutation).

□

P 2.98. *If a, b, c are real numbers such that*

$$a^2+b^2+c^2=3, \qquad a \ge \frac{4}{3},$$

then

$$3(abc+1) \ge 2(ab+bc+ca).$$

(Vasile Cîrtoaje, 2019)

Solution. Since

$$2bc=(b+c)^2+a^2-3,$$

we write the inequality as follows:

$$(3a-2)(2bc)-4a(b+c)+6 \ge 0,$$

$$(3a-2)(b+c)^2-4a(b+c)+3a^3-2a^2-9a+12 \ge 0,$$

$$(3a-2)\left(b+c-\frac{2a}{3a-2}\right)^2+\frac{3(3a^4-4a^3-9a^2+18a-8)}{3a-2} \ge 0,$$

$$(3a-2)\left(b+c-\frac{2a}{3a-2}\right)^2+\frac{3(a-1)^2(3a-4)(a+2)}{3a-2} \ge 0.$$

The equality holds for $a=b=c=1$, and also for $a=\dfrac{4}{3}$ and $b,c=\dfrac{4\pm\sqrt{6}}{6}$.

Remark. The inequality is equivalent to

$$3\left(a-\frac{4}{3}\right)\left(b-\frac{4}{3}\right)\left(c-\frac{4}{3}\right)+\left(a+b+c-\frac{8}{3}\right)^2 \ge 0.$$

□

P 2.99. *If a, b, c are real numbers such that $a \ge \dfrac{8}{7}$ and $a^2+b^2+c^2=3$, then*

$$\frac{3-a-b-c}{1-abc} \ge \frac{49}{100}.$$

(Vasile Cîrtoaje, 2018)

Solution. From

$$3=a^2+b^2+c^2>3\sqrt[3]{a^2b^2c^2},$$

it follows $abc<1$. In addition, since

$$2bc=(b+c)^2-b^2-c^2=(b+c)^2+a^2-3,$$

the inequality can be written as follows:

$$49(abc-1) \ge 100(a+b+c-3),$$

$$49a[(b+c)^2+a^2-3]-98 \geq 200(a+b+c-3),$$

$$49a(b+c)^2-200(b+c)+49a^3-347a+502 \geq 0,$$

$$49a\left(b+c-\frac{100}{49a}\right)^2+\frac{E}{49} \geq 0,$$

where

$$E=(7a)^4-347(7a)^2+3514(7a)-10000.$$

Denoting $x=7a$, $x \geq 8$, we have

$$E=x^4-347x^2+3514x-10000=(x-8)(x^3+8x^2-283x+1250)$$

$$\geq (x-8)(x^3+8x^2-283x+1240)=(x-8)^2(x^2+16x-155)$$

$$\geq (x-8)^2(x^2+16x-161)=(x-8)^2(x-7)(x+21) \geq 0.$$

The equality occurs for $a=\frac{8}{7}$ and

$$b+c=\frac{100}{49a}=\frac{25}{14}, \qquad 2bc=(b+c)^2+a^2-3=\frac{293}{196},$$

therefore for $a=\frac{8}{7}$ and $b,c=\frac{25 \pm \sqrt{39}}{28}$.

□

P 2.100. *If $a,b,c \in [-1,1]$, then*

$$a^3+b^3+c^3+abc \leq \frac{15}{16}(a+b+c)+\frac{19}{16}.$$

(Vasile Cîrtoaje, 2018)

Solution. Since

$$a^3+b^3+c^3=3abc+\frac{1}{2}(a+b+c)[3(a^2+b^2+c^2)-(a+b+c)^2],$$

we may apply the following statement (see Remark 3 from P 2.53): *If $a,b,c \in [-1,1]$, then for fixed $a+b+c$ and $a^2+b^2+c^2$, the product abc is maximal when $a \geq b=c$ or $a=1$.* Thus, we only need to consider these cases.

Case 1: $a=1$. We need to show that

$$16(b^3+c^3+bc)-15(b+c)+18 \leq 0.$$

Denote

$$s=b+c, \quad s \leq 2,$$

and write the inequality as

$$16s^3 - 16bc(3s-1) - 15s - 18 \le 0.$$

We have $4bc \le s^2$ and

$$(b-1)(c-1) \ge 0, \quad bc \ge s-1.$$

For $3s-1 \le 0$, it is enough to show that

$$16s^3 - 4s^2(3s-1) - 15s - 18 \le 0,$$

which is equivalent to

$$4s^3 + 4s^2 - 15s - 18 \le 0,$$

$$(s-2)(2s+3)^2 \le 0.$$

For $3s-1 \ge 0$, it is enough to show that

$$16s^3 - 16(s-1)(3s-1) - 15s - 18 \le 0,$$

which is equivalent to

$$16s^3 - 48s^2 + 49s - 34 \le 0,$$

$$(s-2)(16s^2 - 16s + 17) \le 0,$$

$$(s-2)[8(s-1)^2 + 8s^2 + 9] \le 0.$$

Case 2: $a \ge b = c$. We need to prove that

$$16(a^3 + 2b^2 + ab^2) \le 15(a+2b) + 19.$$

There are two sub-cases: $a \ge 0$ and $a \le 0$.

If $a \ge 0$, write the inequality as

$$a(16a^2 + 32b^2 - 15) + 16b^3 - 30b - 19 \le 0.$$

Since

$$a(16a^2 + 32b^2 - 15) \le a(16 + 32b^2 - 15) = a(1 + 32b^2) \le 1 + 32b^2,$$

it is enough to show that

$$16b^3 + 32b^2 - 30b - 18 \le 0,$$

which is equivalent to

$$2(b-1)(4b+3)^2 \le 0.$$

If $a \le 0$, hence $b \le a \le 0$, substituting $a = -x$ and $b = -y$ $(y \ge x \ge 0)$, the required inequality becomes

$$16(x^3 + 2y^3 + xy^2) + 19 \ge 15(x + 2y).$$

Since $xy^2 \ge 0$, it suffices to show that

$$(16x^3 - 15x + 6) + 2(16y^3 - 15y + 6) + 1 \ge 0.$$

This is true because, by the AM-GM inequality, we have

$$16x^3 - 15x + 6 = (16x^3 + 3 + 3) - 15x \ge 3(2\sqrt[3]{18} - 5)x \ge 0$$

and, similarly,

$$16y^3 - 15y + 6 > 0.$$

The equality occurs for $a = b = c = 1$, and also for $a = 1$ and $b = c = \frac{-3}{4}$ (or any cyclic permutation).

□

P 2.101. *If a, b, c are real numbers, then*

$$(a^3 + b^3 + c^3)^2 \ge (a^4 + b^4 + c^4)(ab + bc + ca).$$

(Vasile Cîrtoaje, 2018)

Solution. Since

$$(a^3 + b^3 + c^3)^2 = (a^4 + b^4 + c^4)(a^2 + b^2 + c^2) - \sum b^2c^2(b - c)^2,$$

the inequality can be rewritten as follows

$$(a^4 + b^4 + c^4)(a^2 + b^2 + c^2 - ab - bc - ca) \ge \sum b^2c^2(b - c)^2,$$

$$(a^4 + b^4 + c^4)\sum(b - c)^2 \ge 2\sum b^2c^2(b - c)^2,$$

$$\sum S_a(b - c)^2 \ge 0,$$

where

$$S_a = a^4 + (b^2 - c^2)^2.$$

The equality occurs for $a = b = c$.

□

P 2.102. *Let $a_1, a_2, \ldots, a_n$ be real numbers such that*

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n.$$

Prove that:

(a) for $n = 3$,

$$\frac{a_1 + a_2 + a_3}{3} + \min_{i \ne j}(a_i - a_j)^2 \le \frac{5}{3};$$

(b) for $n = 5$,

$$\frac{a_1 + a_2 + a_3 + a_4 + a_5}{5} + \min_{i \ne j}(a_i - a_j)^2 \le 1.$$

(Vasile Cîrtoaje, 2019)

Solution. (a) We need to show that

$$s+\min_{i\neq j}(a_i-a_j)^2\le \frac{5}{3},$$

where

$$s=\frac{a_1+a_2+a_3}{3}.$$

According to Lemma below, we have

$$\min_{i\neq j}(a_i-a_j)^2\le \frac{3(1-s^2)}{2}.$$

Thus, we only need to show that

$$s+\frac{3(1-s^2)}{2}\le \frac{5}{3},$$

which is equivalent to

$$(3s-1)^2\ge 0.$$

For $a_1\ge a_2\ge a_3$, the equality holds when $a_1-a_2=a_2-a_3$ and $a_1+a_2+a_3=1$, i.e. when

$$a_1=\frac{1+2\sqrt{3}}{3},\quad a_2=\frac{1}{3},\quad a_3=\frac{1-2\sqrt{3}}{3}.$$

(b) We need to show that

$$s+\min_{i\neq j}(a_i-a_j)^2\le 1,$$

where

$$s=\frac{a_1+a_2+a_3+a_4+a_5}{5}.$$

According to Lemma below, we have

$$\min_{i\neq j}(a_i-a_j)^2\le \frac{3(1-s^2)}{2}.$$

Thus, we only need to show that

$$s+\frac{1-s^2}{2}\le 1,$$

which is equivalent to

$$(s-1)^2\ge 0.$$

For $a_1\ge a_2\ge a_3\ge a_4\ge a_5$, the equality holds when $a_1-a_2=a_2-a_3=a_3-a_4=a_4-a_5$ and $a_1+a_2+a_3+a_4+a_5=5$, i.e. when $a_1=a_2=a_3=a_4=a_5=1$.

Lemma. If $a_1, a_2, \dots, a_n$ are real numbers satisfying

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n, \qquad a_1 + a_2 + \cdots + a_n = ns, \quad s \in [-1, 1],$$

then

$$\min_{i \neq j}(a_i - a_j)^2 \le \frac{12(1-s^2)}{n^2-1}.$$

Proof. Assume that

$$a_1 \ge a_2 \ge \cdots \ge a_n,$$

and denote

$$x = \min_{i \neq j}(a_i - a_j)^2.$$

We have

$$n^2(1-s^2) = n(a_1^2 + a_2^2 + \cdots + a_n^2) - (a_1 + a_2 + \cdots + a_n)^2 = \sum_{i<j}(a_i - a_j)^2$$

$$= \sum_{i=1}^{n-1}(a_i - a_{i+1})^2 + \sum_{i=1}^{n-2}(a_i - a_{i+2})^2 + \sum_{i=1}^{n-3}(a_i - a_{i+3})^2 + \cdots + \sum_{i=1}^{1}(a_i - a_n)^2$$

$$\ge (n-1)x + 2^2(n-2)x + 3^2(n-3)x + \cdots + (n-1)^2 \cdot [n - (n-1)]x$$

$$= n\left[1^2 + 2^2 + 3^3 + \cdots + (n-1)^2\right]x - \left[1^3 + 2^3 + 3^3 + \cdots + (n-1)^3\right]x$$

$$= \frac{n^2(n-1)(2n-1)x}{6} - \frac{n^2(n-1)^2 x}{4} = \frac{n^2(n^2-1)x}{12},$$

hence

$$x \le \frac{12(1-s^2)}{n^2-1}.$$

For $a_1 \ge a_2 \ge \cdots \ge a_n$, the equality occurs when

$$a_1 - a_2 = a_2 - a_3 = \cdots = a_{n-1} - a_n.$$

Remark. The inequality (b) can be generalized as follows:

$$1 - \frac{a_1 + a_2 + \cdots + a_n}{n} \ge \frac{n^2-1}{24} \min_{i \neq j}(a_i - a_j)^2.$$

□

P 2.103. *Let $a_1, a_2, \dots, a_7$ be real numbers such that*

$$a_1^2 + a_2^2 + \cdots + a_7^2 = n.$$

Prove that:

$$(a) \qquad \sqrt{\frac{|a_1+a_2+\cdots+a_7|}{7}} \;+\; \min_{i\neq j}(a_i-a_j)^2 \;\le\; 1;$$

$$(b) \qquad \sqrt{\frac{|a_1+a_2+\cdots+a_7|}{7}}+8\min_{i\neq j}(a_i-a_j)^2 \le \frac{19}{8}.$$

(Vasile Cîrtoaje, 2019)

Solution. Let

$$s=\frac{a_1+a_2+\cdots+a_7}{7}, \qquad |s|\le 1.$$

(a) We need to show that

$$\sqrt{|s|}+\min_{i\neq j}(a_i-a_j)^2 \le 1.$$

According to Lemma from the proof of the preceding P 2.102, we have

$$\min_{i\neq j}(a_i-a_j)^2 \le \frac{1-s^2}{4}.$$

Thus, we only need to show that

$$\sqrt{|s|}+\frac{1-s^2}{4}\le 1,$$

which is equivalent to

$$(1-|s|)(1+|s|)\le 4\left(1-\sqrt{|s|}\right).$$

This is true if

$$\left(1+\sqrt{|s|}\right)(1+|s|)\le 4,$$

which is obvious for $|s|\le 1$. The equality holds for $a_1=a_2=\cdots=a_7=\pm 1$.

(b) We need to show that

$$\sqrt{|s|}+8\min_{i\neq j}(a_i-a_j)^2 \le \frac{19}{8}.$$

As shown at (a), we have

$$\min_{i\neq j}(a_i-a_j)^2 \le \frac{1-s^2}{4}.$$

Thus, we only need to show that

$$\sqrt{|s|}+2(1-|s|^2)\le \frac{19}{8},$$

which is equivalent to

$$16|s|^2+3 \geq 8\sqrt{|s|}.$$

This follows immediately from the AM-GM inequality. For $a_1 \geq a_2 \geq \cdots \geq a_7$, the equality holds when $a_1-a_2=a_2-a_3=\cdots=a_6-a_7$ and $a_1+a_2+\cdots+a_7=\dfrac{\pm 7}{4}$, i.e. when $a_k=\dfrac{2+(4-k)\sqrt{15}}{8}$, and also when $a_k=\dfrac{-2+(4-k)\sqrt{15}}{8}$.

□

P 2.104. *Let f be a differentiable convex function on a closed interval $\mathbb{I}=[a,b]$. If $a_1,a_2,\ldots,a_n \in \mathbb{I}$, then Jensen's difference*

$$D=f(a_1)+f(a_2)+\cdots+f(a_n)-nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

is maximal when all $a_i \in \{a,b\}$.

(Vasile Cîrtoaje, 1990)

Solution. For fixed $a_2,a_3,\ldots,a_n$, let

$$F(x)=f(x)+f(a_2)+\cdots+f(a_n)-nf\left(\frac{x+a_2+\cdots+a_n}{n}\right).$$

Since $f'(x)$ is increasing, from

$$F'(x)=f'(x)-f'\left(\frac{x+a_2+\cdots+a_n}{n}\right),$$

it follows that $F'(x) \leq 0$ for $x \in [a,c]$ and $F'(x) \geq 0$ for $x \in [c,b]$, where

$$c=\frac{a_2+\cdots+a_n}{n-1}.$$

As a consequence, $F(x)$ is decreasing on $[a,c]$ and increasing on $[c,b]$. Thus, $f(x)$ is maximal for $x=a$ or $x=b$, i.e. Jensen's difference

$$D=f(a_1)+f(a_2)+\cdots+f(a_n)-nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

is maximal for $a_1 \in \{a,b\}$. Similarly, Jensen's difference D is maximal for $a_i \in \{a,b\}$, $i=2,3,\ldots,n$.

Remark. The following statement is also valid:

- *Let f be a differentiable convex function on an interval $\mathbb{I}$, and let $a_1,a_2,\ldots,a_n \in \mathbb{I}$ such that $a_1 \geq a_2 \geq \cdots \geq a_n$. For fixed a_1 and a_n, Jensen's difference*

$$D=f(a_1)+f(a_2)+\cdots+f(a_n)-nf\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)$$

is maximal when all $a_i \in \{a_1,a_n\}$, $i=2,3,\ldots,n-1$.

The proof is similar with the above one.

□

P 2.105. *If a,b,c are real numbers, then*

$$2(a^2-a+1)(b^2-b+1)(c^2-c+1)\ge(abc-1)^2.$$

(Vasile Cîrtoaje, 2018)

First Solution. If one of a,b,c is zero, the inequality is true. Thus, if $a=0$, the inequality becomes

$$2(b^2-b+1)(c^2-c+1)\ge 1.$$

Indeed, we have

$$2(b^2-b+1)(c^2-c+1)\ge 2\cdot\frac{3}{4}\cdot\frac{3}{4}>1.$$

Assume now that $abc\neq 0$, denote

$$x=(b^2-b+1)(c^2-c+1),\qquad y=bc,$$

and write the required inequality as $A\ge 0$, where

$$A=(2x-y^2)a^2-2(x-y)a+2x-1.$$

We have

$$2x-y^2\ge 2\cdot\frac{3b^2}{4}\cdot\frac{3c^2}{4}-b^2c^2>0$$

and

$$A=(2x-y^2)\left(a-\frac{x-y}{2x-y^2}\right)^2+B,$$

where

$$B=2x-1-\frac{(x-y)^2}{2x-y^2}=\frac{x[3x-2(y^2-y+1)]}{2x-y^2}.$$

Since

$$3x-2(y^2-y+1)\ge 0,$$

with equality for $b=c=\dfrac{3\pm\sqrt{5}}{2}$ (see P 2.5), we have $B\ge 0$, hence $A\ge 0$.

The equality occurs for $a=\dfrac{x-y}{2x-y^2}$ and $b=c=\dfrac{3\pm\sqrt{5}}{2}$ (when $b^2-3b+1=0$). Since

$$x=(b^2-b+1)^2=(2b)^2=4b^2,$$

$$x-y=4b^2-b^2=3b^2,\qquad 2x-y^2=8b^2-b^4,$$

$$a=\frac{x-y}{2x-y^2}=\frac{3b^2}{8b^2-b^4}=\frac{3}{8-b^2}=\frac{3}{8-(3b-1)}=\frac{1}{3-b}=b,$$

the equality holds for

$$a=b=c=\frac{3\pm\sqrt{5}}{2}.$$

Second Solution (*by KaiRain*). We get the required inequality by multiplying the inequalities

$$3(b^2-b+1)(c^2-c+1)\ge 2(y^2-y+1),$$

$$4(a^2-a+1)(y^2-y+1)\ge 3(ay-1)^2,$$

where $y=bc$. The first inequality is treated in P 2.5, while the second inequality is equivalent to

$$(ay-2a-2y+1)^2\ge 0.$$

□

P 2.106. *If a,b,c are real numbers, then*

$$(1+\sqrt{2})(a^2-a+1)(b^2-b+1)(c^2-c+1)\ge a^2b^2c^2+1.$$

(Vasile Cîrtoaje, 1992)

Solution. For $a=b=c$, the inequality becomes

$$(1+\sqrt{2})(a^2-a+1)^3\ge a^6+1.$$

This inequality is equivalent to

$$[a^2-(1+\sqrt{2})a+1]^2[\sqrt{2}a^2-(\sqrt{2}-1)a+\sqrt{2}]\ge 0.$$

Based on this result, we have

$$(1+\sqrt{2})^3(a^2-a+1)^3(b^2-b+1)^3(c^2-c+1)^3\ge (a^6+1)(b^6+1)(c^6+1).$$

It suffices to show that

$$(a^6+1)(b^6+1)(c^6+1)\ge (a^2b^2c^2+1)^3,$$

which is just Hölder's inequality. The equality occurs for

$$a=b=c=\frac{1}{2}\left(1+\sqrt{2}\pm\sqrt{2\sqrt{2}-1}\right).$$

□

P 2.107. *If a,b,c,d are real numbers, then*

$$(1-a+a^2)(1-b+b^2)(1-c+c^2)(1-d+d^2)\ge\left(\frac{1+abcd}{2}\right)^2.$$

(Vasile Cîrtoaje, 1992)

Solution. For $a=b=c=d$, the inequality can be written as

$$2(1-a+a^2)^2 \geq 1+a^4.$$

It is true, since

$$2(1-a+a^2)^2-1-a^4=(1-a)^4 \geq 0.$$

Using this result, we get

$$4(1-a+a^2)^2(1-b+b^2)^2 \geq (1+a^4)(1+b^4) \geq (1+a^2b^2)^2.$$

Then, the desired inequality follows by multiplying the inequalities

$$2(1-a+a^2)(1-b+b^2) \geq 1+a^2b^2,$$

$$2(1-c+c^2)(1-d+d^2) \geq 1+c^2d^2,$$

$$(1+a^2b^2)(1+c^2d^2) \geq (1+abcd)^2.$$

The equality holds for $a=b=c=d=1$.

□

P 2.108. *If a,b,c,d are real numbers, then*

$$3(a^2-a+1)(b^2-b+1)(c^2-c+1)(d^2-d+1) \geq a^2b^2c^2d^2-abcd+1.$$

Solution. For fixed b,c,d, denote $m=bcd$ and write the inequality as $F(a) \geq 1$, where

$$F(a)=3f(a)(b^2-b+1)(c^2-c+1)(d^2-d+1),$$

$$f(a)=\frac{a^2-a+1}{m^2a^{-}ma+1}.$$

If $abcd=0$ or $abc=1$ or $bcd=1$ or $cda=1$ or $dab=1$, the inequality is true. Indeed, if $a=0$ or $bcd=1$, then $f(a)=1$ and

$$F(a)=3(b^2-b+1)(c^2-c+1)(d^2-d+1) \geq 3\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}>1.$$

Consider now $abcd \neq 0$, $abc \neq 1$, $bcd \neq 1$, $cda \neq 1$, $dab \neq 1$. We have the derivative

$$f'(a)=\frac{(m-1)[ma^2-2(m+1)a+1]}{(m^2a^2-ma+1)^2},$$

with $f'(a_1)=f'(a_2)=0$, where

$$a_1=\frac{m+1-\sqrt{m^2+m+1}}{m}, \quad a_2=\frac{m+1+\sqrt{m^2+m+1}}{m}.$$

For $a = a_1$ and $a = a_2$, we have

$$abcd = \frac{2a-1}{a-2}.$$

Case 1: $m > 1$. $f(a)$ is increasing on $(-\infty, a_1] \cup [a_2, \infty)$ and decreasing on $[a_1, a_2]$. Thus, it suffices to consider $a = -\infty$ and $a = a_2$. For $a = -\infty$, we have

$$F(-\infty) = \frac{3}{b^2c^2d^2}(b^2-b+1)(c^2-c+1)(d^2-d+1) \geq \frac{3}{b^2c^2d^2} \cdot \frac{3b^2}{4} \cdot \frac{3c^2}{4} \cdot \frac{3d^2}{4} = \frac{81}{64} > 1.$$

It remains the case $a = a_2$.

Case 2: $0 < m < 1$. $f(a)$ is increasing on $[a_1, a_2]$ and decreasing on $(-\infty, a_1] \cup [a_2, \infty)$. Thus, it suffices to consider $a = a_1$ and $a = \infty$. For $a = \infty$, we have $F(\infty) > 1$. It remains the case $a = a_1$.

Case 3: $m < 0$. $f(a)$ is increasing on $(-\infty, a_2] \cup [a_1, \infty)$ and decreasing on $[a_2, a_1]$. Thus, it suffices to consider $a = -\infty$ and $a = a_1$. For $a = -\infty$, we have $F(-\infty) > 1$. It remains the case $a = a_1$.

Due to symmetry, it suffices to consider the cases $a \in \{a_1, a_2\}$, $b \in \{b_1, b_2\}$, $c \in \{c_1, c_2\}$ and $d \in \{d_1, d_2\}$, when

$$abcd = \frac{2a-1}{a-2} = \frac{2b-1}{b-2} = \frac{2c-1}{c-2} = \frac{2d-1}{d-2}.$$

From these relations, we get $a = b = c = d = -1$, or

$$a = b = c = d = \frac{1}{4}\left(3 + \sqrt{5} \pm \sqrt{6\sqrt{5} - 2}\right)$$

(which follows from $a + \dfrac{1}{a} = \dfrac{3+\sqrt{5}}{2}$). In the first case, we have

$$F(a) = \frac{3(a^2 - a + 1)^4}{a^8 - a^4 + 1} = 243 > 1,$$

and in the second case

$$F(a) = \frac{3(a^2 - a + 1)^4}{a^8 - a^4 + 1} = 1.$$

Therefore, the equality occurs for

$$a = b = c = d = \frac{1}{4}\left(3 + \sqrt{5} \pm \sqrt{6\sqrt{5} - 2}\right).$$

□

P 2.109. *If a, b, c, d are real numbers, then*

$$(a^2 - a + 2)(b^2 - b + 2)(c^2 - c + 2)(d^2 - d + 2) \geq (a + b + c + d)^2.$$

(Vasile Cîrtoaje, 1994)

Solution. . We denote

$$x = a + b - 1, \qquad y = c + d - 1,$$

and use the inequality

$$(a^2 - a + 2)(b^2 - b + 2) \geq (a + b - 1)^2 + 3,$$

which is equivalent to

$$(2ab - a - b)^2 + 3(a - b)^2 \geq 0.$$

Thus, it suffices to show that

$$(x^2 + 3)(y^2 + 3) \geq (x + y + 2)^2,$$

which is equivalent to

$$x^2y^2 + 2(x^2 + y^2 - xy) - 4(x + y) + 5 \geq 0,$$

$$(xy - 1)^2 + 2(x^2 + y^2) - 4(x + y) + 4 \geq 0,$$

$$(xy - 1)^2 + (x - y)^2 + (x + y)^2 - 4(x + y) + 4 \geq 0,$$

$$(xy - 1)^2 + (x - y)^2 + (x + y - 2)^2 \geq 0,$$

The equality occurs for $a = b = c = d = 1$.

Remark. From the given proof, the following stronger inequality holds:

$$(a^2 - a + 2)(b^2 - b + 2)(c^2 - c + 2)(d^2 - d + 2) \geq (a + b + c + d)^2 + (x + y - 2)^2,$$

that is

$$(a^2 - a + 2)(b^2 - b + 2)(c^2 - c + 2)(d^2 - d + 2) \geq (a + b + c + d)^2 + (a + b + c + d - 4)^2.$$

□

P 2.110. *If a, b, c, d are real numbers such that*

$$a + b + c + d \geq a^2 + b^2 + c^2 + d^2,$$

then

$$4abcd + 3(a^2 + b^2 + c^2 + d^2) + 24 \geq 10(a + b + c + d).$$

Solution. Consider the nontrivial case

$$a+b+c+d \geq a^2+b^2+c^2+d^2 > 0,$$

when

$$0 < \frac{a+b+c+d}{4} \leq \frac{a^2+b^2+c^2+d^2}{a+b+c+d} \leq 1,$$

and consider the function

$$f(x) = 24x^4 - 10(a+b+c+d)x^3 + 3(a^2+b^2+c^2+d^2)x^2 + 4abcd,$$

defined for

$$\frac{a+b+c+d}{4} \leq x \leq 1.$$

We have

$$f'(x) = 6xg(x), \qquad g(x) = 16x^2 - 5(a+b+c+d)x + a^2+b^2+c^2+d^2.$$

Since

$$g'(x) = 32x - 5(a+b+c+d) \geq 8(a+b+c+d) - 5(a+b+c+d) > 0$$

g is increasing, therefore

$$g(x) \geq g\left(\frac{a+b+c+d}{4}\right)$$

$$= a^2+b^2+c^2+d^2 - \frac{1}{4}(a+b+c+d)^2] \geq 0.$$

From $f'(x) \geq 0$, it follows that f is increasing, hence

$$f(1) \geq f\left(\frac{a^2+b^2+c^2+d^2}{a+b+c+d}\right).$$

Thus, it is enough to prove the homogeneous inequality

$$f\left(\frac{a^2+b^2+c^2+d^2}{a+b+c+d}\right) \geq 0,$$

which is equivalent to

$$24x^4 - 7x^3y^2 + 4abcdy^4 \geq 0,$$

where

$$x = a^2+b^2+c^2+d^2, \qquad y = a+b+c+d.$$

Write the inequality as

$$6x^3(4x-y^2) \geq y^2(x^3 - 4abcdy^2).$$

Since $4x \geq y^2$, it is enough to show that

$$3x^2(4x - y^2) \geq 2(x^3 - 4abcdy^2).$$

For $d = 0$,the inequality is true if $10x - 3y^2 \geq 0$. Indeed, we have

$$10x - 3y^2 = 10(a^2 + b^2 + c^2) - 3(a + b + c)^2$$

$$= a^2 + b^2 + c^2 + 3(a^2 + b^2 + c^2 - ab - bc - ca) > 0.$$

According to Remark 4 from P 2.53, for fixed x and y, the product $abcd$ is minimal only if the set (a, b, c, d) has at most two distinct elements. Thus, it is enough to to consider the case $a = b$ and $c = d$, and the case $a = b = c$. Due to homogeneity, we can set $d = 1$.

Case 1: $a = b$ and $c = d = 1$. Since

$$3x^2(4x - y^2) = 48(a^2 + 1)^2(a - 1)^2,$$

$$2(x^3 - 4abcdy^2) = 16[(a^2 + 1)^3 - 2a^2(a + 1)^2]$$
$$= 16(a - 1)^2(a^4 + 2a^3 + 4a^2 + 2a + 1),$$

we need to show that

$$3(a^2 + 1)^3 \geq a^4 + 2a^3 + 4a^2 + 2a + 1,$$

that is

$$a^4 - a^3 + a^2 - a + 1 \geq 0,$$

$$(a - 1)^2(a^2 + a + 1) + a^2 \geq 0.$$

Case 2: $a = b = c$ and $d = 1$. Since

$$3x^2(4x - y^2) = 9(3a^2 + 1)^2(a - 1)^2,$$

$$2(x^3 - 4abcdy^2) = 2[(3a^2 + 1)^3 - 4a^3(3a + 1)^2]$$
$$= 2(a - 1)^2(27a^4 + 18a^3 + 12a^2 + 2a + 1),$$

we need to show that

$$9(3a^2 + 1)^2 \geq 2(27a^4 + 18a^3 + 12a^2 + 2a + 1),$$

that is

$$27a^4 - 36a^3 + 30a^2 - 4a + 7 \geq 0,$$

$$9a^4 + 8a^2 + 3 + 2(9a^2 + 2)(a - 1)^2 \geq 0.$$

The equality holds for $a = b = c = d = 0$ and for $a = b = c = d = 1$.

□

P 2.111. *Let a,b,c,d be real numbers such that $abcd>0$. Prove that*

$$\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)\left(d+\frac{1}{d}\right)\ge(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

(Vasile Cîrtoaje, 2011)

First Solution. Write the inequality as $A\ge B$, where

$$\begin{aligned}A&=(a^2+1)(b^2+1)(c^2+1)(d^2+1)\\&=(1+a^2c^2)(1+b^2d^2)+\sum a^2+\sum a^2b^2+\sum a^2b^2c^2,\\B&=\left(\sum a\right)\left(\sum abc\right)=4abcd+\sum a^2(bc+cd+bd).\end{aligned}$$

Then,

$$\begin{aligned}A-B=&(1-abcd)^2+(ac-bd)^2+\frac{1}{2}\sum a^2(1-bc)^2+\frac{1}{2}\sum a^2(1-cd)^2\\&+\sum a^2b^2-\sum a^2bd,\end{aligned}$$

and hence

$$A-B\ge\sum a^2b^2-\sum a^2bd=\frac{1}{2}\sum a^2(b-d)^2\ge 0.$$

The equality holds for $a=b=c=d=1$.

Second Solution. Since

$$(a+b)(b+c)(c+d)(d+a)-(a+b+c+d)(bcd+cda+dab+abc)=(ac-bd)^2\ge 0,$$

it suffices to show that

$$(a^2+1)(b^2+1)(c^2+1)(d^2+1)\ge(a+b)(b+c)(c+d)(d+a).$$

By the Cauchy-Schwarz inequality, we have

$$(a^2+1)(1+b^2)\ge(a+b)^2,$$

$$(b^2+1)(1+c^2)\ge(b+c)^2,$$

$$(c^2+1)(1+d^2)\ge(c+d)^2,$$

$$(d^2+1)(1+a^2)\ge(d+a)^2.$$

Multiplying these inequalities, we get

$$\begin{aligned}(a^2+1)(b^2+1)(c^2+1)(d^2+1)&\ge|(a+b)(b+c)(c+d)(d+a)|\\&\ge(a+b)(b+c)(c+d)(d+a).\end{aligned}$$

□

P 2.112. *Let a, b, c, d be real numbers such that*

$$a+b+c+d=4, \qquad a^2+b^2+c^2+d^2=7.$$

Prove that

$$a^3+b^3+c^3+d^3 \le 16.$$

(Vasile Cîrtoaje, 2010)

First Solution. Assume that $a \le b \le c \le d$, and denote

$$s=a+b, \quad p=ab, \qquad s \le 2, \quad 4p \le s^2.$$

Since

$$2(a^3+b^3)=2(s^3-3ps),$$

$$c+d=4-s, \qquad c^2+d^2=7-(a^2+b^2)=7-s^2+2p,$$

$$2(c^3+d^3)=(c+d)[3(c^2+d^2)-(c+d)^2]=(4-s)(-4s^2+8s+5+6p),$$

we have

$$\begin{aligned}2(a^3+b^3+c^3+d^3-16) &= 12p(2-s)+6s^3-24s^2+27s-12\\ &\le 3s^2(2-s)+6s^3-24s^2+27s-12\\ &= 3(s-1)^2(s-4) \le 0.\end{aligned}$$

This completes the proof. The equality holds for $a=b=c=1/2$ and $d=5/2$ (or any cyclic permutation).

Second Solution (by *Vo Quoc Ba Can*). From

$$7=a^2+b^2+c^2+d^2 \ge a^2+\frac{1}{3}(b+c+d)^2=a^2+\frac{1}{3}(4-a)^2,$$

it follows that

$$a \in \left[\frac{-1}{2}, \frac{5}{2}\right].$$

Similarly, we have

$$b, c, d \in \left[\frac{-1}{2}, \frac{5}{2}\right].$$

On the other hand,

$$\begin{aligned}a^3+b^3+c^3+d^3 &= \frac{5}{2}\sum a^2+\sum\left(a^3-\frac{5}{2}a^2\right)\\ &= \frac{35}{2}-\frac{1}{2}\sum a^2(5-2a)\end{aligned}$$

and, by virtue of the Cauchy-Schwarz inequality,

$$\sum a^2(5-2a) \ge \frac{\left[\sum a(5-2a)\right]^2}{\sum(5-2a)}=\frac{\left(5\sum a-2\sum a^2\right)^2}{20-2\sum a}=3.$$

Therefore,

$$a^3+b^3+c^3+d^3 \le \frac{35}{2}-\frac{3}{2}=16.$$

Remark. In the same manner as in the second solution, we can prove the following generalization.

- *If $a_1,a_2,\ldots,a_n$ are real numbers such that*

$$a_1+a_2+\cdots+a_n=2n, \quad a_1^2+a_2^2+\cdots+a_n^2=n(n+3),$$

then

$$a_1^3+a_2^3+\cdots+a_n^3 \le n(n^2+3n+4),$$

with equality for $a_1=\ldots=a_{n-1}=1$ and $a_n=n+1$ (or any cyclic permutation).

□

P 2.113. *Let a,b,c,d be real numbers such that*

$$a+b+c+d=0.$$

Prove that

$$12(a^4+b^4+c^4+d^4) \le 7(a^2+b^2+c^2+d^2)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. Assume that $a^2=\max\{b^2,c^2,d^2\}$ and denote

$$x=\sqrt{\frac{b^2+c^2+d^2}{3}}, \quad x^2 \le a^2.$$

Since

$$x^2=\frac{b^2+c^2+d^2}{3} \ge \left(\frac{b+c+d}{3}\right)^2=\frac{a^2}{9},$$

we have

$$\frac{a^2}{9} \le x^2 \le a^2.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
b^4+c^4+d^4 &= (b^2+c^2+d^2)^2-2(b^2c^2+c^2d^2+d^2b^2)\\
&= 9x^4-2(b^2c^2+c^2d^2+d^2b^2)\\
&\le 9x^4-\frac{2}{3}(bc+cd+db)^2\\
&= 9x^4-\frac{1}{6}[(b+c+d)^2-b^2-c^2-d^2]^2\\
&= 9x^4-\frac{1}{6}(a^2-3x^2)^2=\frac{45x^4+6a^2x^2-a^4}{6},
\end{aligned}$$

hence

$$a^4+b^4+c^4+d^4 \le \frac{45x^4+6a^2x^2+5a^4}{6}.$$

Therefore, it suffices to prove that

$$2(45x^4+6a^2x^2+5a^4) \le 7(a^2+3x^2)^2,$$

which is equivalent to the obvious inequality

$$(x^2-a^2)(9x^2-a^2) \le 0.$$

The equality holds for $-a/3=b=c=d$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization.

- *If $a_1,a_2,\dots,a_n$ are real numbers such that*

$$a_1+a_2+\cdots+a_n=0,$$

then

$$\frac{(a_1^2+a_2^2+\cdots+a_n^2)^2}{a_1^4+a_2^4+\cdots+a_n^4} \ge \frac{n(n-1)}{n^2-3n+3},$$

with equality for $-a_1/(n-1)=a_2=\cdots=a_n$ (or any cyclic permutation).

□

P 2.114. *Let a,b,c,d be real numbers such that*

$$a+b+c+d=0.$$

Prove that

$$(a^2+b^2+c^2+d^2)^3 \ge 3(a^3+b^3+c^3+d^3)^2.$$

(Vasile Cîrtoaje, 2011)

Solution. Applying the AM-GM inequality and the identity

$$(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a),$$

we have

$$\begin{aligned}(a^2+b^2+c^2+d^2)^3 &= [a^2+b^2+c^2+(a+b+c)^2]^3\\ &= [(a+b)^2+(b+c)^2+(c+a)^2]^3\\ &\ge 27(a+b)^2(b+c)^2(c+a)^2\\ &= 3[(a+b+c)^3-a^3-b^3-c^3]^2\\ &= 3(a^3+b^3+c^3+d^3)^2.\end{aligned}$$

The equality holds for $a = b = c = -d/3$ (or any cyclic permutation).

Remark. The following generalization holds (*Vasile Cirtoaje*, 2011).

- *If $a_1, a_2, \ldots, a_n$ are real numbers such that*

$$a_1 + a_2 + \cdots + a_n = 0,$$

then

$$(a_1^2 + a_2^2 + \cdots + a_n^2)^3 \geq \frac{n(n-1)}{(n-2)^2}(a_1^3 + a_2^3 + \cdots + a_n^3)^2.$$

Moreover,

- *If $k \geq 3$ is an odd number, and $a_1, a_2, \ldots, a_n$ are real numbers such that*

$$a_1 + a_2 + \cdots + a_n = 0,$$

then

$$(a_1^2 + a_2^2 + \cdots + a_n^2)^k \geq \frac{n^k(n-1)^{k-2}}{[(n-1)^{k-1} - 1]^2}(a_1^k + a_2^k + \cdots + a_n^k)^2,$$

with equality for $a_1 = \cdots = a_{n-1} = -a_n/(n-1)$ (or any cyclic permutation).

□

P 2.115. *If a, b, c, d are real numbers such that*

$$a + b + c + d = 0,$$

then

$$a^4 + b^4 + c^4 + d^4 + 28abcd \geq 0.$$

(Adrian Zahariuc, 2015)

First Solution. Assume that a, b, c, d are nonzero and $a \geq b \geq c \geq d$. Since the statement remains unchanged by changing the sign of all numbers, it suffices to consider the cases

$$a \geq b \geq c \geq d > 0,$$

$$a \geq b \geq c > 0 > d,$$

$$a \geq b > 0 > c \geq d.$$

Clearly, the first and the third case are trivial. For

$$a \geq b \geq c > 0 > d,$$

we rewrite the inequality as

$$a^4 + b^4 + c^4 + (a + b + c)^4 \geq 28abc(a + b + c).$$

By the AM-GM inequality, we have

$$(a+b+c)^4 \ge 27abc(a+b+c).$$

Thus, it suffices to show that

$$a^4+b^4+c^4 \ge abc(a+b+c).$$

This inequality follows from

$$a^4+b^4+c^4 \ge a^2b^2+b^2c^2+c^2a^2 \ge abc(a+b+c).$$

The equality holds for $a=b=c=-d/3$ (or any cyclic permutation).

Second Solution Write the inequality as $f_4(a,b,c) \ge 0$, where

$$f_4(a,b,c) = a^4+b^4+c^4+(a+b+c)^4 - 28abc(a+b+c).$$

According to P 2.60, it suffices to show that $f_4(a,1,1) \ge 0$. Indeed, we have

$$f_4(a,1,1) = 2(a-1)^2(a+3)^2 \ge 0.$$

□

P 2.116. *If a,b,c,d are real numbers such that*

$$abcd = 1.$$

Prove that

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2) \ge (a+b+c+d)^2.$$

(Pham Kim Hung, 2006)

Solution. Substituting a,b,c,d by $|a|,|b|,|c|,|d|$, respectively, the left side of the inequality remains unchanged, while the right side either remains unchanged or increases. Therefore, it suffices to prove the inequality only for $a,b,c,d \ge 0$. Among a,b,c,d there are two numbers less than or equal to 1, or greater than or equal to 1. Let b and d be these numbers; that is,

$$(1-b)(1-d) \ge 0.$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(1+a^2)(1+b^2)(1+c^2)(1+d^2) &= (1+a^2+b^2+a^2b^2)(c^2+1+d^2+c^2d^2)\\ &\ge (c+a+bd+abcd)^2 = (c+a+bd+1)^2.\end{aligned}$$

So, it suffices to show that

$$c+a+bd+1 \ge a+b+c+d,$$

which is equivalent to $(1-b)(1-d) \ge 0$. The equality holds for $a=b=c=d=1$.

□

P 2.117. *Let a,b,c,d be real numbers such that*

$$a^2+b^2+c^2+d^2=4.$$

Prove that

$$(abc)^3+(bcd)^3+(cda)^3+(dab)^3 \le 4.$$

(Vasile Cîrtoaje, 2004)

Solution. Substituting a,b,c,d by $|a|,|b|,|c|,|d|$, respectively, the hypothesis and the right side of the inequality remains unchanged, while the left side either remains unchanged or decreases. Therefore, it suffices to prove the inequality only for $a,b,c,d \ge 0$. Setting

$$x=a^2, \quad y=b^2, \quad z=c^2, \quad t=d^2,$$

we need to prove that

$$(xyz)^{3/2}+(yzt)^{3/2}+(ztx)^{3/2}+(txy)^{3/2} \le 4$$

for

$$x+y+z+t=4.$$

By the AM-GM inequality, we have

$$4\sqrt[4]{xyz} \le 1+x+y+z=5-t,$$

$$(xyz)^{3/2}=xyz\sqrt{xyz} \le \frac{xyz(5-t)^2}{16};$$

analogously,

$$(yzt)^{3/2} \le \frac{yzt(5-x)^2}{16}, \quad (ztx)^{3/2} \le \frac{ztx(5-y)^2}{16}, \quad (txy)^{3/2} \le \frac{txy(5-z)^2}{16}.$$

Therefore, it suffices to show that

$$xyz(5-t)^2+yzt(5-x)^2+ztx(5-y)^2+txy(5-z)^2 \le 64,$$

which is equivalent to $E(x,y,z,t) \ge 0$, where

$$E(x,y,z,t)=36xyzt-25(xyz+yzt+ztx+txy)+64.$$

To prove this inequality, we use the mixing variables method. Without loss of generality, assume that $x \ge y \ge z \ge t \ge 0$. Setting

$$u=\frac{x+y+z}{3},$$

we have

$$3u+t=4, \quad t \le u \le \frac{4}{3}, \quad u^3 \ge xyz.$$

We will show that

$$E(x,y,z,t) \ge E(u,u,u,t) \ge 0.$$

The left inequality is equivalent to

$$25[(u^3 - xyz) + t(3u^2 - xy - yz - zx)] \ge 36t(u^3 - xyz),$$

$$(25 - 36t)(u^3 - xyz) + 25t(3u^2 - xy - yz - zx) \ge 0.$$

By Schur's inequality

$$(x+y+z)^3 + 9xyz \ge 4(x+y+z)(xy+yz+zx),$$

we get

$$9u^3 + 3xyz \ge 4u(xy+yz+zx),$$

hence

$$3u^2 - xy - yz - zx \ge 3u^2 - \frac{9u^3 + 3xyz}{4u} = \frac{3(u^3 - xyz)}{4u}.$$

Therefore, it suffices to prove that

$$25 - 36t + \frac{75t}{4u} \ge 0.$$

Write this inequality as

$$25(4u + 3t) \ge 144ut,$$

then in the homogeneous form

$$25(4u + 3t)(3u + t) \ge 576ut,$$

or, equivalently,

$$75(4u^2 + t^2) \ge 251ut.$$

This inequality is true, since

$$75(4u^2 + t^2) - 251ut \ge 75(4u^2 + t^2 - 4ut) = 75(2u - t)^2 \ge 0.$$

The right inequality $E(u,u,u,t) \ge 0$ holds also, since

$$\begin{aligned} E(u,u,u,t) &= (36u^3 - 75u^2)t - 25u^3 + 64 \\ &= (36u^3 - 75u^2)(4 - 3u) - 25u^3 + 64 \\ &= 4(16 - 75u^2 + 86u^3 - 27u^4) \\ &= 4(1-u)^2(16 + 32u - 27u^2) \\ &= 4(1-u)^2[4(4-u) + 9u(4-3u)] \ge 0. \end{aligned}$$

This completes the proof. The equality holds for $a = b = c = d = 1$.

□

P 2.118. *Let a, b, c, d be real numbers such that*

$$a^2+b^2+c^2+d^2=1.$$

Prove that

$$(1-a)^4+(1-b)^4+(1-c)^4+(1-d)^4 \ge a^4+b^4+c^4+d^4.$$

(Vasile Cîrtoaje, 2007)

Solution. The desired inequality follows by summing the inequalities

$$(1-a)^4+(1-b)^4 \ge c^4+d^4,$$

$$(1-c)^4+(1-d)^4 \ge a^4+b^4.$$

Since

$$(1-a)^4+(1-b)^4 \ge 2(1-a)^2(1-b)^2$$

and

$$c^4+d^4 \ge \frac{1}{2}(c^2+d^2)^2,$$

the former inequality holds if

$$2(1-a)(1-b) \ge c^2+d^2.$$

Indeed,

$$2(1-a)(1-b)-c^2-d^2 = 2(1-a)(1-b)+a^2+b^2-1 = (a+b-1)^2 \ge 0.$$

The equality holds for $a=b=c=d=\dfrac{1}{2}$.

□

P 2.119. *If $a,b,c,d \ge \dfrac{-1}{2}$ such that*

$$a+b+c+d=4,$$

then

$$\frac{1-a}{1-a+a^2}+\frac{1-b}{1-b+b^2}+\frac{1-c}{1-c+c^2}+\frac{1-d}{1-d+d^2} \ge 0.$$

(Vasile Cîrtoaje, 2014)

Solution *(by Nguyen Van Quy)*. Assume that $a \le b \le c \le d$ and consider two cases: $a > 0$ and $a \le 0$.

Case 1: $a > 0$. Write the inequality as

$$\frac{a^2}{1-a+a^2} + \frac{b^2}{1-b+b^2} + \frac{c^2}{1-c+c^2} + \frac{d^2}{1-d+d^2} \le 4.$$

We have

$$\frac{a^2}{1-a+a^2} + \frac{b^2}{1-b+b^2} + \frac{c^2}{1-c+c^2} + \frac{d^2}{1-d+d^2} \le \frac{a^2}{a} + \frac{b^2}{b} + \frac{c^2}{c} + \frac{d^2}{d} = 4.$$

Case 2: $-1/2 \le a \le 0$. We can check that the equality holds for $a = -1/2$ and $b = c = d = 3/2$ (or any cyclic permutation). Define the function

$$f(x) = \frac{1-x}{1-x+x^2} + k_1 x + k_2, \quad x \ge \frac{-1}{2},$$

such that

$$f(3/2) = f'(3/2) = 0.$$

We get

$$k_1 = \frac{12}{49}, \quad k_2 = \frac{-4}{49},$$

when

$$f(x) = \frac{1-x}{1-x+x^2} + \frac{12x-4}{49} = \frac{(2x-3)^2(3x+5)}{49(1-x+x^2)}.$$

Since $f(x) \ge 0$ for $x \ge -1/2$, we have

$$\frac{1-x}{1-x+x^2} \ge \frac{4-12x}{49}.$$

Therefore,

$$\frac{1-b}{1-b+b^2} + \frac{1-c}{1-c+c^2} + \frac{1-d}{1-d+d^2} \ge \frac{12-12(b+c+d)}{49} = \frac{12(a-3)}{49}.$$

Thus, it suffices to show that

$$\frac{1-a}{1-a+a^2} + \frac{12(a-3)}{49} \ge 0.$$

Indeed,

$$\frac{1-a}{1-a+a^2} + \frac{12(a-3)}{49} = \frac{(2a+1)(6a^2-27a+13)}{49(1-a+a^2)} \ge 0.$$

The proof is completed. The equality holds for $a = b = c = d = 1$, and also for $a = -1/2$ and $b = c = d = 3/2$ (or any cyclic permutation).

□

P 2.120. *If a,b,c,d are real numbers such that $a \ge b \ge c \ge d$, $b+c \ge 0$ and*

$$a^2+b^2+c^2+d^2=4,$$

then

$$a^2c^2+b^2d^2 \le 2.$$

(Vasile Cîrtoaje, 2020)

Solution. We have

$$2-a^2c^2-b^2d^2 = 2+(a^2-b^2)(b^2-c^2)-b^2(a^2+c^2+d^2-b^2)$$

$$= 2+(a^2-b^2)(b^2-c^2)-b^2(4-2b^2)$$

$$= (a^2-b^2)(b^2-c^2)+2(b^2-1)^2$$

$$= (a-b)(b-c)(a+b)(b+c)+2(b^2-1)^2 \ge 0.$$

The equality holds for $a=b=1 \ge c \ge d$ and $c^2+d^2=2$, and for $a \ge b=c=1 \ge d$ and $a^2+d^2=2$.

□

P 2.121. *Let a,b,c,d be real numbers such that*

$$a^2+b^2+c^2+d^2=4.$$

If $a \ge b \ge c \ge d$, then

$$1-abcd \le (a-d)^2.$$

(Vasile Cîrtoaje, 2019)

Solution. There are five cases to consider.

Case 1: $a \ge b \ge c \ge d \ge 0$. For $a=d$, the inequality is a trivial equality. Consider next that $a>d$, give up the condition $b \ge c$ (consider only that $b,c \in [d,a]$) and write the inequality in the homogeneous form $f \ge 0$, where

$$f = 16abcd+4(a-d)^2(a^2+b^2+c^2+d^2)-(a^2+b^2+c^2+d^2)^2.$$

For fixed a, c and d, f is a function of b, $b \in [d,a]$. We have

$$f'(b) = 16acd+8(a-d)^2b-4b(a^2+b^2+c^2+d^2) = 4bh(b),$$

where

$$h(b) = \frac{4acd}{b}+2(a-d)^2-(a^2+b^2+c^2+d^2).$$

Since $h(b)$ is a decreasing function, there are three possible cases: (1) $h(b) \ge 0$ for $b \in [d,a]$, hence $f(b)$ is increasing on $[d,a]$; (2) $h(b) \ge 0$ for $b \in [d,d_1]$ and

$h(b) \le 0$ for $b \in [d_1, a]$, hence $f(b)$ is increasing on $[d, d_1]$ and decreasing on $[d_1, a]$; (3) $h(b) \le 0$ for $b \in [d, a]$, hence $f(b)$ is decreasing on $[d, a]$. In all these cases $f(b)$ is minimal when $b \in \{a, d\}$. As a consequence, we only need to prove the required inequality for $b \in \{a, d\}$. Similarly, we only need to prove the required inequality for $c \in \{a, d\}$. So, we need to show that

$$16a^k d^{4-k} + 4(a-d)^2[ka^2 + (4-k)d^2] - [ka^2 + (4-k)d^2]^2 \ge 0,$$

where

$$k \in \{1, 2, 3\}.$$

For $d = 0$, the inequality reduces to

$$k(4-k)a^4 \ge 0,$$

which is true for $k \in \{1, 2, 3\}$. Next, due to homogeneity, we may set $d = 1$ (which involves $a > 1$). The required inequality becomes

$$4(a-1)^2(ka^2 + 4 - k) \ge (ka^2 + 4 - k)^2 - 16a^k.$$

For$k = 1$, we need to show that

$$4(a-1)^2(a^2 + 3) \ge (a^2 + 3)^2 - 16a,$$

that is

$$4(a-1)^2(a^2 + 3) \ge (a-1)^2(a^2 + 2a + 9),$$

$$(a-1)^2(3a^2 - 2a + 3) \ge 0.$$

The last inequality is clearly true.

For$k = 2$, we need to show that

$$2(a-1)^2(a^2 + 1) \ge (a^2 + 1)^2 - 4a^2,$$

that is

$$2(a-1)^2(a^2 + 1) \ge (a^2 - 1)^2,$$

$$(a-1)^4 \ge 0.$$

For$k = 3$, we need to show that

$$4(a-1)^2(3a^2 + 1) \ge (3a^2 + 1)^2 - 16a^3,$$

that is

$$4(a-1)^2(3a^2 + 1) \ge (a-1)^2(9a^2 + 2a + 1),$$

$$(a-1)^2(3a^2 - 2a + 3) \ge 0.$$

Case 2: $a \ge b \ge c \ge 0 \ge d$. Replacing d by $-d$, we need to show that

$$(a+d)^2 \ge 1 + abcd$$

for $a \ge b \ge c \ge 0$ and $d \ge 0$ satisfying $a^2+b^2+c^2+d^2=4$. It is enough to show that

$$2(a+d)^2 \ge 2+ad(b^2+c^2),$$

that is

$$2(a+d)^2 \ge 2+ad(4-a^2-d^2),$$

$$(a^2+d^2)(2+ad) \ge 2.$$

Since

$$4=a^2+b^2+c^2+d^2 \le 3a^2+d^2,$$

we have

$$(a^2+d^2)(2+ad) \ge 2(a^2+d^2)=\frac{4a^2+4d^2}{2} \ge \frac{3a^2+d^2}{2} \ge 2.$$

Case 3: $a \ge b \ge 0 \ge c \ge d$. Replacing c by $-c$ and d by $-d$, we need to show that

$$abcd+(a+d)^2 \ge 1$$

for $a \ge b \ge 0$ and $d \ge c \ge 0$ satisfying $a^2+b^2+c^2+d^2=4$. It is enough to show that

$$(a+d)^2 \ge 1.$$

Since

$$4=a^2+b^2+c^2+d^2 \le 2a^2+2d^2,$$

we have

$$(a+d)^2 \ge a^2+d^2 \ge 2.$$

Case 4: $a \ge 0 \ge b \ge c \ge d$. Replacing a,b,c,d with $-d,-c,-b,-a$, respectively, this case reduces to the case 1.

Case 5: $0 \ge a \ge b \ge c \ge d$. Replacing a,b,c,d with $-d,-c,-b,-a$, respectively, this case reduces to the case $a \ge b \ge c \ge d \ge 0$.

The proof is completed. The equality occurs for $a=b=c=d=1$.

□

P 2.122. *If $a,b,c,d,e \ge -3$ such that*

$$a+b+c+d+e=5,$$

then

$$\frac{1-a}{1+a+a^2}+\frac{1-b}{1+b+b^2}+\frac{1-c}{1+c+c^2}+\frac{1-d}{1+d+d^2}+\frac{1-e}{1+e+e^2} \ge 0.$$

(Vasile Cîrtoaje, 2014)

***Solution*.** Assume that $a \le b \le c \le d \le e$ and consider two cases: $a \ge 0$ and $a \le 0$.

Case 1: $a \ge 0$. For any $x \ge 0$, we have

$$\frac{1-x}{1+x+x^2} - \frac{1-x}{3} = \frac{(x-1)^2(x+2)}{3(1+x+x^2)} \ge 0.$$

Therefore, it suffices to show that

$$\frac{1-a}{3} + \frac{1-b}{3} + \frac{1-c}{3} + \frac{1-d}{3} + \frac{1-e}{3} \ge 0,$$

which is an identity.

Case 2: $-3 \le a \le 0$. We can check that the equality holds for $a = -3$ and $b = c = d = e = 2$. Based on this, define the function

$$f(x) = \frac{1-x}{1+x+x^2} + k_1 x + k_2, \quad x \ge -3,$$

such that

$$f(2) = f'(2) = 0.$$

We get

$$k_1 = \frac{2}{49}, \quad k_2 = \frac{3}{49},$$

when

$$f(x) = \frac{1-x}{1+x+x^2} + \frac{2x+3}{49} = \frac{(x-2)^2(2x+13)}{49(1+x+x^2)}.$$

Since $f(x) \ge 0$ for $x \ge -3$, we have

$$\frac{1-x}{1+x+x^2} \ge \frac{-2x-3}{49}.$$

Thus, it suffices to show that

$$\frac{1-a}{1+a+a^2} - \frac{2b+3}{49} - \frac{2c+3}{49} - \frac{2d+3}{49} - \frac{2e+3}{49} \ge 0,$$

which is equivalent to

$$\frac{1-a}{1+a+a^2} - \frac{2(b+c+d+e)+12}{49} \ge 0,$$

$$\frac{1-a}{1+a+a^2} - \frac{2(5-a)+12}{49} \ge 0,$$

$$\frac{(a+3)(2a^2-26a+9)}{49(1+a+a^2)} \ge 0.$$

Clearly, the last inequality is true for $-3 \le a \le 0$.

The proof is completed. The equality holds for $a = b = c = d = e = 1$, and also for $a = -3$ and $b = c = d = e = 2$ (or any cyclic permutation).

□

P 2.123. *Let a,b,c,d,e be real numbers such that*

$$a+b+c+d+e=0.$$

Prove that

$$30(a^4+b^4+c^4+d^4+e^4)\geq 7(a^2+b^2+c^2+d^2+e^2)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. Write the inequality as $E(a,b,c,d,e)\geq 0$, where

$$E(a,b,c,d,e)=30(a^4+b^4+c^4+d^4+e^4)-7(a^2+b^2+c^2+d^2+e^2)^2.$$

Among the numbers a,b,c,d,e there exist three with the same sign. Let a,b,c be these numbers. We will show that

$$E(a,b,c,d,e)\geq E(a,b,c,x,x)\geq 0,$$

where

$$x=\frac{d+e}{2}=\frac{-(a+b+c)}{2}.$$

The inequality $E(a,b,c,d,e)\geq E(a,b,c,x,x)$ is equivalent to

$$30(d^4+e^4-2x^4)\geq 7(d^2+e^2-2x^2)(2a^2+2b^2+2c^2+d^2+e^2+2x^2).$$

Since

$$d^4+e^4-2x^4=\frac{(d-e)^2(7d^2+10de+7e^2)}{8}$$

and

$$d^2+e^2-2x^2=\frac{(d-e)^2}{2},$$

we need to show that

$$15(7d^2+10de+7e^2)\geq 14(2a^2+2b^2+2c^2+d^2+e^2+2x^2),$$

which reduces to

$$21(d^2+e^2)+34de\geq 7(a^2+b^2+c^2).$$

Since a,b,c have the same sign, we have

$$a^2+b^2+c^2\leq (a+b+c)^2=(d+e)^2.$$

Thus, it suffices to prove that

$$21(d^2+e^2)+34de\geq 7(d+e)^2,$$

which is equivalent to the obvious inequality

$$4(d^2+e^2)+10(d+e)^2\geq 0.$$

The inequality $E(a,b,c,x,x) \ge 0$, where $x = -(a+b+c)/2$, can be written as

$$23\sum a^4 + 2\left(\sum a\right)^4 \ge 7\left(\sum a^2\right)\left(\sum a\right)^2 + 14\sum a^2b^2.$$

Since

$$\left(\sum a\right)^2 - 3\sum ab = \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2] \ge 0,$$

it suffices to prove that

$$23\sum a^4 + 6\left(\sum ab\right)\left(\sum a\right)^2 \ge 7\left(\sum a^2\right)\left(\sum a\right)^2 + 14\sum a^2b^2.$$

This is equivalent to

$$\sum a^4 + abc\sum a \ge \frac{1}{2}\sum ab(a^2+b^2) + \sum a^2b^2,$$

which follows by summing Schur's inequality of degree four

$$\sum a^4 + abc\sum a \ge \sum ab(a^2+b^2)$$

and the obvious inequality

$$\frac{1}{2}\sum ab(a^2+b^2) \ge \sum a^2b^2.$$

This completes the proof. The equality holds for $a = b = c = 2$ and $d = e = -3$ (or any permutation thereof).

Remark. Notice that the following generalization holds (*Vasile Cîrtoaje*, 2010).

- *If n is an odd positive integer and $a_1, a_2, \dots, a_n$ are real numbers such that*

$$a_1 + a_2 + \cdots + a_n = 0,$$

then

$$\frac{(a_1^2 + a_2^2 + \cdots + a_n^2)^2}{a_1^4 + a_2^4 + \cdots + a_n^4} \le \frac{n(n^2-1)}{n^2+3},$$

with equality when $(n+1)/2$ of $a_1, a_2, \dots, a_n$ are equal to $(n-1)/2$ and the other $(n-1)/2$ numbers are equal to $-(n+1)/2$.

□

P 2.124. *If a,b,c,d,e are real numbers such that $a+b+c+d+e = 5$, then*

$$(a^2-a+1)(b^2-b+1)(c^2-c+1)(d^2-d+1)(e^2-e+1) \ge 1.$$

(Vasile Cîrtoaje, 2015)

Solution *(by KaiRain)*. Without loss of generality, consider

$$e = \min\{a, b, c, d, e\}, \qquad e \le 1.$$

Write the inequality as

$$A(e^2 - e + 1) \ge 4,$$

where

$$A = 4(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1),$$

and use the inequality

$$2(a^2 - a + 1)(b^2 - b + 1) \ge (a + b - 1)^2 + 1,$$

which is equivalent to

$$(2ab - a - b)^2 + (a - b)^2 \ge 0.$$

According to this result and the Cauchy-Schwarz inequality, we have

$$A \ge [(a + b - 1)^2 + 1][1 + (c + d - 1)^2] \ge (a + b - 1 + c + d - 1)^2 = (3 - e)^2.$$

Thus, it suffices to show that

$$(3 - e)^2(e^2 - e + 1) \ge 4,$$

which is equivalent to

$$(e - 1)^2(e^2 - 5e + 5) \ge 0.$$

Since

$$e^2 - 5e + 5 \ge 5(1 - e) \ge 0,$$

the proof is completed. The equality holds for $a = b = c = d = e = 1$.

□

P 2.125. *If a, b, c, d, e are real numbers, then*

$$4(a^2 - a + 1)(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1) \ge (abcde - 1)^2.$$

(Vasile Cîrtoaje, 2015)

Solution. Firstly, we consider the cases $abcde = 1$, $a = 0$ and $bcde = 2$. For $abcde = 1$, the inequality is trivial. For $a = 0$, we need to show that

$$4(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1) \ge 1.$$

Indeed, we have

$$4(b^2 - b + 1)(c^2 - c + 1)(d^2 - d + 1)(e^2 - e + 1) \ge 4 \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} > 1.$$

For $bcde = 2$, the inequality becomes

$$4(a^2-a+1)(b^2-b+1)(c^2-c+1)(d^2-d+1)(e^2-e+1) \geq (2a-1)^2.$$

Since

$$(b^2-b+1)(c^2-c+1)(d^2-d+1)(e^2-e+1) \geq |b|\cdot|c|\cdot|d|\cdot|e| = 2,$$

it suffices to show that

$$8(a^2-a+1) \geq (2a-1)^2,$$

which is clearly true. Otherwise, we write the inequality as $F(a) \geq 1$, where

$$F(a) = 4f(a)(b^2-b+1)(c^2-c+1)(d^2-d+1)(e^2-e+1),$$

$$f(a) = \frac{a^2-a+1}{(ga-1)^2}, \quad g = bcde,$$

$$f'(a) = \frac{(g-2)a-2g+1}{(ag-1)^3}.$$

If $g \neq 2$, we have $f'(a_1) = 0$ for

$$a_1 = \frac{2g-1}{g-2}.$$

Case 1: $g \in (-\infty, 0) \cup (2, \infty)$. $f(a)$ is increasing on $\left(-\infty, \frac{1}{g}\right) \cup [a_1, \infty)$ and decreasing on $\left(\frac{1}{g}, a_1\right]$. Thus, it suffices to consider $a = -\infty$ and $a = a_1$. For $a = -\infty$, we have

$$F(a) = \frac{4}{g^2}(b^2-b+1)(c^2-c+1)(d^2-d+1)(e^2-e+1) \geq \frac{4}{g^2}\cdot\frac{3b^2}{4}\cdot\frac{3c^2}{4}\cdot\frac{3d^2}{4}\cdot\frac{3e^2}{4} > 1.$$

It remains the case $a = a_1$.

Case 2: $0 < g < 2$. $f(a)$ is increasing on $\left[a_1, \frac{1}{g}\right)$ and decreasing on $(-\infty, a_1] \cup \left(\frac{1}{g}, \infty\right)$. Thus, it suffices to consider $a = \infty$ and $a = a_1$ and . For $a = \infty$, the inequality holds. It remains the case $a = a_1$.

For $a = a_1$, we have

$$abcde = \frac{a(2a-1)}{a-2}.$$

Due to symmetry, it suffices to consider the case $a = a_1$, $b = b_1$, $c = c_1$, $d = d_1$, $e = e_1$, when

$$abcde = \frac{a(2a-1)}{a-2} = \frac{b(2b-1)}{b-2} = \frac{c(2c-1)}{c-2} = \frac{d(2d-1)}{d-2} = \frac{e(2e-1)}{e-2}. \quad (*)$$

From

$$\frac{a(2a-1)}{a-2}=\frac{b(2b-1)}{b-2},$$

we get

$$(a-b)(ab-2a-2b+1)=0.$$

The case $ab-2a-2b+1=0$ implies

$$b=\frac{2a-1}{a-2},$$

and from

$$abcde=\frac{a(2a-1)}{a-2},$$

we get $abcde=ab$, hence $cde=1$. Since

$$(c^2-c+1)(d^2-d+1)(e^2-e+1)\geq |c|\cdot|d|\cdot|e|=1$$

and

$$4(a^2-a+1)(b^2-b+1)-3(ab-1)^2=(ab-2a-2b+1)^2\geq 0,$$

we have

$$F(a)\geq \frac{4(a^2-a+1)(b^2-b+1)}{(ab-1)^2}>1.$$

As a consequence, it suffices to consider the case $a=b=c=d=e=f$, when (*) implies

$$a=b=c=d=e=\frac{1}{4}\left(3+\sqrt{5}\pm\sqrt{6\sqrt{5}-2}\right)$$

(from $a+\dfrac{1}{a}=\dfrac{3+\sqrt{5}}{2}$). In this case, we have

$$F(a)=\frac{4(a^2-a+1)^5}{(a^5-1)^2}=1.$$

Note that

$$4(a^2-a+1)^5-(a^5-1)^2=(a^4-3a^3+3a^2-3a+1)^2(3a^2-2a+3).$$

The equality occurs for

$$a=b=c=d=e=\frac{1}{4}\left(3+\sqrt{5}\pm\sqrt{6\sqrt{5}-2}\right).$$

□

P 2.126. *If $a_1, a_2, \ldots, a_5$ are real numbers such that*

$$a_1^3 + a_2^3 + a_3^3 + a_4^3 + a_5^3 = 0,$$

then

$$\sum_{i<j} a_i a_j \le 0.$$

(Vasile Cîrtoaje, 2019)

Solution. Since the statement remain unchanged by replacing all a_i with $-a_i$, there are two cases to consider: $a_1 \le 0$, $a_2, a_3, a_4, a_5 \ge 0$, and $a_1, a_2 \le 0$, $a_3, a_4, a_5 \ge 0$.

In the first case, we need to show that if $x, a, b, c, d \ge 0$ such that

$$x^3 = a^3 + b^3 + c^3 + d^3,$$

then

$$(a+b+c+d)x \ge ab + ac + ad + bc + bd + cd.$$

Write the inequality as

$$2(a+b+c+d)x + a^2 + b^2 + c^2 + d^2 \ge (a+b+c+d)^2.$$

Denoting

$$s = \frac{a+b+c+d}{4},$$

the inequality becomes

$$8sx + a^2 + b^2 + c^2 + d^2 \ge 16s^2.$$

Since

$$a^2 + b^2 + c^2 + d^2 \ge 4s^2$$

and

$$x^3 \ge 4s^3 \ge \frac{27}{8}s^3 \qquad x \ge \frac{3}{2}\,s,$$

we have

$$8sx + a^2 + b^2 + c^2 + d^2 - 16s^2 \ge 12s^2 + 4s^2 - 16s^2 = 0.$$

In the second case, we need to show that if $x, y, a, b, c \ge 0$ such that

$$x^3 + y^3 = a^3 + b^3 + c^3,$$

then

$$(a+b+c)(x+y) \ge xy + ab + bc + ca.$$

From the known inequalities

$$(x+y)^3 \ge x^3 + y^3, \qquad (x+y)^3 \le 4(x^3 + y^3),$$

we get

$$x+y \geq \sqrt[3]{x^3+y^3} = \sqrt[3]{a^3+a^3+c^3},$$

$$x+y \leq \sqrt[3]{4x^3+4y^3} = \sqrt[3]{4(a^3+b^3+c^3)}.$$

Denoting $t = x+y$, we have

$$t_1 \leq t \leq t_2,$$

where

$$t_1 = \sqrt[3]{a^3+b^3+c^3}, \qquad t_2 = \sqrt[3]{4(a^3+b^3+c^3)}.$$

On the other hand, it is enough to prove the inequality

$$(a+b+c)(x+y) \geq \frac{(x+y)^2}{4} + \frac{(a+b+c)^2}{3},$$

or, better, the inequality

$$(a+b+c)(x+y) \geq \frac{5(x+y)^2}{12} + \frac{(a+b+c)^2}{3},$$

which is equivalent to

$$5t^2 - 12(a+b+c)t + 4(a+b+c)^2 \leq 0,$$

$$[t-2(a+b+c))[5t-2(a+b+c)] \leq 0.$$

This is true if $t_1 \geq \frac{2}{5}(a+b+c)$ and $t_2 \leq 2(a+b+c$. Thus, we need to show that

$$a^3+b^3+c^3 \geq \frac{8}{125}(a+b+c)^3$$

and

$$a^3+b^3+c^3 \leq 2(a+b+c)^3.$$

Indeed, we have

$$a^3+b^3+c^3 \geq \frac{1}{9}(a+b+c)^3 \geq \frac{8}{125}(a+b+c)^3$$

and

$$a^3+b^3+c^3 \leq (a+b+c)^3 \leq 2(a+b+c)^3.$$

Thus, the proof is completed. The equality holds for $a_1 = a_2 = a_3 = a_4 = a_5 = 0$.

□

P 2.127. *If $a_1, a_2, \ldots, a_{13}$ are real numbers such that*

$$a_1 + a_2 + \cdots + a_{13} = \frac{13}{2},$$

then

$$\frac{8a_1+7}{a_1^2-a_1+1} + \frac{8a_2+7}{a_2^2-a_2+1} + \cdots + \frac{8a_{13}+7}{a_{13}^2-a_{13}+1} \le \frac{572}{3}.$$

(Vasile Cîrtoaje, 2018)

Solution. Since

$$\frac{8a_1+7}{a_1^2-a_1+1} - 16 = \frac{-(4a_1-3)^2}{a_1^2-a_1+1},$$

we may rewrite the inequality as

$$\sum \frac{(4a_1-3)^2}{a_1^2-a_1+1} \ge \frac{52}{3}.$$

Substituting

$$a_i = x_i + \frac{1}{2}, \qquad i = 1, 2, \ldots, 13,$$

we need to show that

$$\sum \frac{(4x_1-1)^2}{4x_1^2+3} \ge \frac{13}{3}$$

for

$$x_1 + x_2 + \cdots + x_{13} = 0.$$

Let

$$S = x_1^2 + x_2^2 + \cdots + x_{13}^2.$$

Since

$$13(4x_1^2+3) = 48x_1^2+4(x_2+\cdots+x_{13})^2+39 \le 48x_1^2+48(x_2^2+\cdots+x_{13}^2)+39 = 48S+39,$$

it is enough to show that

$$\sum \frac{(4x_1-1)^2}{48S+39} \ge \frac{1}{3},$$

which is an identity.

The equality occurs for $a_1 = a_2 = \cdots = a_{13} = \frac{1}{2}$, and for $a_1 = \frac{-5}{2}$, $a_2 = \cdots = a_{13} = \frac{3}{4}$ (or any cyclic permutation).

□

P 2.128. *Let* $a_1, a_2, \dots, a_n \geq -1$ *such that*

$$a_1 + a_2 + \cdots + a_n = 0.$$

Prove that

$$(n-2)(a_1^2 + a_2^2 + \cdots + a_n^2) \geq a_1^3 + a_2^3 + \cdots + a_n^3.$$

(Vasile Cîrtoaje, 2005)

Solution. Without loss of generality, assume that

$$a_1 \geq a_2 \geq \cdots \geq a_n.$$

Write the inequality as

$$\sum_{i=1}^{n} a_i f(a_i) \geq 0,$$

where

$$f(x) = (n-2)x - x^2.$$

Since

$$\begin{aligned} f(a_i) - f(a_{i+1}) &= (a_i - a_{i+1})(n - 2 - a_i - a_{i+1}) \\ &\geq (a_i - a_{i+1})(n - 2 - a_1 - a_2) \\ &= (a_i - a_{i+1})(n - 2 + a_3 + \cdots + a_n) \\ &= (a_i - a_{i+1})[(1 + a_3) + \cdots + (1 + a_n)] \geq 0, \end{aligned}$$

we have $a_1 \geq a_2 \geq \cdots \geq a_n$ and

$$f(a_1) \geq f(a_2) \geq \cdots \geq f(a_n).$$

Therefore, by Chebyshev's inequality, we get

$$n \sum_{i=1}^{n} a_i f(a_i) \geq (a_1 + a_2 + \cdots + a_n)[f(a_1) + f(a_2) + \cdots + f(a_n)] = 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 0$, and for $a_1 = n-1$ and $a_2 = \cdots = a_n = -1$ (or any cyclic permutation).

□

P 2.129. *Let* $a_1, a_2, \dots, a_n \geq -1$ *such that*

$$a_1 + a_2 + \cdots + a_n = 0.$$

Prove that

$$(n-2)(a_1^2 + a_2^2 + \cdots + a_n^2) + (n - 1(a_1^3 + a_2^3 + \cdots + a_n^3) \geq 0.$$

(Vasile Cîrtoaje, 2005)

Solution. For the nontrivial case $a_1^2+a_2^2+\cdots+a_n^2 \neq 0$, write the inequality as

$$(n-1)\frac{S_3}{S_2}+n-2 \geq 0,$$

where

$$S_2 = a_1^2+a_2^2+\cdots+a_n^2, \quad S_3 = a_1^3+a_2^3+\cdots+a_n^3.$$

Without loss of generality, assume that

$$a_1 \leq a_2 \leq \cdots \leq a_n, \quad a_1 < 0.$$

For any $p > 0$ such that $a_1 + p \geq 0$, by the Cauchy-Schwarz inequality, we have

$$\sum_{i=1}^n (a_i+1)^2(a_i+p) \geq \frac{[\sum_{i=1}^n (a_i+1)(a_i+p)]^2}{\sum_{i=1}^n (a_i+p)},$$

which is equivalent to

$$\sum_{i=1}^n a_i^3 + (p+2)\sum_{i=1}^n a_i^2 + np \geq \frac{(\sum_{i=1}^n a_i^2 + np)^2}{np},$$

$$\frac{S_3}{S_2} \geq \frac{S_2}{np} - p.$$

Thus, it suffices to show that

$$S_2 + \frac{n(n-2)}{n-1}p \geq np^2.$$

Case 1: $S_2 \geq \dfrac{n}{n-1}$. Choosing $p = 1$, we have

$$a_1 + p \geq 0$$

and

$$S_2 + \frac{n(n-2)}{n-1}p - np^2 = S_2 - \frac{n}{n-1} \geq 0.$$

Case 2: $0 < S_2 \leq \dfrac{n}{n-1}$. We set

$$p = \sqrt{\frac{n-1}{n}S_2}.$$

From

$$\begin{aligned} p^2 - a_1^2 &= \frac{n-1}{n}S_2 - a_1^2 \\ &= \frac{n-1}{n}(a_2^2+\cdots+a_n^2) - \frac{1}{n}a_1^2 \\ &\geq \frac{(a_2+\cdots+a_n)^2}{n} - \frac{a_1^2}{n} = 0, \end{aligned}$$

we get $a_1 + p \ge 0$. In addition,

$$S_2 + \frac{n(n-2)}{n-1}p - np^2 = S_2 + (n-2)\sqrt{\frac{n}{n-1}S_2} - (n-1)S_2$$
$$= (n-2)\sqrt{S_2}\left(\sqrt{\frac{n}{n-1}} - \sqrt{S_2}\right) \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 0$, and also for $a_1 = -1$ and $a_2 = \cdots = a_n = 1/(n-1)$ (or any cyclic permutation).

□

P 2.130. *Let* $a_1, a_2, \ldots, a_n \ge n - 1 - \sqrt{n^2 - n + 1}$ *be nonzero real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \ge n.$$

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that

$$a_1 \le a_2 \le \cdots \le a_n.$$

There are two cases to consider: $a_1 > 0$ and $a_1 < 0$.

Case 1: $a_1 > 0$. By the Cauchy-Schwarz inequality, we have

$$\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2} \ge \frac{1}{n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)^2$$
$$\ge \frac{1}{n}\left(\frac{n^2}{a_1 + a_2 + \cdots a_n}\right)^2 = n.$$

Case 2: $a_1 < 0$. There exists k, $1 \le k \le n-1$, such that

$$a_1 \le \cdots \le a_k < 0 < a_{k+1} \le \cdots \le a_n.$$

Let us denote

$$x = \frac{a_1 + \cdots + a_k}{k}, \quad y = \frac{a_{k+1} + \cdots + a_n}{n-k}.$$

We have

$$-1 < n - 1 - \sqrt{n^2 - n + 1} \le x < 0, \quad y > 1, \quad kx + (n-k)y = n.$$

From $k \ge 1$ and $k(y - x) = n(y - 1) > 0$, we get

$$y - x \le n(y - 1),$$

hence

$$y \ge \frac{n-x}{n-1}.$$

In addition, from $n-1-\sqrt{n^2-n+1} \le x$, we get

$$n+2(n-1)x-x^2 \ge 0.$$

By the Cauchy-Schwarz inequality, we have

$$\frac{1}{a_1^2}+\cdots+\frac{1}{a_k^2} \ge \frac{1}{k}\left(\frac{1}{-a_1}+\cdots+\frac{1}{-a_k}\right)^2 \ge \frac{1}{k}\left(\frac{k^2}{-a_1-\cdots-a_k}\right)^2 = \frac{k}{x^2}$$

and

$$\frac{1}{a_{k+1}^2}+\cdots+\frac{1}{a_n^2} \ge \frac{1}{n-k}\left(\frac{1}{a_{k+1}}+\cdots+\frac{1}{a_n}\right)^2 \ge \frac{1}{n-k}\left[\frac{(n-k)^2}{a_{k+1}+\cdots+a_n}\right]^2 = \frac{n-k}{y^2}.$$

Then, it suffices to prove that

$$\frac{k}{x^2}+\frac{n-k}{y^2} \ge n,$$

which is equivalent to

$$k(y-x)(y+x) \ge nx^2(y-1)(y+1).$$

Since $k(y-x)=n(y-1)>0$, we need to show that $y+x \ge x^2(y+1)$, which is equivalent to

$$(1-x)(x+y+xy) \ge 0.$$

This is true, since $1-x>0$ and

$$x+y+xy \ge x+\frac{(1+x)(n-x)}{n-1} = \frac{n+2(n-1)x-x^2}{n-1} \ge 0.$$

The equality holds when $a_1=a_2=\cdots=a_n=1$, as well as when one of $a_1,a_2,\ldots,a_n$ is $n-1-\sqrt{n^2-n+1}$ and the others are $\dfrac{1+\sqrt{n^2-n+1}}{n-1}$.

□

P 2.131. *Let* $a_1,a_2,\ldots,a_n \le \dfrac{n}{n-2}$ *be real numbers such that*

$$a_1+a_2+\cdots+a_n=n.$$

If k *is a positive integer,* $k \ge 2$*, then*

$$a_1^k+a_2^k+\cdots+a_n^k \ge n.$$

(Vasile Cîrtoaje, 2012)

Solution. First we show that at most one of a_i is negative. Assume, for the sake of contradiction, that $a_{n-1} < 0$ and $a_n < 0$. Then,

$$a_{n-1} + a_n = n - (a_1 + \cdots + a_{n-2}) \geq n - (n-2) \cdot \frac{n}{n-2} = 0,$$

which is a contradiction. There are two cases to consider.

Case 1: $a_1, a_2, \ldots, a_n \geq 0$. The desired inequality is just Jensen's inequality applied to the convex function $f(x) = x^k$.

Case 2: $a_1, a_2, \ldots, a_{n-1} \geq 0$ and $a_n < 0$. Let us denote

$$x = \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}, \quad y = -a_n.$$

We have

$$(n-1)x - y = n, \quad x \geq y > 0.$$

The condition $x \geq y$ follows from

$$x - y = n - (n-2)x \geq 0.$$

By Jensen's inequality, we have

$$a_1^k + a_2^k + \cdots + a_{n-1}^k \geq (n-1)x^k.$$

In addition, $a_n^k \geq -y^k$. Thus, it suffices to show that

$$(n-1)x^k - y^k \geq n.$$

We will use the inequality

$$x^k - y^k > (x-y)^k,$$

which is equivalent to

$$1 - \left(\frac{y}{x}\right)^k > \left(1 - \frac{y}{x}\right)^k.$$

Indeed,

$$1 - \left(\frac{y}{x}\right)^k - \left(1 - \frac{y}{x}\right)^k > 1 - \left(\frac{y}{x}\right) - \left(1 - \frac{y}{x}\right) = 0.$$

Therefore, it suffices to show that

$$(n-2)x^k + (x-y)^k \geq n.$$

By Jensen's inequality, we have

$$(n-2)x^k + (x-y)^k \geq [(n-2)+1]\left[\frac{(n-2)x + (x-y)}{(n-2)+1}\right]^k = n\left(\frac{n}{n-1}\right)^{k-1} > n.$$

This completes the proof. The equality holds for $a_1 = a_2 = \cdots a_n = 1$.

□

P 2.132. *If* $a_1, a_2, \ldots, a_n \ge \dfrac{-n}{n-2}$, $n \ge 3$, *then*

$$\frac{1}{a_1^2}+\frac{1}{a_2^2}+\cdots+\frac{1}{a_n^2} \ge \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}.$$

(Vasile Cîrtoaje, 2012)

Solution. Let

$$f(x)=\frac{1}{x^2}-\frac{1}{x}, \quad x \ge \frac{-n}{n-2}, \ \ x \neq 0.$$

We need to show that

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge 0.$$

Consider

$$a_1,\ldots,a_k<0, \quad a_{k+1},\ldots,a_n>0, \quad k=0,1,\ldots,n-1.$$

Since $a_1+\cdots+a_k \ge \dfrac{-kn}{n-2}$, it follows that

$$a_{k+1}+\cdots+a_n = n-(a_1+\cdots+a_n) \le \frac{n(n+k-2)}{n-2}.$$

Denote

$$x=\frac{1}{a_{k+1}}+\cdots+\frac{1}{a_n}.$$

Since

$$f(a_i) \ge \frac{2(n-2)(n-1)}{n^2}, \ \ i=1,2,\ldots,k,$$

we get

$$f(a_1)+f(a_2)+\cdots+f(a_n) \ge \frac{2k(n-2)(n-1)}{n^2}+f(a_{k+1})+\cdots+f(a_n)$$

$$=\frac{2k(n-2)(n-1)}{n^2}+\left(\frac{1}{a_{k+1}^2}+\cdots+\frac{1}{a_n^2}\right)-x \ge \frac{2k(n-2)(n-1)}{n^2}+\frac{x^2}{n-k}-x.$$

Thus, we need to show that

$$nx^2-n(n-k)x+\frac{2k(n-k)(n-2)(n-1)}{n} \ge 0.$$

For $9k \ge \dfrac{n^3}{(3n-4)^2}$, we have

$$nx^2+\frac{2k(n-k)(n-2)(n-1)}{n}-n(n-k)x \ge$$

$$\ge [2\sqrt{2k(n-k)(n-2)(n-1)}-n(n-k)]x$$

$$= \sqrt{n-k}[2\sqrt{2k(n-2)(n-1)} - n\sqrt{n-k}]x \geq 0.$$

So, it remains to study the case $9k \leq \dfrac{n^3}{(3n-4)^2}$, when it suffices to show that

$$x \geq \frac{n(n-k) + \sqrt{(n-k)[n^3 - k(3n-4)^2]}}{2n}.$$

Because

$$x \geq \frac{(n-k)^2}{a_{k+1} + \cdots + a_n} \geq \frac{(n-k)^2(n-2)}{n(n+k-2)},$$

it suffices to show that

$$\frac{(n-k)^2(n-2)}{n+k-2} \geq \frac{n(n-k) + \sqrt{(n-k)[n^3 - k(3n-4)^2]}}{2},$$

that is equivalent to

$$\frac{(n-k)[n^2 - 2n - (3n-4)k])}{n+k-2} \geq \sqrt{(n-k)[n^3 - k(3n-4)^2)]},$$

$$(n-k)[n^2 - 2n - (3n-4)k]^2 \geq (k+n-2)^2[n^3 - k(3n-4)^2],$$

$$32k(k-1)(n-1)^2(n-2) \geq 0.$$

The equality occurs for $k = 0$, that means for $a_1 = a_2 = \cdots = a_n = 1$, and for $k = 1$, that means $a_1 = \dfrac{-n}{n+2}$ and $a_2 = \cdots = a_n = \dfrac{n}{n-2}$ (or any cyclic permutation).

□

P 2.133. *If* $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ *are real numbers such that*

$$a_1, a_2, \ldots, a_n \geq \frac{-(3n-2)}{n-2}, \quad a_1 + a_2 + \cdots + a_n = n,$$

then

$$\frac{1-a_1}{(1+a_1)^2} + \frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_n}{(1+a_n)^2} \geq 0.$$

(Vasile Cîrtoaje, 2014)

Solution. Since the inequality holds for $n = 3$ (see P 2.25), consider further that $n \geq 4$. Assume that $a_1 \leq a_2 \leq \cdots \leq a_n$ and consider two cases: $a_1 \geq 0$ and $a_1 \leq 0$.

Case 1: $a_1 \geq 0$. For $x \geq 0$, from

$$\frac{1-x}{(1+x)^2} + \frac{x-1}{4} = \frac{(x-1)^2(x+3)}{4(1+x)^2},$$

we get

$$\frac{1-x}{(1+x)^2} \geq \frac{1-x}{4}.$$

Therefore, it suffices to show that

$$\frac{1-a_1}{4} + \frac{1-a_2}{4} + \cdots + \frac{1-a_n}{4} \geq 0,$$

which is an identity.

Case 2: $-(3n-2)/(n-2) \leq a_1 \leq 0$. We can check that the equality holds for

$$a_1 = \frac{-(3n-2)}{n-2}, \quad a_2 = a_3 = \cdots = a_n = \frac{n+2}{n-2}.$$

Based on this, define the function

$$f(x) = \frac{1-x}{(1+x)^2} + k_1 x + k_2, \quad x \geq \frac{-(3n-2)}{n-2}, \ \ x \neq -1,$$

such that

$$f\left(\frac{n+2}{n-2}\right) = f'\left(\frac{n+2}{n-2}\right) = 0.$$

We get

$$k_1 = \frac{(n-4)(n-2)^2}{4n^3},$$

$$k_2 = \frac{(n-2)(-n^2+6n+8)}{4n^3},$$

$$f(x) = \frac{[(n-2)x-n-2]^2[(n-4)x+3n-4]}{4n^3(1+x)^2}.$$

Since $f(x) \geq 0$ for $n \geq 4$ and $x \geq -(3n-2)/(n-2)$, $x \neq -1$, we have

$$\frac{1-x}{(1+x)^2} \geq -k_1 x - k_2.$$

Based on this result, we get

$$\frac{1-a_2}{(1+a_2)^2} + \cdots + \frac{1-a_n}{(1+a_n)^2} \geq -k_1(a_2+a_3+\cdots+a_n) - (n-1)k_2.$$

Thus, it suffices to show that

$$\frac{1-a_1}{(1+a_1)^2} - k_1(a_2+a_3+\cdots+a_n) - (n-1)k_2 \geq 0,$$

which is equivalent to

$$\frac{1-a_1}{(1+a_1)^2} - k_1(n-a_1) - (n-1)k_2 \geq 0,$$

$$[(n-2)a_1+3n-2][(n-4)(n-2)a_1^2-2(n^2+4n-8)a_1+n^2-2n+8]\ge 0.$$

The last inequality is clearly true for $n \ge 4$ and $-(3n-2)/(n-2) \le a_1 \le 0$. This completes the proof. The equality holds for $a_1 = a_2 = \cdots a_n = 1$, and also for $a_1 = \dfrac{-(3n-2)}{n-2}$ and $a_2 = \cdots = a_n = \dfrac{n+2}{n-2}$ (or any cyclic permutation).

□

P 2.134. *Let $a_1, a_2, \dots, a_n$ be real numbers.*

(a) *If $k \ge n$, then*

$$\frac{(a_1+a_2+\cdots+a_n+k-n)^2}{(a_1^2+k-1)(a_2^2+k-1)\cdots(a_n^2+k-1)} \le \frac{1}{k^{n-2}};$$

(b) *If $k \ge \dfrac{n}{2}$, then*

$$\frac{a_1+a_2+\cdots+a_n+k-n}{(a_1^2+2k-1)(a_2^2+2k-1)\cdots(a_n^2+2k-1)} \le \frac{1}{2(2k)^{n-1}};$$

(c)
$$\frac{(a_1+a_2+\cdots+a_n)^2}{(a_1^2+n-1)(a_2^2+n-1)\cdots(a_n^2+n-1)} \le \frac{1}{n^{n-2}};$$

(d)
$$\frac{a_1+a_2+\cdots+a_n}{(a_1^2+2n-1)(a_2^2+2n-1)\cdots(a_n^2+2n-1)} \le \frac{1}{2(2n)^{n-1}}.$$

(Vasile Cîrtoaje, 1994)

Solution *(by Gabriel Dospinescu)*. a) Assume that

$$a_1^2 \le \cdots \le a_j^2 \le 1 \le a_{j+1}^2 \le \cdots \le a_n^2,$$

where $0 \le j \le n$. By Bernoulli's inequality and Cauchy-Schwarz inequality, we have

$$\frac{1}{k^n}\prod_{i=1}^{n}(a_i^2+k-1) = \prod_{i=1}^{n}\left(1+\frac{a_i^2-1}{k}\right)$$

$$= \prod_{i=1}^{j}\left(1+\frac{a_i^2-1}{k}\right)\prod_{i=j+1}^{n}\left(1+\frac{a_i^2-1}{k}\right)$$

$$\ge \left(1+\sum_{i=1}^{j}\frac{a_i^2-1}{k}\right)\left(1+\sum_{i=j+1}^{n}\frac{a_i^2-1}{k}\right)$$

$$= \frac{1}{k^2}\Big[(a_1^2+\cdots+a_j^2)+(n-j)+(k-n)\Big]\Big[j+(a_{j+1}^2+\cdots+a_n^2)+(k-n)\Big]$$

$$\geq \frac{1}{k^2}\left[(a_1+\cdots+a_j)+(a_{j+1}+\cdots+a_n)+(k-n)\right]^2$$

$$=\frac{1}{k^2}(a_1+a_2+\cdots+a_n+k-n]^2.$$

Thus, the proof is completed. The equality occurs for $a_1=a_2=\cdots=a_n=1$.

(b)Replacing k by $2k$ ($k\geq n/2$), the inequality in (a) becomes

$$\frac{(a_1+a_2+\cdots+a_n+2k-n)^2}{(a_1^2+2k-1)(a_2^2+2k-1)\cdots(a_n^2+2k-1)}\leq\frac{1}{(2k)^{n-2}}.$$

Thus, we only need to show that

$$4k(a_1+a_2+\cdots+a_n+k-n)\leq(a_1+a_2+\cdots+a_n+2k-n)^2,$$

which is equivalent to

$$(a_1+a_2+\cdots+a_n-n)^2\geq 0.$$

The equality occurs for $a_1=a_2=\cdots=a_n=1$.

(c) The inequality follows from (a) for $k=n$.

(d) The inequality follows from (b) for $k=n$.

□

P 2.135. *Let $a_1,a_2,\ldots,a_n$ be real numbers.*

(a) *If $k\geq\frac{n}{4}$, then*

$$\frac{(a_1+a_2+\cdots+a_n+2k-n)^2}{(a_1^2-a_1+k)(a_2^2-a_2+k)\cdots(a_n^2-a_n+k)}\leq\frac{4}{k^{n-2}};$$

(b)

$$\frac{(a_1+a_2+\cdots+a_n)^2}{(a_1^2-a_1+\frac{n}{2})(a_2^2-a_2+\frac{n}{2})\cdots(a_n^2-a_n+\frac{n}{2})}\leq\frac{2^n}{n^{n-2}}.$$

(Vasile Cîrtoaje, 1994)

Solution. (a) The inequality follows from the inequality (a) in P 2.134 by replacing all a_i with $2a_i-1$ and k with $4k$. The equality occurs for $a_1=a_2=\cdots=a_n=1$.

(b) The inequality follows from (a) for $k=\frac{n}{2}$.

□

P 2.136. *Let $a_1, a_2, \ldots, a_n$ be real numbers.*

(a) *If $k \ge n$, then*

$$\frac{(a_1+a_2+\cdots+a_n)^2+n(k-n)}{(a_1^2+k-1)(a_2^2+k-1)\cdots(a_n^2+k-1)} \le \frac{n}{k^{n-1}};$$

(b)

$$\frac{(a_1+a_2+\cdots+a_n)^2+n^2}{(a_1^2+2n-1)(a_2^2+2n-1)\cdots(a_n^2+2n-1)} \le \frac{n}{(2n)^{n-1}}.$$

(Vasile Cîrtoaje, 2018)

Solution. (a) We will use the induction method. For $n=1$, the inequality is an identity. Assume that the inequality is true for $n-1$ variables. Since $k \ge n > n-1$, we have

$$\frac{(n-1)x^2+k-n+1}{(a_1^2+k-1)(a_2^2+k-1)\cdots(a_{n-1}^2+k-1)} \le \frac{1}{k^{n-2}},$$

where

$$x = \frac{a_1+a_2+\cdots+a_{n-1}}{n-1}.$$

We need to show that

$$\frac{[(n-1)x+a_n]^2+n(k-n)}{(a_1^2+k-1)(a_2^2+k-1)\cdots(a_n^2+k-1)} \le \frac{n}{k^{n-1}}$$

for $k \ge n$. Using the induction hypothesis, it suffices to prove that

$$\frac{[(n-1)x+a_n]^2+n(k-n)}{n(a_n^2+k-1)} \le \frac{(n-1)x^2+k-n+1}{k},$$

which is equivalent to

$$nx^2a_n^2+(k-n)(x^2+a_n^2)-2kxa_n+n \ge 0,$$

$$n(xa_n-1)^2+(k-n)(x-a_n)^2 \ge 0.$$

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

(b) The inequality follows from (a) for $k = 2n$.

□

Chapter 3

Symmetric Polynomial Inequalities in Nonnegative Variables

3.1 Applications

3.1. If a, b, c are positive real numbers, then

$$a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab + bc + ca).$$

3.2. Let a, b, c be nonnegative real numbers. If $0 \leq k \leq \sqrt{2}$, then

$$a^2 + b^2 + c^2 + kabc + 2k + 3 \geq (k + 2)(a + b + c).$$

3.3. If a, b, c are positive real numbers, then

$$abc(a + b + c) + 2(a^2 + b^2 + c^2) + 3 \geq 4(ab + bc + ca).$$

3.4. If a, b, c are positive real numbers, then

$$a(b^2 + c^2) + b(c^2 + a^2) + c(a^2 + b^2) + 3 \geq 3(ab + bc + ca).$$

3.5. If a, b, c are positive real numbers, then

$$\left(\frac{a^2 + b^2 + c^2}{3}\right)^3 \geq a^2b^2c^2 + (a - b)^2(b - c)^2(c - a)^2.$$

3.6. If a, b, c are positive real numbers, then

$$(a+b+c-3)(ab+bc+ca-3) \geq 3(abc-1)(a+b+c-ab-bc-ca).$$

3.7. If a, b, c are positive real numbers, then

(a) $$a^3+b^3+c^3+ab+bc+ca+9 \geq 5(a+b+c);$$

(b) $$a^3+b^3+c^3+4(ab+bc+ca)+18 \geq 11(a+b+c).$$

3.8. If a, b, c are positive real numbers, then

(a) $$a^3+b^3+c^3+abc+8 \geq 4(a+b+c);$$

(b) $$4(a^3+b^3+c^3)+15abc+54 \geq 27(a+b+c).$$

3.9. Let a, b, c be nonnegative real numbers such that

$$a+b+c = a^2+b^2+c^2.$$

Prove that

$$ab+bc+ca \geq a^2b^2+b^2c^2+c^2a^2.$$

3.10. If a, b, c are nonnegative real numbers, then

$$(a^2+2bc)(b^2+2ca)(c^2+2ab) \geq (ab+bc+ca)^3.$$

3.11. If a, b, c are nonnegative real numbers, then

$$(2a^2+bc)(2b^2+ca)(2c^2+ab) \geq (ab+bc+ca)^3.$$

3.12. Let a, b, c be nonnegative real numbers such that

$$a+b+c=2.$$

Prove that

(a) $$(a^2+b^2)(b^2+c^2)(c^2+a^2) \leq (a+b)(b+c)(c+a);$$

(b) $$(a^2+b^2)(b^2+c^2)(c^2+a^2) \leq 2.$$

3.13. Let a, b, c be nonnegative real numbers such that

$$a + b + c = 2.$$

Prove that

$$(a^3 + b^3)(b^3 + c^3)(c^3 + a^3) \le 2.$$

3.14. Let a, b, c be nonnegative real numbers such that

$$a^2 + b^2 + c^2 = 2.$$

Prove that

$$(a^3 + b^3)(b^3 + c^3)(c^3 + a^3) \le 2.$$

3.15. If a, b, c are nonnegative real numbers such that

$$a + b + c = 2,$$

then

$$(3a^2 - 2ab + 3b^2)(3b^2 - 2bc + 3c^2)(3c^2 - 2ca + 3a^2) \le 36.$$

3.16. Let a, b, c be nonnegative real numbers such that

$$a + b + c = 3.$$

Prove that

$$(a^2 - 4ab + b^2)(b^2 - 4bc + c^2)(c^2 - 4ca + a^2) \le 3.$$

3.17. If a, b, c are positive real numbers such that

$$a + b + c = 3,$$

then

$$abc + \frac{12}{ab + bc + ca} \ge 5.$$

3.18. If a, b, c are positive real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$5(a + b + c) + \frac{3}{abc} \ge 18.$$

3.19. If a, b, c are positive real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$12 + 9abc \geq 7(ab + bc + ca).$$

3.20. If a, b, c are positive real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$21 + 18abc \geq 13(ab + bc + ca).$$

3.21. If a, b, c are positive real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$(2 - ab)(2 - bc)(2 - ca) \geq 1.$$

3.22. Let a, b, c be positive real numbers such that

$$abc = 1.$$

Prove that

$$\left(\frac{a + b + c}{3}\right)^5 \geq \frac{a^2 + b^2 + c^2}{3}.$$

3.23. If a, b, c are positive real numbers such that

$$abc = 1,$$

then

$$a^3 + b^3 + c^3 + a^{-3} + b^{-3} + c^{-3} + 21 \geq 3(a + b + c)(a^{-1} + b^{-1} + c^{-1}).$$

3.24. If a, b, c are positive real numbers such that

$$abc = 1,$$

then

$$a^2 + b^2 + c^2 - ab - bc - ca \geq \frac{9}{4}(a + b + c - 3).$$

3.25. If a, b, c are positive real numbers such that

$$abc = 1,$$

then

$$a^2 + b^2 + c^2 + a + b + c \geq 2(ab + bc + ca).$$

3.26. If a, b, c are positive real numbers such that

$$abc = 1,$$

then

$$a^2 + b^2 + c^2 + 15(ab + bc + ca) \geq 16(a + b + c).$$

3.27. If a, b, c are positive real numbers such that

$$abc = 1,$$

then

$$\frac{2}{a + b + c} + \frac{1}{3} \geq \frac{3}{ab + bc + ca}.$$

3.28. If a, b, c are positive real numbers such that

$$abc = 1,$$

then

$$ab + bc + ca + \frac{6}{a + b + c} \geq 5.$$

3.29. If a, b, c are positive real numbers such that

$$abc = 1,$$

then

$$\sqrt[3]{(1 + a)(1 + b)(1 + c)} \geq \sqrt[4]{4(1 + a + b + c)}.$$

3.30. If a, b, c are positive real numbers, then

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 18(a^2 - bc)(b^2 - ca)(c^2 - ab).$$

3.31. If a, b, c are positive real numbers such that

$$a + b + c = 3,$$

then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \ge a^2 + b^2 + c^2.$$

3.32. If a, b, c are positive real numbers such that

$$ab + bc + ca = 3,$$

then

$$a^3 + b^3 + c^3 + 7abc \ge 10.$$

3.33. If a, b, c are nonnegative real numbers such that

$$a^3 + b^3 + c^3 = 3,$$

then

$$a^4b^4 + b^4c^4 + c^4a^4 \le 3.$$

3.34. If a, b, c are nonnegative real numbers, then

$$(a+1)^2(b+1)^2(c+1)^2 \ge 4(a+b+c)(ab+bc+ca) + 28abc.$$

3.35. If a, b, c are positive real numbers such that

$$a + b + c = 3,$$

then

$$1 + 8abc \ge 9\min\{a, b, c\}.$$

3.36. If a, b, c are positive real numbers such that

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 + 4abc \ge 5\min\{a, b, c\}.$$

3.37. If a, b, c are positive real numbers such that

$$a+b+c=abc,$$

then

$$(1-a)(1-b)(1-c)+(\sqrt{3}-1)^3 \geq 0.$$

3.38. If a, b, c are nonnegative real numbers such that

$$a+b+c=2,$$

then

$$(a^2+bc)(b^2+ca)(c^2+ab) \leq 1.$$

3.39. If a, b, c are nonnegative real numbers, then

$$(8a^2+bc)(8b^2+ca)(8c^2+ab) \leq (a+b+c)^6.$$

3.40. If a, b, c are positive real numbers such that

$$a^2b^2+b^2c^2+c^2a^2=3,$$

then

$$a+b+c \geq abc+2.$$

3.41. Let a, b, c be nonnegative real numbers such that

$$a+b+c=5.$$

Prove that

$$(a^2+3)(b^2+3)(c^2+3) \geq 192.$$

3.42. If a, b, c are nonnegative real numbers, then

$$a^2+b^2+c^2+abc+2 \geq a+b+c+ab+bc+ca.$$

3.43. If a, b, c are nonnegative real numbers, then

$$\sum a^3(b+c)(a-b)(a-c) \geq 3(a-b)^2(b-c)^2(c-a)^2.$$

3.44. Find the greatest real number k such that

$$a+b+c+4abc \geq k(ab+bc+ca)$$

for all $a,b,c \in [0,1]$.

3.45. If $a,b,c \geq \frac{2}{3}$ such that

$$a+b+c=3,$$

then

$$a^2b^2+b^2c^2+c^2a^2 \geq ab+bc+ca.$$

3.46. If a,b,c are positive real numbers such that

$$a \leq 1 \leq b \leq c, \quad a+b+c=3,$$

then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq a^2+b^2+c^2.$$

3.47. If a,b,c are positive real numbers such that

$$a \leq 1 \leq b \leq c, \quad a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

then

$$a^2+b^2+c^2 \leq \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

3.48. If a,b,c are positive real numbers such that

$$a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

then

$$(abc-1)\left(a^n+b^n+c^n-\frac{1}{a^n}-\frac{1}{b^n}-\frac{1}{c^n}\right) \leq 0$$

for any integer $n \geq 2$.

3.49. Let a, b, c be positive real numbers, and let

$$E(a,b,c) = a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b).$$

Prove that

(a) $$(a+b+c)E(a,b,c) \ge ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2;$$

(b) $$2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)E(a,b,c) \ge (a-b)^2 + (b-c)^2 + (c-a)^2.$$

3.50. Let $a \ge b \ge c$ be nonnegative real numbers. Schur's inequalities of third and fourth degree state that

(a) $$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \ge 0;$$

(b) $$a^2(a-b)(a-c) + b^2(b-c)(b-a) + c^2(c-a)(c-b) \ge 0.$$

Prove that (a) is sharper than (b) if

$$\sqrt{b} + \sqrt{c} \le \sqrt{a},$$

and (b) is sharper than (a) if

$$\sqrt{b} + \sqrt{c} \ge \sqrt{a}.$$

3.51. If a, b, c are nonnegative real numbers such that

$$(a+b)(b+c)(c+a) = 8,$$

then

$$\sqrt{a} + \sqrt{b} + \sqrt{c} \ge ab + bc + ca.$$

3.52. If

$$a, b, c \in [1,\ 4+3\sqrt{2}],$$

then

$$9(ab+bc+ca)(a^2+b^2+c^2) \ge (a+b+c)^4.$$

3.53. If a,b,c are nonnegative real numbers such that

$$a+b+c+abc=4,$$

then

(a) $$a^2+b^2+c^2+12 \ge 5(ab+bc+ca);$$

(b) $$3(a^2+b^2+c^2)+13(ab+bc+ca) \ge 48.$$

3.54. Let a,b,c be the lengths of the sides of a triangle. If

$$a+b+c=3,$$

then

$$a^2b^2+b^2c^2+c^2a^2 \ge ab+bc+ca.$$

3.55. Let a,b,c be the lengths of the sides of a triangle. If

$$a^2+b^2+c^2=3,$$

then

$$ab+bc+ca \ge 1+2abc.$$

3.56. Let a,b,c be the lengths of the sides of a triangle. If

$$a+b+c=3,$$

then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{41}{6} \ge 3(a^2+b^2+c^2).$$

3.57. Let $a \ge b \ge c$ such that

$$a+b+c=p, \qquad ab+bc+ca=q,$$

where p and q are fixed real numbers satisfying $p^2 \ge 3q$.

(a) If a,b,c are nonnegative real numbers, then the product $r=abc$ is minimal only when $a=b$ or $c=0$, and is maximal only when $b=c$;

(b) If a,b,c are the lengths of the sides of a triangle (non-degenerate or degenerate), then the product $r=abc$ is minimal only when $a=b \ge c$, and is maximal only when $b=c \ge \dfrac{a}{2}$ or $b+c=a$.

3.58. Let $a \geq b \geq c > 0$ be positive real numbers such that

$$a + b + c = p, \quad abc = r,$$

where p and r are fixed positive numbers satisfying $p^3 \geq 27r$. Prove that

$$q = ab + bc + ca$$

is minimal only when $b = c$, and is maximal only when $a = b$.

3.59. If a, b, c are positive real numbers such that

$$a + b + c = 3,$$

then

$$\frac{9}{abc} + 16 \geq \frac{75}{ab + bc + ca}.$$

3.60. If a, b, c are positive real numbers such that

$$a + b + c = 3,$$

then

$$8\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + 9 \geq 10(a^2 + b^2 + c^2).$$

3.61. If a, b, c are positive real numbers such that

$$a + b + c = 3,$$

then

$$7(a^2 + b^2 + c^2) + 8(a^2b^2 + b^2c^2 + c^2a^2) + 4a^2b^2c^2 \geq 49.$$

3.62. If a, b, c are nonnegative real numbers, then

$$(a^3 + b^3 + c^3 + abc)^2 \geq 2(a^2 + b^2)(b^2 + c^2)(c^2 + a^2).$$

3.63. If a, b, c are nonnegative real numbers, then

$$[ab(a + b) + bc(b + c) + ca(c + a)]^2 \geq 4(ab + bc + ca)(a^2b^2 + b^2c^2 + c^2a^2).$$

3.64. Let a,b,c be nonnegative real numbers such that

$$ab+bc+ca=3.$$

Prove that

$$4(a^3+b^3+c^3)+7abc+125\geq 48(a+b+c).$$

3.65. If $a,b,c\in[0,1]$, then

(a) $$a\sqrt{a}+b\sqrt{b}+c\sqrt{c}+4abc\geq 2(ab+bc+ca);$$

(b) $$a\sqrt{a}+b\sqrt{b}+c\sqrt{c}\geq \frac{3}{2}(ab+bc+ca-abc);$$

(c) $$3(a\sqrt{a}+b\sqrt{b}+c\sqrt{c})+\frac{500}{81}abc\geq 5(ab+bc+ca).$$

3.66. If

$$a,b,c\geq \frac{13-4\sqrt{10}}{3}\approx 0.117$$

such that $a+b+c=9$, then

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\geq \sqrt{ab+bc+ca}.$$

3.67. Let a,b,c be the lengths of the sides of a triangle. If

$$a^2+b^2+c^2=3,$$

then

$$a+b+c\geq 2+abc.$$

3.68. Let $f_n(a,b,c)$ be a symmetric homogeneous polynomial of degree $n\leq 5$. Prove that

(a) the inequality $f_n(a,b,c)\geq 0$ holds for all nonnegative real numbers a,b,c if and only if $f_n(a,1,1)\geq 0$ and $f_n(0,b,c)\geq 0$ for all nonnegative real numbers a,b,c;

(b) the inequality $f_n(a,b,c)\geq 0$ holds for all the lengths a,b,c of the sides of a non-degenerate or degenerate triangle if and only if $f_n(x,1,1)\geq 0$ for $0\leq x\leq 2$, and $f_n(y+z,y,z)\geq 0$ for all $y,z\geq 0$.

3.69. If a, b, c are nonnegative real numbers such that

$$a + b + c = 3,$$

then

$$4(a^4 + b^4 + c^4) + 45 \geq 19(a^2 + b^2 + c^2).$$

3.70. Let a, b, c be nonnegative real numbers. If $k \leq 2$, then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) \geq 0.$$

3.71. Let a, b, c be nonnegative real numbers. If $k \in \mathbf{R}$, then

$$\sum (b+c)(a-b)(a-c)(a-kb)(a-kc) \geq 0.$$

3.72. If a, b, c are nonnegative real numbers, then

$$\sum a(a-2b)(a-2c)(a-5b)(a-5c) \geq 0.$$

3.73. If a, b, c are the lengths of the sides of a triangle, then

$$a^4 + b^4 + c^4 + 9abc(a+b+c) \leq 10(a^2b^2 + b^2c^2 + c^2a^2).$$

3.74. If a, b, c are the lengths of the sides of a triangle, then

$$3(a^4 + b^4 + c^4) + 7abc(a+b+c) \leq 5\sum ab(a^2 + b^2).$$

3.75. If a, b, c are the lengths of the sides of a triangle, then

$$\frac{b^2 + c^2 - 6bc}{a} + \frac{c^2 + a^2 - 6ca}{b} + \frac{a^2 + b^2 - 6ab}{c} + 4(a+b+c) \leq 0.$$

3.76. Let $f_6(a,b,c)$ be a sixth degree symmetric homogeneous polynomial written in the form

$$f_6(a,b,c) = Ar^2 + B(p,q)r + C(p,q), \quad A \le 0,$$

where

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

Prove that

(a) the inequality $f_6(a,b,c) \ge 0$ holds for all nonnegative real numbers a,b,c if and only if $f_6(a,1,1) \ge 0$ and $f_6(0,b,c) \ge 0$ for all nonnegative real numbers a,b,c;

(b) the inequality $f_6(a,b,c) \ge 0$ holds for all lengths a,b,c of the sides of a non-degenerate or degenerate triangle if and only if $f_6(x,1,1) \ge 0$ for $0 \le x \le 2$, and $f_6(y+z,y,z) \ge 0$ for all $y,z \ge 0$.

3.77. If a,b,c are nonnegative real numbers, then

$$\sum a(b+c)(a-b)(a-c)(a-2b)(a-2c) \ge (a-b)^2(b-c)^2(c-a)^2.$$

3.78. Let a,b,c be nonnegative real numbers.

(a) If $2 \le k \le 6$, then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) + \frac{4(k-2)(a-b)^2(b-c)^2(c-a)^2}{a+b+c} \ge 0;$$

(b) If $k \ge 6$, then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) + \frac{(k+2)^2(a-b)^2(b-c)^2(c-a)^2}{4(a+b+c)} \ge 0.$$

3.79. If a,b,c are nonnegative real numbers, then

$$(3a^2+2ab+3b^2)(3b^2+2bc+3c^2)(3c^2+2ca+3a^2) \ge 8(a^2+3bc)(b^2+3ca)(c^2+3ab).$$

3.80. Let a,b,c be nonnegative real numbers such that

$$a+b+c = 2.$$

If

$$\frac{-2}{3} \le k \le \frac{11}{8},$$

then

$$(a^2+kab+b^2)(b^2+kbc+c^2)(c^2+kca+a^2) \le k+2.$$

3.81. Let a, b, c be nonnegative real numbers such that

$$a+b+c=2.$$

Prove that

$$(2a^2+bc)(2b^2+ca)(2c^2+ab)\le 4.$$

3.82. Let a, b, c be nonnegative real numbers, no two of which are zero. Then,

$$\sum(a-b)(a-c)(a-2b)(a-2c)\ge \frac{5(a-b)^2(b-c)^2(c-a)^2}{ab+bc+ca}.$$

3.83. If a, b, c are positive real numbers such that

$$abc=1,$$

then

$$ab+bc+ca+\frac{50}{a+b+c+5}\ge \frac{37}{4}.$$

3.84. If a, b, c are positive real numbers, then

$$(a+b+c-3)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-3\right)+abc+\frac{1}{abc}\ge 2.$$

3.85. If a, b, c are positive real numbers such that

$$abc=1,$$

then

(a) $$\frac{3}{7}\left(ab+bc+ca-\frac{2}{3}\right)\ge \sqrt{\frac{2}{3}(a+b+c)-1};$$

(b) $$ab+bc+ca-3\ge \frac{46}{27}(\sqrt{a+b+c-2}-1).$$

3.86. Let a,b,c be positive real numbers.

(a) If $abc=2$, then

$$(a+b+c-3)^2+1\geq \frac{a^2+b^2+c^2}{3};$$

(b) If $abc=\frac{1}{2}$, then

$$a^2+b^2+c^2+3(3-a-b-c)^2\geq 3.$$

3.87. If a,b,c are positive real numbers such that

$$a+b+c=3,$$

then

$$4\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)+9abc\geq 21.$$

3.88. If a,b,c are nonnegative real numbers such that

$$ab+bc+ca=abc+2,$$

then

$$a^2+b^2+c^2+abc\geq 4.$$

3.89. If a,b,c are nonnegative real numbers such that $a+b+c=3$, then

$$(a+b)(b+c)(c+a)\geq (a+bc)(b+ca)(c+ab).$$

3.90. Let a,b,c be positive numbers such that

$$a+b+c\leq 3\sqrt[4]{abc}.$$

Prove that

$$a^2+b^2+c^2\leq 3.$$

3.91. If a,b,c are positive real numbers, then

$$\left(\frac{b+c}{a}-2-\sqrt{2}\right)^2+\left(\frac{c+a}{b}-2-\sqrt{2}\right)^2+\left(\frac{a+b}{c}-2-\sqrt{2}\right)^2\geq 6.$$

3.92. If a, b, c are positive real numbers, then

$$2(a^3+b^3+c^3)+9(ab+bc+ca)+39 \geq 24(a+b+c).$$

3.93. If a, b, c are positive real numbers such that $a^2+b^2+c^2=3$, then

$$a^3+b^3+c^3-3 \geq |(a-b)(b-c)(c-a)|.$$

3.94. Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2=3$. Prove that

$$1-abc \geq \frac{5}{3}\min\{(a-b)^2,(b-c)^2,(c-a)^2\}.$$

3.95. If a, b, c are nonnegative real numbers, then

$$a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2 \geq 2|a^3b+b^3c+c^3a-ab^3-bc^3-ca^3|.$$

3.96. If a, b, c are nonnegative real numbers, then

$$a^4+b^4+c^4-abc(a+b+c) \geq 2\sqrt{2}\,|a^3b+b^3c+c^3a-ab^3-bc^3-ca^3|.$$

3.97. If a, b, c are nonnegative real numbers such that $a+b+c=3$, then

$$(a^3b+b^3c+c^3a-3abc)(ab^3+bc^3+ca^3-3abc) \geq (a^2b^2+b^2c^2+c^2a^2-3abc)^2.$$

3.98. If $a, b, c \geq -5$ such that

$$a+b+c=3,$$

then

$$\frac{1-a}{1+a+a^2}+\frac{1-b}{1+b+b^2}+\frac{1-c}{1+c+c^2} \geq 0.$$

3.99. Let $a, b, c \neq \frac{1}{k}$ be nonnegative real numbers such that

$$a+b+c=3.$$

If $k \geq \frac{4}{3}$, then

$$\frac{1-a}{(1-ka)^2}+\frac{1-b}{(1-kb)^2}+\frac{1-c}{(1-kc)^2} \geq 0.$$

3.100. If a,b,c are positive real numbers such that

$$abc = 1,$$

then

$$3(2a^2+1)(2b^2+1)(2c^2+1) \le (a+b+c)^4.$$

3.101. If a,b,c are positive real numbers such that

$$a+b+c=\sqrt{3},$$

then

$$(3\sqrt{3}-5)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{b}\right) \ge a^2+b^2+c^2.$$

3.102. If $a,b,c \ge 1$ such that

$$a+b+c=4,$$

then

$$12\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{b}\right) \ge 5(a^2+b^2+c^2).$$

3.103. If a,b,c are positive real numbers such that

$$a+b+c=3, \qquad c \le \frac{15}{32},$$

then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge a^2+b^2+c^2.$$

3.104. If $a \ge b \ge c \ge 0$ and $ab+bc+ca=3$, then

(a) $\qquad b+c \le 2;$

(b) $\qquad b^2+bc+c^2 \le 3.$

3.105. If $a,b,c \in \left[0, 1+\frac{1}{\sqrt{2}}\right]$ and $a^2+b^2+c^2=3$, then

$$a+b+c \ge abc+2.$$

3.106. Let $a, b, c \geq \frac{1}{6}$ be real numbers such that $a^2 + b^2 + c^2 = 3$. Then,

$$a + b + c \geq abc + 2.$$

3.107. If a, b, c are nonnegative real numbers such that

$$ab + bc + ca + 6abc = 9,$$

then

$$2(a + b + c) \geq ab + bc + ca + 3.$$

3.108. If a, b, c are nonnegative real numbers such that

$$ab + bc + ca + abc = 4,$$

then

$$4(a + b + c) + 3a^2b^2c^2 \geq 15.$$

3.109. If $a, b, c \in \left[0, \frac{5}{3}\right]$ such that $a + b + c = 3$, then

$$(a + b)(b + c)(c + a) \geq 8\sqrt[3]{abc}.$$

3.110. If a, b, c are nonnegative real numbers such that $a \geq b \geq c$ and

$$a^2 + b^2 + c^2 = 3,$$

then

$$a + b + c + \left(1 - \frac{1}{\sqrt{3}}\right)(a - c)^2 \geq 3.$$

3.111. If a, b, c are nonnegative real numbers such that $a \geq b \geq c$ and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \leq \sqrt{\frac{2}{3}}\,(a - c).$$

3.112. If a, b, c are nonnegative real numbers such that $a \ge b \ge c$ and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \le \frac{7}{10}(a-c)^2.$$

3.113. If $a \ge b \ge c \ge \frac{1}{3}$ and $a^2 + b^2 + c^2 = 3$, then

$$1 - abc \le \frac{11}{18}(a-c).$$

3.114. If a, b, c are nonnegative real numbers such that $a \ge b \ge c$ and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - \sqrt{abc} \le \frac{2}{3}(a-c)^2.$$

3.115. If a, b, c are nonnegative real numbers such that $a \ge b \ge c$ and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \le \frac{2}{3}a(a-c)^2.$$

3.116. If a, b, c are nonnegative real numbers such that $a \ge b \ge c$ and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \le \frac{1}{9}(5a+c)(a-c)^2.$$

3.117. If a, b, c are nonnegative real numbers such that $a \ge b \ge c$ and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \ge \frac{2}{3}(b-c)^2.$$

3.118. Let a,b,c be nonnegative real numbers such that

$$a^2+b^2+c^2=3.$$

If $a\ge b\ge c$, then

$$1-abc \ \ge\ \frac{2}{3}(1+\sqrt{2})(a-b)(b-c).$$

3.119. If a,b,c are nonnegative real numbers such that $a\ge b\ge c$ and

$$a^2+b^2+c^2=3,$$

then

(a) $$1-abc \ \ge\ 2b(a-b)(b-c);$$

(b) $$1-abc \ \ge\ (a-c)(a-b)(b-c);$$

(c) $$1-abc \ \ge\ a(a-b)(b-c);$$

(d) $$1-abc \ \ge\ (a+c)(a-b)(b-c).$$

3.120. If a,b,c are nonnegative real numbers such that $a\ge b\ge c$ and

$$a^2+b^2+c^2=3,$$

then

(a) $$1-abc \ \ge\ \frac{2}{3}\,b(a-b)^2;$$

(b) $$1-abc \ \ge\ \frac{2}{27}\,(2a+7b)(a-b)^2.$$

3.121. If a,b,c are nonnegative real numbers such that $a\ge b\ge c$ and

$$a^2+b^2+c^2=3,$$

then

(a) $$1-abc\ge\frac{1}{3}\,(b+c)(b-c)^2;$$

(b) $$1-abc\ge\frac{2}{27}\,(7b+2c)(b-c)^2.$$

3.122. If a, b, c are nonnegative real numbers such that $a^2 + b^2 + c^2 = 3$, then

(a) $$1 - \sqrt{abc} \ \geq \ (a-b)(b-c);$$

(b) $$1 - \sqrt[3]{a^2b^2c^2} \ \geq \ \frac{4}{3}(a-b)(b-c).$$

3.123. If a, b, c are nonnegative real numbers such that $a \geq b \geq c$ and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \ \geq \ \frac{2}{3}b(\sqrt{a} - \sqrt{c})^2.$$

3.124. If a, b, c are nonnegative real numbers such that $a \geq b \geq c$ and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \ \geq \ \frac{8}{3}b(\sqrt{a} - \sqrt{b})^2.$$

3.125. If a, b, c are nonnegative real numbers such that $a \geq b \geq c$ and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \ \geq \ \frac{1}{3}(a+3c)(\sqrt{a} - \sqrt{c})^2.$$

3.126. If a, b, c are nonnegative real numbers such that $a \geq b \geq c$ and

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \ \geq \ \frac{2}{3}(a+3c)(\sqrt{b} - \sqrt{c})^2.$$

3.127. Let

$$F(a,b,c) = 3(a^2+b^2+c^2) - (a+b+c)^2,$$

where a,b,c are positive real numbers such that $a \le b \le c$ and

$$a^2(b^2+c^2) \ge 2.$$

Then,

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

3.128. Let

$$F(a,b,c) = a+b+c-3\sqrt[3]{abc},$$

where a,b,c are positive real numbers. If

$$\min\{a,b,c\} \ge \frac{1}{abc},$$

then

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

3.129. Let

$$F(a,b,c) = a+b+c-3\sqrt[3]{abc},$$

where a,b,c are positive real numbers such that $a \le b \le c$ and

$$a(b+c) \ge 2.$$

Then,

$$F(a,b,c) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

3.130. Let

$$F(a,b,c,d) = a+b+c+d-4\sqrt[4]{abcd},$$

where a,b,c,d are positive real numbers. If

$$\min\{a^2,b^2,c^2,d^2\} \ge \frac{1}{abcd},$$

then

$$F(a,b,c,d) \ge F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}\right).$$

3.131. Let a,b,c,d be nonnegative real numbers such that

$$a^2+b^2+c^2+d^2=1.$$

Prove that

$$(1-a)(1-b)(1-c)(1-d) \ \geq \ abcd.$$

3.132. Let a,b,c,d and x be positive real numbers such that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}=\frac{4}{x^2}.$$

If $x \geq 2$, then

$$(a-1)(b-1)(c-1)(d-1) \ \geq \ (x-1)^4.$$

3.133. If a,b,c,d are positive real numbers, then

$$\frac{(1+a^3)(1+b^3)(1+c^3)(1+d^3)}{(1+a^2)(1+b^2)(1+c^2)(1+d^2)} \ \geq \ \frac{1+abcd}{2}.$$

3.134. Let a,b,c,d be positive real numbers such that

$$a+b+c+d=4.$$

Prove that

$$\left(a+\frac{1}{a}-1\right)\left(b+\frac{1}{b}-1\right)\left(c+\frac{1}{c}-1\right)\left(d+\frac{1}{d}-1\right)+3 \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

3.135. If a,b,c,d are nonnegative real numbers, then

$$4(a^3+b^3+c^3+d^3)+15(abc+bcd+cda+dab) \geq (a+b+c+d)^3.$$

3.136. Let a,b,c,d be positive real numbers such that

$$a+b+c+d=4.$$

Prove that

$$1+2(abc+bcd+cda+dab) \geq 9\min\{a,b,c,d\}.$$

3.137. Let a,b,c,d be nonnegative real numbers such that

$$a+b+c+d=4.$$

Prove that

$$5(a^2+b^2+c^2+d^2)\geq a^3+b^3+c^3+d^3+16.$$

3.138. Let a,b,c,d be nonnegative real numbers such that

$$a+b+c+d=4.$$

Prove that

$$3(a^2+b^2+c^2+d^2)+4abcd\geq 16.$$

3.139. Let a,b,c,d be nonnegative real numbers such that

$$a+b+c+d=4.$$

Prove that

$$27(abc+cd+cda+dab)\leq 44abcd+64.$$

3.140. Let a,b,c,d be positive real numbers such that

$$a+b+c+d=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

Prove that

$$(1-abcd)\left(a^2+b^2+c^2+d^2-\frac{1}{a^2}-\frac{1}{b^2}-\frac{1}{c^2}-\frac{1}{d^2}\right)\geq 0.$$

3.141. Let a,b,c,d be positive real numbers such that

$$a+b+c+d=1.$$

Prove that

$$(1-a)(1-b)(1-c)(1-d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\geq\frac{81}{16}.$$

3.142. Let a,b,c,d be nonnegative real numbers such that

$$a+b+c+d=a^3+b^3+c^3+d^3=2.$$

Prove that

$$a^2+b^2+c^2+d^2\ge\frac{7}{4}.$$

3.143. Let $a,b,c,d\in(0,4]$ such that

$$abcd=1.$$

Prove that

$$(1+2a)(1+2b)(1+2c)(1+2d)\ge(5-2a)(5-2b)(5-2c)(5-2d).$$

3.144. If $a,b,c,d\in\left[0,1+\frac{1}{\sqrt{6}}\right]$ and $a^2+b^2+c^2+d^2=4$, then

$$a+b+c+d\ge abcd+3.$$

3.145. Let a,b,c,d be positive real numbers such that

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\le(1+\sqrt{10}\,)^2.$$

Prove that any three of a,b,c,d are the lengths of the sides of a triangle (non-degenerate or degenerate).

3.146. Let a,b,c,d be positive real numbers such that

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\le\frac{119}{6}.$$

Prove that there exist three of a,b,c,d which are the lengths of the sides of a triangle (non-degenerate or degenerate).

3.147. Let a,b,c,d be positive real numbers such that

$$3(a+b+c+d)^2\ge 11(a^2+b^2+c^2+d^2).$$

Prove that any three of a,b,c,d are the lengths of the sides of a triangle (non-degenerate or degenerate).

3.148. Let a, b, c, d be positive real numbers such that

$$15(a+b+c+d)^2 \geq 49(a^2+b^2+c^2+d^2).$$

Prove that there exist three of a, b, c, d which are the lengths of the sides of a triangle (non-degenerate or degenerate).

3.149. Let a, b, c, d be nonnegative real numbers such that

$$a^2+b^2+c^2+d^2=4.$$

If $a \geq b \geq c \geq d$, then

$$a+b+c+d+(2-\sqrt{2})(a-d)^2 \geq 4.$$

3.150. Let a, b, c, d be nonnegative real numbers such that

$$a^2+b^2+c^2+d^2=4.$$

If $a \geq b \geq c \geq d$, then

$$1-abcd \leq \frac{\sqrt{3}}{2}(a-d).$$

3.151. Let a, b, c, d be nonnegative real numbers such that

$$a^2+b^2+c^2+d^2=4.$$

If $a \geq b \geq c \geq d$, then

$$1-\sqrt{abcd} \leq \frac{3}{4}(a-d)^2.$$

3.152. If a, b, c, d are nonnegative real numbers such that $a \geq b \geq c \geq d$ and

$$a^2+b^2+c^2+d^2=4,$$

then

$$1-(abcd)^{3/4} \leq \frac{3}{4}(a-d)^2.$$

3.153. Let a,b,c,d be nonnegative real numbers such that

$$a^2+b^2+c^2+d^2=4.$$

If $a \ge b \ge c \ge d$, then

(a) $$1-\sqrt{abcd} \ \ge \ \frac{1}{2}(b-c)^2;$$

(b) $$1-\sqrt{abcd} \ \ge \ \frac{1}{4}(a-d)^2.$$

3.154. Let a,b,c,d be nonnegative real numbers such that

$$a^2+b^2+c^2+d^2=4.$$

If $a \ge b \ge c \ge d$, then

$$1-abcd \ge \frac{3}{4}(c-d)^2.$$

3.155. Let a,b,c,d be nonnegative real numbers such that

$$a^2+b^2+c^2+d^2=4.$$

If $a \ge b \ge c \ge d$, then

(a) $$1-abcd \ \ge \ (a-b)(c-d);$$

(b) $$1-abcd \ \ge \ \frac{1+\sqrt{3}}{2}\,(a-b)(c-d).$$

3.156. Let a,b,c,d be nonnegative real numbers such that

$$a^2+b^2+c^2+d^2=4.$$

If $a \ge b \ge c \ge d$, then

$$1-abcd \ \ge \ 3(a-b)(b-c)(c-d)(a-d).$$

3.157. Let a,b,c,d be nonnegative real numbers such that

$$a^2+b^2+c^2+d^2=4.$$

If $a \ge b \ge c \ge d$, then

(a) $$1-\sqrt{abcd} \ \ge \ \frac{1}{3}(b-d)^2;$$

(b) $$1-(abcd)^{3/4} \ \ge \ \frac{1}{2}(b-d)^2.$$

3.158. Let a,b,c,d be nonnegative real numbers such that

$$a^4+b^4+c^4+d^4=4.$$

If $a\ge b\ge c\ge d$, then

(a) $$1-\sqrt{abcd}\ \ge\ \frac{1}{2}(ac-bd)^2;$$

(b) $$1-abcd\ \ge\ \frac{1}{\sqrt{2}}\,(ac-bd)^2.$$

3.159. If a,b,c,d are nonnegative real numbers such that $a\ge b\ge c\ge d$ and

$$a^4+b^4+c^4+d^4=4,$$

then

$$1-abcd\ \ge\ \frac{3}{4}(ad-bc)^2.$$

3.160. If a,b,c,d are nonnegative real numbers such that $a\ge b\ge c\ge d$ and

$$a+b+c+d=4,$$

then

(a) $$\frac{a^4+b^4+c^4+d^4}{4}-abcd\ \ge 2(b-c)^2,$$

(b) $$\frac{a^4+b^4+c^4+d^4}{4}-abcd\ \ge \frac{3}{2}(a-b)^2;$$

(c) $$\frac{a^4+b^4+c^4+d^4}{4}-abcd\ \ge \frac{4}{3}(a-c)^2;$$

(d) $$\frac{a^4+b^4+c^4+d^4}{4}-abcd\ \ge \frac{4}{3}(c-d)^2.$$

3.161. If a,b,c,d are nonnegative real numbers such that $a\ge b\ge c\ge d$ and $a+d\ge b+c$, then

$$a+b+c+d-4\sqrt[4]{abcd}\ \le 2\left(\sqrt{a}-\sqrt{d}\right)^2.$$

3.162. If a,b,c,d are nonnegative real numbers such that $a \ge b \ge c \ge d$ and

$$a+kd \ge b+c, \qquad k=(1+\sqrt{2}\,)^4 \approx 33.970,$$

then

$$a+b+c+d-4\sqrt[4]{abcd} \ \le 2\left(\sqrt{a}-\sqrt{d}\right)^2.$$

3.163. If a,b,c,d are nonnegative real numbers such that $a \ge b \ge c \ge d$ and

$$a+d \ge 2c,$$

then

$$a+b+c+d-4\sqrt[4]{abcd} \ \le \frac{5}{2}\left(\sqrt{a}-\sqrt{d}\right)^2.$$

3.164. If a,b,c,d are nonnegative real numbers such that $a \ge b \ge c \ge d$ and

$$a+kd \ge 2c, \qquad k=(3+2\sqrt{3}\,)^4 \approx 1745.95,$$

then

$$a+b+c+d-4\sqrt[4]{abcd} \ \le \frac{5}{2}\left(\sqrt{a}-\sqrt{d}\right)^2.$$

3.165. If a,b,c,d are nonnegative real numbers such that $a \ge b \ge c \ge d$ and

$$a+b+c+d=4,$$

then

$$(a-d)^2 \ \le \frac{a^4+b^4+c^4+d^4}{4}-abcd \ \le 4(a-d)^2.$$

3.166. Let a,b,c,d,e be nonnegative real numbers.

(a) If $a+b+c=3(d+e)$, then

$$4(a^4+b^4+c^4+d^4+e^4) \ge (a^2+b^2+c^2+d^2+e^2)^2;$$

(b) If $a+b+c=d+e$, then

$$12(a^4+b^4+c^4+d^4+e^4) \le 7(a^2+b^2+c^2+d^2+e^2)^2.$$

3.167. Let a,b,c,d,e be nonnegative real numbers such that

$$a+b+c+d+e=5.$$

Prove that

$$a^4+b^4+c^4+d^4+e^4+150 \le 31(a^2+b^2+c^2+d^2+e^2).$$

3.168. Let a,b,c,d,e be positive real numbers such that

$$a^2+b^2+c^2+d^2+e^2=5.$$

Prove that

$$abcde(a^4+b^4+c^4+d^4+e^4) \le 5.$$

3.169. Let a,b,c,d,e be positive real numbers such that

$$a+b+c+d+e=5.$$

Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{20}{a^2+b^2+c^2+d^2+e^2} \ge 9.$$

3.170. If $a,b,c,d,e \ge 1$, then

$$\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)\left(d+\frac{1}{d}\right)\left(e+\frac{1}{e}\right)+68 \ge$$

$$\ge 4(a+b+c+d+e)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right).$$

3.171. If a,b,c and x,y,z are nonnegative real numbers such that

$$x^3+y^3+z^3=a^3+b^3+c^3,$$

then

$$(a+b+c)(x+y+z) \ge ab+bc+ca+xy+yz+zx.$$

3.172. Let a,b,c,d,e,f be nonnegative real numbers such that

$$a^2+b^2+c^2+d^2+e^2+f^2=6.$$

If $a \ge b \ge c \ge d \ge e \ge f$, then

$$1-abcdef \le \frac{3}{2}(a-f)^2.$$

3.173. Let $a_1,a_2,\dots,a_n$ and $b_1,b_2,\dots,b_n$be nonnegative real numbers such that

$$a_1^2+a_2^2+\cdots+a_n^2=b_1^2+b_2^2+\cdots+b_n^2.$$

Then, for $n=3$ and $n=4$, the following inequalities holds:

$$(n-1)(a_1+a_2+\cdots+a_n)(b_1+b_2+\cdots+b_n) \ge n\left(\sum_{i<j}a_ia_j+\sum_{i<j}b_ib_j\right).$$

3.174. Let a, b, c and x, y, z be positive real numbers such that

$$(a+b+c)(x+y+z)=(a^2+b^2+c^2)(x^2+y^2+z^2)=4.$$

Prove that

$$abcxyz < \frac{1}{36}.$$

3.175. Let a_1, a_2, $\cdots$, a_n $(n \ge 3)$ be positive real numbers such that

$$a_1+a_2+\cdots+a_n=a_1^2+a_2^2+\cdots+a_n^2=n-1.$$

Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n} \ge \frac{n^2(2n-3)}{2(n-1)(n-2)}.$$

3.176. Let a_1, a_2, $\cdots$, a_n be positive real numbers such that

$$a_1+a_2+\cdots+a_n=n.$$

Prove that

$$n^2\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n\right) \ge 4(n-1)(a_1^2+a_2^2+\cdots+a_n^2-n).$$

3.177. If $a_1, a_2, \ldots, a_n$ are positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = n, \qquad a_2, a_3, \ldots, a_n \geq 1,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq a_1^2 + a_2^2 + \cdots + a_n^2.$$

3.178. Let a_1, a_2, $\cdots$, a_n be nonnegative real numbers such that

$$a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$(n+1)(a_1^2 + a_2^2 + \cdots + a_n^2) \geq n^2 + a_1^3 + a_2^3 + \cdots + a_n^3.$$

3.179. Let a_1, a_2, $\cdots$, a_n be nonnegative real numbers such that

$$a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$(n-1)(a_1^3 + a_2^3 + \cdots + a_n^3) + n^2 \geq (2n-1)(a_1^2 + a_2^2 + \cdots + a_n^2).$$

3.180. Let a_1, a_2, $\ldots$, a_n $(n \geq 3)$ be positive real numbers such that

$$a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \geq \frac{n}{n-1}(1 - a_1 a_2 \cdots a_n).$$

3.181. If $a_1, a_2, \ldots, a_n$ are positive numbers such that $a_1 + a_2 + \cdots + a_n = n$ and

$$a_1 a_2 \cdots a_n \leq \frac{1}{(n-1)^{n-2}},$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq a_1^2 + a_2^2 + \cdots + a_n^2.$$

3.182. If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that

$$a_1 \le a_2 \le \cdots \le a_n,$$

then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} - \sqrt[n]{a_1 a_2 \cdots a_n} \le \left(1 - \frac{1}{n}\right)\left(\sqrt{a_n} - \sqrt{a_1}\right)^2.$$

3.183. Let a_1, a_2, ..., a_n $(n \ge 3)$ be positive real numbers such that

$$a_1 \le a_2 \le \cdots \le a_n,$$

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) = k.$$

(a) If $n^2 < k \le n^2 + \dfrac{i(n-i)}{2}$, $i \in \{2, 3, \cdots, n-1\}$, then a_{i-1}, a_i and a_{i+1} are the lengths of the sides of a non-degenerate or degenerate triangle;

(b) If $n^2 < k \le \alpha_n$, where $\alpha_n = \dfrac{9n^2}{8}$ for even n, and $\alpha_n = \dfrac{9n^2 - 1}{8}$ for odd n, then there exist three numbers a_i which are the lengths of the sides of a non-degenerate or degenerate triangle.

3.184. Let a_1, a_2, ..., a_n $(n \ge 3)$ be positive real numbers such that

$$a_1 \le a_2 \le \cdots \le a_n,$$

$$(a_1 + a_2 + \cdots + a_n)^2 = k(a_1^2 + a_2^2 + \cdots + a_n^2).$$

(a) If $\dfrac{(2n-i)^2}{4n-3i} \le k < n$, $i \in \{2, 3, \cdots, n-1\}$, then a_{i-1}, a_i and a_{i+1} are the lengths of the sides of a non-degenerate or degenerate triangle;

(b) If $\dfrac{8n+1}{9} \le k < n$, then there exist three numbers a_i which are the lengths of the sides of a non-degenerate or degenerate triangle.

3.2 Solutions

P 3.1. *If a, b, c are positive real numbers, then*

$$a^2 + b^2 + c^2 + 2abc + 1 \geq 2(ab + bc + ca).$$

(Darij Grinberg, 2004)

First Solution. Setting

$$a = x^3, \quad b = y^3, \quad c = z^3, \qquad x, y, z > 0,$$

we need to prove that

$$x^6 + y^6 + z^6 + 2x^3y^3z^3 + 1 \geq 2(x^3y^3 + y^3z^3 + z^3x^3).$$

Using Schur's inequality and then the AM-GM inequality, we have

$$x^6 + y^6 + z^6 + 3x^2y^2z^2 \geq \sum x^2y^2(x^2 + y^2) \geq 2\sum x^3y^3.$$

Thus, it suffices to show that

$$2x^3y^3z^3 - 3x^2y^2z^2 + 1 \geq 0,$$

which is equivalent to

$$(xyz - 1)^2(2xyz + 1) \geq 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. Among the numbers $1 - a$, $1 - b$ and $1 - c$ there are always two with the same sign; let

$$(1 - b)(1 - c) \geq 0.$$

We have

$$\begin{aligned} &a^2 + b^2 + c^2 + 2abc + 1 - 2(ab + bc + ca) = \\ &= (a - 1)^2 + (b - c)^2 + 2a + 2abc - 2a(b + c) \\ &= (a - 1)^2 + (b - c)^2 + 2a(1 - b)(1 - c) \geq 0. \end{aligned}$$

Remark. The following generalization holds:

- *Let a, b, c be positive real numbers. If $0 \leq k \leq 1$, then*

$$a^2 + b^2 + c^2 + 2kabc + k \geq (k + 1)(ab + bc + ca).$$

Since the both sides of the inequality are linear of k, it suffices to prove it for only $k = 0$ and $k = 1$. For $k = 0$, the inequality reduces to

$$a^2 + b^2 + c^2 \geq ab + bc + ca,$$

which is well-known.

□

P 3.2. *Let a, b, c be nonnegative real numbers. If $0 \le k \le \sqrt{2}$, then*

$$a^2 + b^2 + c^2 + kabc + 2k + 3 \ge (k+2)(a+b+c).$$

Solution. Since the both sides of the inequality are linear of k, it suffices to prove the inequality for $k = 0$ and $k = \sqrt{2}$. For $k = 0$, the inequality reduces to

$$(a-1)^2 + (b-1)^2 + (c-1)^2 \ge 0.$$

Consider further that $k = \sqrt{2}$, and write the inequality as

$$(a-1)^2 + (b-1)^2 + (c-1)^2 \ge \sqrt{2}\,(a+b+c-2-abc).$$

Using the substitution

$$x = a-1, \quad y = b-1, \quad z = c-1,$$

we need to show that

$$x^2 + y^2 + z^2 + \sqrt{2}\,(xyz + xy + yz + zx) \ge 0$$

for $x, y, z \ge -1$. Among the numbers x, y and z there are always two of them with the same sign; let us say

$$yz \ge 0.$$

Since

$$y^2 + z^2 \ge \frac{1}{2}(y+z)^2$$

and

$$xyz + xy + yz + zx = (x+1)yz + x(y+z) \ge x(y+z),$$

it suffices to prove that

$$x^2 + \frac{1}{2}(y+z)^2 + \sqrt{2}\,x(y+z) \ge 0,$$

which is equivalent to

$$\left[x + \frac{1}{\sqrt{2}}(y+z)\right]^2 \ge 0.$$

Thus, the proof is completed. The equality holds for $a = b = c = 1$. In addition, if $k = \sqrt{2}$, then the equality holds also for $a = 0$ and $b = c = 1 + 1/\sqrt{2}$ (or any cyclic permutation).

□

P 3.3. *If a, b, c are positive real numbers, then*

$$abc(a+b+c) + 2(a^2 + b^2 + c^2) + 3 \ge 4(ab + bc + ca).$$

First Solution. Applying the AM-GM inequality two times, we get

$$abc(a+b+c)+3 \ge 2\sqrt{3abc(a+b+c)} \ge \frac{18abc}{a+b+c}.$$

Therefore, it suffices to prove that

$$a^2+b^2+c^2+\frac{18abc}{a+b+c} \ge 2(ab+bc+ca),$$

which is just Schur's inequality of third degree. The equality holds for $a=b=c=1$.

Second Solution. Applying the AM-GM, we get

$$\begin{aligned} abc(a+b+c)+3 &= (a^2bc+1)+(ab^2c+1)+(abc^2+1) \\ &\ge 2a\sqrt{bc}+2b\sqrt{ca}+2c\sqrt{ab}. \end{aligned}$$

Thus, it suffices to prove that

$$a^2+b^2+c^2+a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab} \ge 2(ab+bc+ca).$$

Substituting

$$x=\sqrt{a}, \quad y=\sqrt{b}, \quad z=\sqrt{c},$$

we need to show that

$$x^4+y^4+z^4+xyz(x+y+z) \ge 2(x^2y^2+y^2z^2+z^2x^2).$$

This inequality can be obtained by summing Schur's inequality of degree four

$$x^4+y^4+z^4+xyz(x+y+z) \ge xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2)$$

to

$$xy(x^2+y^2)+yz(y^2+z^2)+zx(z^2+x^2) \ge 2(x^2y^2+y^2z^2+z^2x^2).$$

The last inequality is equivalent to

$$xy(x-y)^2+yz(y-z)^2+zx(z-x)^2 \ge 0.$$

□

P 3.4. *If a, b, c are positive real numbers, then*

$$a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3 \ge 3(ab+bc+ca).$$

Solution. Write the inequality as follows

$$(a+b+c)(ab+bc+ca)+3 \geq 3(abc+ab+bc+ca).$$

$$(a+b+c-3)(ab+bc+ca)+3 \geq 3abc.$$

Using the known inequality

$$(a+b+c)(ab+bc+ca) \geq 9abc,$$

it suffices to show that

$$3(a+b+c-3)(ab+bc+ca)+9 \geq (a+b+c)(ab+bc+ca),$$

which is equivalent to

$$[2(a+b+c)-9](ab+bc+ca)+9 \geq 0.$$

For the nontrivial case $2(a+b+c)-9<0$, using the known inequality

$$(a+b+c)^2 \geq 3(ab+bc+ca),$$

it is enough to show that

$$[2(a+b+c)-9](a+b+c)^2+27 \geq 0.$$

This inequality is equivalent to the obvious inequality

$$(a+b+c-3)^2[2(a+b+c)+3] \geq 0.$$

The equality holds for $a=b=c=1$.

□

P 3.5. *If a,b,c are positive real numbers, then*

$$\left(\frac{a^2+b^2+c^2}{3}\right)^3 \geq a^2b^2c^2+(a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2011)

Solution *(by Vo Quoc Ba Can)*. Assume that

$$a=\min\{a,b,c\}.$$

By virtue of the AM-GM inequality, we have

$$\begin{aligned}\left(\frac{a^2+b^2+c^2}{3}\right)^3 &= \left[\frac{(a^2+b^2+c^2-2bc)+bc+bc}{3}\right]^3 \\ &\geq (a^2+b^2+c^2-2bc)b^2c^2 \\ &= a^2b^2c^2+(b-c)^2b^2c^2.\end{aligned}$$

Thus, it suffices to prove that

$$(b-c)^2b^2c^2 \geq (b-c)^2(b-a)^2(c-a)^2.$$

This is obvious, because

$$b^2 > (b-a)^2, \quad c^2 > (c-a)^2.$$

The equality occurs for $a = b = c$.

□

P 3.6. *If a, b, c are nonnegative real numbers, then*

$$[ab(a+b)+bc(b+c)+ca(c+a)]^2 \geq 4(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2).$$

(Vasile Cîrtoaje and Vo Quoc Ba Can, 2011)

First Solution. Assume that $a \geq b \geq c$. For the nontrivial case $b > 0$, by the AM-GM inequality, we have

$$4(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2) \leq \left[b(ab+bc+ca)+\frac{a^2b^2+b^2c^2+c^2a^2}{b}\right]^2.$$

Thus, it suffices to prove that

$$ab(a+b)+bc(b+c)+ca(c+a) \geq b(ab+bc+ca)+\frac{a^2b^2+b^2c^2+c^2a^2}{b}.$$

This inequality reduces to the obvious form

$$ac(a-b)(b-c) \geq 0.$$

The equality holds for $a = b = c$, for $b = c = 0$ (or any cyclic permutation), and for $a = 0$ and $b = c$ (or any cyclic permutation).

Second Solution. We will prove the stronger inequality

$$[ab(a+b)+bc(b+c)+ca(c+a)]^2 \geq 4(ab+bc+ca)(a^2b^2+b^2c^2+c^2a^2)+A,$$

where

$$A = (a-b)^2(b-c)^2(c-a)^2.$$

Let

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$

Since

$$(a-b)^2(b-c)^2(c-a)^2 = -27r^2+2(9pq-2p^3)r+p^2q^2-4q^3,$$

we can write this inequality as

$$(pq-3r)^2 \ge 4q(q^2-2pr)-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3,$$

which reduces to

$$r(p^3+9r-4pq) \ge 0.$$

This is true since

$$p^3+9r-4pq \ge 0$$

is just the third degree Schur's inequality.

□

P 3.7. *If a,b,c are positive real numbers, then*

(a) $$a^3+b^3+c^3+ab+bc+ca+9 \ge 5(a+b+c);$$

(b) $$a^3+b^3+c^3+4(ab+bc+ca)+18 \ge 11(a+b+c).$$

(Vasile Cîrtoaje, 2010)

Solution. Let

$$p=a+b+c, \quad q=ab+bc+ca.$$

From

$$a(a-1)^2+b(b-1)^2+c(c-1)^2 \ge 0,$$

we get

$$a^3+b^3+c^3 \ge 2(a^2+b^2+c^2)-a-b-c=2p^2-p-4q.$$

(a) Using the result above and the known inequality $p^2 \ge 3q$, we have

$$\begin{aligned} a^3+b^3+c^3+ab+bc+ca+9-5(a+b+c) &\ge \\ &\ge (2p^2-p-4q)+q+9-5p \\ &= 2p^2-6p+9-3q \\ &\ge 2p^2-6p+9-p^2 \\ &= (p-3)^2 \ge 0. \end{aligned}$$

The equality holds for $a=b=c=1$.

(b) Using the result above, we have

$$\begin{aligned} a^3+b^3+c^3+4(ab+bc+ca)+18-11(a+b+2 &\ge \\ &\ge (2p^2-p-4q)+4q+18-11p \\ &= 2(p-3)^3 \ge 0. \end{aligned}$$

The equality holds for $a=b=c=1$.

□

P 3.8. *If a,b,c are positive real numbers, then*

(a) $$a^3+b^3+c^3+abc+8 \ge 4(a+b+c);$$

(b) $$4(a^3+b^3+c^3)+15abc+54 \ge 27(a+b+c).$$

Solution. Let

$$p=a+b+c, \quad q=ab+bc+ca.$$

By Schur's inequality of third degree, we have

$$p^3+9abc \ge 4pq,$$

$$abc \ge \frac{p(4q-p^2)}{9}.$$

a) We get

$$\begin{aligned} a^3+b^3+c^3+abc &= 4abc+p(p^2-3q) \\ &\ge \frac{4p(4q-p^2)}{9}+p(p^2-3q) \\ &= \frac{p(5p^2-11q)}{9}. \end{aligned}$$

Then, it suffices to prove that

$$\frac{p(5p^2-11q)}{9}+8 \ge 4p,$$

which is equivalent to

$$5p^3-36p+72 \ge 11pq.$$

Since $p^2 \ge 3q$, we have

$$\begin{aligned} 3(5p^3-36p+72-11pq) &\ge 3(5p^3-36p+72)-11p^2 \\ &= 4(p^3-27p+54) \\ &= 4(p-3)^2(p+6) \ge 0. \end{aligned}$$

The equality holds for $a=b=c=1$.

(b) We get

$$\begin{aligned} 4(a^3+b^3+c^3)+15abc &= 27abc+4p(p^2-3q) \\ &\ge 3p(4q-p^2)+4p(p^2-3q)=p^3. \end{aligned}$$

Then, it suffices to prove that

$$p^3+54 \ge 27p,$$

which is equivalent to the obvious inequality

$$(p-3)^2(p+6) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization (*Vasile Cîrtoaje*, 2010):

- *Let a, b, c be nonnegative real numbers. If $0 \leq k \leq 27/4$, then*

$$a^3 + b^3 + c^3 + (k-3)abc + 2k \geq k(a+b+c).$$

□

P 3.9. *Let a, b, c be nonnegative real numbers such that*

$$a+b+c = a^2+b^2+c^2.$$

Prove that

$$ab+bc+ca \geq a^2b^2+b^2c^2+c^2a^2.$$

(Vasile Cîrtoaje, 2006)

Solution *(by Michael Rozenberg)*. From the hypothesis condition, by squaring, we get

$$a^4+b^4+c^4-a^2-b^2-c^2 = 2(ab+bc+ca-a^2b^2-b^2c^2-c^2a^2).$$

Therefore, we can write the required inequality as

$$a^4+b^4+c^4 \geq a^2+b^2+c^2.$$

This inequality has the homogeneous form

$$(a+b+c)^2(a^4+b^4+c^4) \geq (a^2+b^2+c^2)^3,$$

which follows immediately from Hölder's inequality. The equality holds for $a = b = c = 1$, for $a = b = c = 0$, for $(a, b, c) = (0, 1, 1)$ (or any cyclic permutation), and for $(a, b, c) = (1, 0, 0)$ (or any cyclic permutation).

□

P 3.10. *If a, b, c are nonnegative real numbers, then*

$$(a^2+2bc)(b^2+2ca)(c^2+2ab) \geq (ab+bc+ca)^3.$$

(Vasile Cîrtoaje, 2006)

Solution. We have

$$(a^2+2bc)(b^2+2ca)(c^2+2ab)=9a^2b^2c^2+2\sum a^3b^3+4abc\sum a^3$$

and

$$(ab+bc+ca)^3=6a^2b^2c^2+\sum a^3b^3+3abc\sum ab(a+b).$$

So, we can rewrite the inequality as

$$3a^2b^2c^2+\sum a^3b^3+4abc\sum a^3\ge 3abc\sum ab(a+b).$$

Since $\sum a^3b^3\ge 3a^2b^2c^2$ (by the AM-GM inequality), it suffices to prove that

$$6abc+4\sum a^3\ge 3\sum ab(a+b).$$

We can get this inequality by summing the inequalities

$$\frac{1}{3}\sum a^3\ge abc$$

and

$$3abc+\sum a^3\ge \sum ab(a+b).$$

The first inequality follows from the AM-GM inequality, while the second is just the third degree Schur's inequality. The equality holds when $a=b=c$, and also when two of a,b,c are zero.

Remark. Similarly, we can also prove the following inequality

$$(2a^2+7bc)(2b^2+7ca)(2c^2+7ab)\ge 27(ab+bc+ca)^3.$$

□

P 3.11. *If a,b,c are nonnegative real numbers, then*

$$(2a^2+bc)(2b^2+ca)(2c^2+ab)\ge (ab+bc+ca)^3.$$

(Vasile Cîrtoaje, 2006)

First Solution. Since

$$(2a^2+bc)(2b^2+ca)(2c^2+ab)=9a^2b^2c^2+4\sum a^3b^3+2abc\sum a^3$$

and

$$(ab+bc+ca)^3=6a^2b^2c^2+\sum a^3b^3+3abc\sum ab(a+b),$$

the inequality is equivalent to

$$3a^2b^2c^2+3\sum a^3b^3+2abc\sum a^3\ge 3abc\sum ab(a+b).$$

We can get this inequality by summing

$$\frac{2}{3}abc\sum a^3 \ge 2a^2b^2c^2$$

and

$$\sum a^3b^3 + 3a^2b^2c^2 \ge abc\sum ab(a+b).$$

The first inequality follows from the AM-GM inequality, while the second is just the third degree Schur's inequality applied to the numbers ab, bc and ca. The equality holds when $a=b=c$, and also when two of a,b,c are zero.

Second Solution. By Hölder's inequality, we have

$$(a^2+bc+a^2)(b^2+b^2+ca)(ab+c^2+c^2) \ge (ab+bc+ca)^3,$$

from which the desired inequality follows.

Remark. Using the first method, we can also prove the following inequality

$$(5a^2+bc)(5b^2+ca)(5c^2+ab) \ge 8(ab+bc+ca)^3.$$

□

P 3.12. *Let a,b,c be nonnegative real numbers such that*

$$a+b+c=2.$$

Prove that

(a) $$(a^2+b^2)(b^2+c^2)(c^2+a^2) \le (a+b)(b+c)(c+a);$$

(b) $$(a^2+b^2)(b^2+c^2)(c^2+a^2) \le 2.$$

Solution. Assume that

$$a=\min\{a,b,c\}.$$

It is easy to check that the equality holds in both inequalities for $a=0$ and $b=c=1$.

(a) Since

$$a^2+b^2 \le b(a+b)$$

and

$$c^2+a^2 \le c(c+a),$$

it suffices to show that

$$bc(b^2+c^2) \le b+c.$$

By the AM-GM inequality, we have

$$2bc(b^2+c^2) \le \left[\frac{2bc+(b^2+c^2)}{2}\right]^2 = \frac{(b+c)^4}{4} \le 2(b+c).$$

The equality holds for $a=0$ and $b=c=1$ (or any cyclic permutation).

(b) ***First Solution.*** Since

$$a^2+b^2 \le b(a+b)$$

and

$$c^2+a^2 \le c(a+c),$$

it suffices to show that

$$bc(a+b)(a+c)(b^2+c^2) \le 2.$$

By the AM-GM inequality, we have

$$4bc(a+b)(a+c)(b^2+c^2) \le \left[\frac{2b(a+c)+2c(a+b)+(b^2+c^2)}{3}\right]^3.$$

Therefore, we only need to show that

$$b^2+c^2+4bc+2ab+2ac \le 6.$$

This is true since

$$\begin{aligned} &12-2(b^2+c^2+4bc+2ab+2ac) = \\ &\quad = 3(a+b+c)^2-2(b^2+c^2+4bc+2ab+2ac) \\ &\quad = 2a^2+b^2+c^2-2bc+2ab+2ac \\ &\quad = 2a(a+b+c)+(b-c)^2 \ge 0. \end{aligned}$$

The equality holds for $a=0$ and $b=c=1$ (or any cyclic permutation).

Second Solution. Let us denote

$$F(a,b,c) = (a^2+b^2)(b^2+c^2)(c^2+a^2).$$

We will show that

$$F(a,b,c) \le F(0, b+a/2, c+a/2) \le 2.$$

The left inequality,

$$(a^2+b^2)(b^2+c^2)(c^2+a^2) \le (b+a/2)^2[(b+a/2)^2+(c+a/2)^2](c+a/2)^2,$$

is true since

$$a^2+b^2 \le (b+a/2)^2,$$

$$b^2+c^2 \le (b+a/2)^2+(c+a/2)^2,$$
$$c^2+a^2 \le (c+a/2)^2.$$

The right inequality holds if the original inequality holds for $a=0$; that is,

$$b^2c^2(b^2+c^2) \le 2$$

for $b+c=2$. Indeed, by virtue of the AM-GM inequality, we have

$$bc \le \left(\frac{b+c}{2}\right)^2 = 1,$$

hence

$$\begin{aligned}2b^2c^2(b^2+c^2) &\le 2bc(b^2+c^2)\\ &\le \left[\frac{2bc+(b^2+c^2)}{2}\right]^2\\ &= \frac{(b+c)^4}{4} = 4.\end{aligned}$$

□

P 3.13. *Let a,b,c be nonnegative real numbers such that*

$$a+b+c=2.$$

Prove that

$$(a^3+b^3)(b^3+c^3)(c^3+a^3) \le 2.$$

Solution. Due to symmetry, we may assume that

$$a=\min\{a,b,c\}.$$

It is easy to check that the equality holds for $a=0$ and $b=c=1$. Write the inequality as

$$\left[\prod(a+b)\right]\left[\prod(a^2-ab+b^2)\right] \le 2.$$

Since

$$\prod(a+b) \le (a+b+c)(ab+bc+ca) = 2(ab+bc+ca),$$

it suffices to show that

$$(ab+bc+ca)\prod(a^2-ab+b^2) \le 1.$$

Since

$$a^2-ab+b^2 \le b^2$$

and

$$c^2 - ca + a^2 \le c^2,$$

it suffices to show that

$$b^2c^2(ab + bc + ca)(b^2 - bc + c^2) \le 1.$$

In virtue of the AM-GM inequality, we have

$$b^2c^2(ab + bc + ca)(b^2 - bc + c^2) \le \left[\frac{bc + bc + (ab + bc + ca) + (b^2 - bc + c^2)}{4}\right]^4.$$

Therefore, it remains to show that

$$b^2 + c^2 + 2bc + ab + ca \le 4.$$

This is true since

$$\begin{aligned} 4 - (b^2 + c^2 + 2bc + ab + ca) &= (a + b + c)^2 - (b^2 + c^2 + 2bc + ab + ca) \\ &= a(a + b + c) \ge 0. \end{aligned}$$

The equality holds for $a = 0$ and $b = c = 1$ (or any cyclic permutation).

□

P 3.14. *Let a, b, c be nonnegative real numbers such that*

$$a^2 + b^2 + c^2 = 2.$$

Prove that

$$(a^3 + b^3)(b^3 + c^3)(c^3 + a^3) \le 2.$$

(Vasile Cîrtoaje, 2011)

Solution. Let

$$x = a^2, \quad y = b^2, \quad z = c^2, \qquad x + y + z = 2.$$

Since

$$(a^3 + b^3)^2 \le (a^2 + b^2)(a^4 + b^4) = (x + y)(x^2 + y^2),$$

it suffices to prove that

$$(x + y)(y + z)(z + x)(x^2 + y^2)(y^2 + z^2)(z^2 + x^2) \le 4.$$

Due to symmetry, we may assume that

$$x = \min\{x, y, z\}.$$

It is easy to check that the equality holds for $x = 0$ and $y = z = 1$. Since

$$(x + y)(y + z)(z + x) \le (x + y + z)(xy + yz + zx) = 2(xy + yz + zx)$$

and

$$x^2+y^2 \le y(x+y), \quad z^2+x^2 \le z(x+z),$$

it suffices to show that

$$yz(xy+yz+zx)(x+y)(x+z)(y^2+z^2) \le 2.$$

Write this inequality as

$$(2yz)[2(xy+yz+zx)][2(x+y)(x+z)](y^2+z^2) \le 16.$$

By the AM-GM inequality, it suffices to show that

$$\left[\frac{2yz+2(xy+yz+zx)+2(x+y)(x+z)+(y^2+z^2)}{4}\right]^4 \le 16.$$

This inequality is equivalent to

$$2x^2+y^2+z^2+6yz+4xy+4zx \le 8,$$

$$2x^2+y^2+z^2+6yz+4xy+4zx \le 2(x+y+z)^2,$$

$$(y-z)^2 \ge 0.$$

The equality holds for $a=0$ and $b=c=1$ (or any cyclic permutation).

□

P 3.15. *If a,b,c are nonnegative real numbers such that*

$$a+b+c=2,$$

then

$$(3a^2-2ab+3b^2)(3b^2-2bc+3c^2)(3c^2-2ca+3a^2) \le 36.$$

(Vasile Cîrtoaje, 2011)

Solution. Due to symmetry, assume that

$$a=\min\{a,b,c\}.$$

On the other hand, we can check that the equality holds for $(a,b,c)=(0,1,1)$. Since

$$0 \le 3a^2-2ab+3b^2 \le b(a+3b)$$

and

$$0 < 3c^2-2ca+3a^2 \le c(a+3c),$$

it suffices to show that

$$bc(a+3b)(a+3c)(3b^2-2bc+3c^2) \le 36.$$

Write this inequality as

$$[4b(a+3c)][4c(a+3b)][3(3b^2-2bc+3c^2)] \le 12^3.$$

By virtue of the AM-GM inequality, it suffices to show that

$$\left[\frac{4b(a+3c)+4c(a+3b)+3(3b^2-2bc+3c^2)}{3}\right]^3 \le 12^3.$$

This is equivalent to

$$9(b+c)^2+4a(b+c) \le 36.$$

We have

$$\begin{aligned}36-9(b+c)^2-4a(b+c) &= 9(a+b+c)^2-9(b+c)^2-4a(b+c)\\ &= a(9a+14b+14c) \ge 0.\end{aligned}$$

The equality holds for $a=0$ and $b=c=1$ (or any cyclic permutation).

Remark. Similarly, we can prove the following more general statement.

- *Let a,b,c be nonnegative real numbers. If $\frac{2}{3} \le k \le 2$, then*

$$(a^2-kab+b^2)(b^2-kbc+c^2)(c^2-kca+a^2) \le \frac{4}{27(2+k)^2}(a+b+c)^6,$$

with equality for $a=0$ and $\frac{b}{c}+\frac{c}{b}=1+\frac{3k}{2}$ (or any cyclic permutation).

□

P 3.16. *Let a,b,c be nonnegative real numbers such that*

$$a+b+c=3.$$

Prove that

$$(a^2-4ab+b^2)(b^2-4bc+c^2)(c^2-4ca+a^2) \le 3.$$

(Vasile Cîrtoaje, 2011)

Solution. Assume that

$$a \le b \le c.$$

If $c^2-4ca+a^2 \le 0$, then

$$b \le c \le (2+\sqrt{3})a \le (2+\sqrt{3})b.$$

From

$$b \le (2+\sqrt{3})a, \qquad c \le (2+\sqrt{3})b,$$

it follows that

$$a^2-4ab+b^2 \le 0, \quad b^2-4bc+c^2 \le 0.$$

Since

$$(a^2-4ab+b^2)(b^2-4bc+c^2)(c^2-4ca+a^2) \le 0,$$

the desired inequality is trivial. Consider further that $c^2-4ca+a^2 \ge 0$. There are only two cases when the left hand side of the desired inequality is nonnegative:

$$a^2-4ab+b^2 \ge 0, \quad b^2-4bc+c^2 \ge 0, \quad c^2-4ca+a^2 \ge 0$$

and

$$a^2-4ab+b^2 \le 0, \quad b^2-4bc+c^2 \le 0, \quad c^2-4ca+a^2 \ge 0.$$

Case 1: $a^2-4ab+b^2 \ge 0$, $b^2-4bc+c^2 \ge 0$, $c^2-4ca+a^2 \ge 0$. Since

$$a^2-4ab+b^2 \le b^2, \quad c^2-4ca+a^2 \le c^2,$$

it suffices to prove that

$$b^2c^2(b^2-4bc+c^2) \le 3.$$

It is easy to show that the homogeneous inequality

$$(a^2-4ab+b^2)(b^2-4bc+c^2)(c^2-4ca+a^2) \le 3\left(\frac{a+b+c}{3}\right)^6$$

becomes an equality for

$$a=0, \quad b^2+c^2=7bc.$$

Thus, we apply the AM-GM inequality as follows:

$$\begin{aligned} b^2c^2(b^2-4bc+c^2) &= \frac{1}{9}(3bc)(3bc)(b^2-4bc+c^2) \\ &\le \frac{1}{9}\left[\frac{3bc+3bc+(b^2-4bc+c^2)}{3}\right]^3 \\ &= 3\left(\frac{b+c}{3}\right)^6 \le 3\left(\frac{a+b+c)}{3}\right)^6 = 3. \end{aligned}$$

Case 2: $a^2-4ab+b^2 \le 0$, $b^2-4bc+c^2 \le 0$, $c^2-4ca+a^2 \ge 0$. By the AM-GM inequality, we have

$$(4ab-a^2-b^2)(4bc-b^2-c^2)(c^2-4ca+a^2) \le$$

$$\le \left[\frac{(4ab-a^2-b^2)+(4bc-b^2-c^2)+(c^2-4ca+a^2)}{3}\right]^3$$

$$= \frac{8}{27}(2ab+2bc-2ca-b^2)^3;$$

therefore, it suffices to prove that

$$(2ab + 2bc - 2ca - b^2)^3 \le \frac{81}{8},$$

which is equivalent to the homogeneous inequality

$$2\sqrt[3]{9}(2ab + 2bc - 2ca - b^2) \le (a + b + c)^2.$$

Since $2\sqrt[3]{9} < \dfrac{21}{5}$, we only need to show that

$$21(2ab + 2bc - 2ca - b^2) \le 5(a + b + c)^2.$$

Write this inequality as $f(a) \ge 0$, where

$$f(a) = 5a^2 + 4(13c - 8b)a + 26b^2 + 5c^2 - 32bc.$$

From $a^2 - 4ab + b^2 \le 0$, it follows that

$$a \ge (2 - \sqrt{3})b,$$

which involves $4a \ge b$ and $5a^2 \ge \dfrac{b^2}{20}$; therefore,

$$f(a) \ge \frac{b^2}{20} + (13c - 8b)b + 26b^2 + 5c^2 - 32bc = \frac{1}{20}(19b - 10c)^2 \ge 0.$$

This completes the proof. The equality holds for $a = 0$, $b = \dfrac{3 - \sqrt{5}}{2}$ and $c = \dfrac{3 + \sqrt{5}}{2}$ (or any permutation).

□

P 3.17. *If a, b, c are positive real numbers such that*

$$a + b + c = 3,$$

then

$$abc + \frac{12}{ab + bc + ca} \ge 5.$$

Solution. By the third degree Schur's inequality

$$(a + b + c)^3 + 9abc \ge 4(a + b + c)(ab + bc + ca),$$

we get $3abc \ge 4(ab + bc + ca) - 9$. Thus, it suffices to prove that

$$4(ab + bc + ca) - 9 + \frac{36}{ab + bc + ca} \ge 15.$$

This inequality is equivalent to

$$(ab + bc + ca - 3)^2 \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.18. *If a, b, c are positive real numbers such that*

$$a^2 + b^2 + c^2 = 3,$$

then

$$5(a + b + c) + \frac{3}{abc} \geq 18.$$

Solution. Let

$$x = \frac{a + b + c}{3}.$$

From

$$2(ab + bc + ca) = (a + b + c)^2 - (a^2 + b^2 + c^2) = 3(3x^2 - 1),$$

we get $x > 1/\sqrt{3}$. By the known inequality

$$(ab + bc + ca)^2 \geq 3abc(a + b + c),$$

we get

$$\frac{1}{abc} \geq \frac{4x}{(3x^2 - 1)^2}.$$

Then, it suffices to prove that

$$5x + \frac{4x}{(3x^2 - 1)^2} \geq 6,$$

which is equivalent to

$$15x^5 - 18x^4 - 10x^3 + 12x^2 + 3x - 2 \geq 0,$$

$$(x - 1)^2(15x^3 + 12x^2 - x - 2) \geq 0.$$

We still have to show that

$$15x^3 + 12x^2 - x - 2 \geq 0.$$

Since $x > 1/\sqrt{3}$, we get

$$15x^3 + 12x^2 - x - 2 > x^2(12 - \frac{1}{x} - \frac{2}{x^2}) > x^2(12 - \sqrt{3} - 6) > 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.19. *If a, b, c are positive real numbers such that*

$$a^2 + b^2 + c^2 = 3,$$

then

$$12 + 9abc \geq 7(ab + bc + ca).$$

(Vasile Cîrtoaje, 2005)

Solution. Denote $x = (a+b+c)/3$. Since

$$2(ab+bc+ca) = (a+b+c)^2 - (a^2+b^2+c^2) = 3(3x^2-1),$$

we can write the inequality as

$$5 + 2abc \geq 7x^2.$$

By Schur's inequality of degree three, we get

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

$$3x^3 + abc \geq 2x(3x^2-1),$$

$$abc \geq 3x^3 - 2x.$$

Then,

$$5 + 2abc - 7x^2 \geq 5 + 2(3x^3-2x) - 7x^2 = (x-1)^2(6x+5) \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.20. *If a, b, c are positive real numbers such that*

$$a^2 + b^2 + c^2 = 3,$$

then

$$21 + 18abc \geq 13(ab+bc+ca).$$

(Vasile Cîrtoaje, 2005)

Solution. Let

$$p = a+b+c, \qquad q = ab+bc+ca.$$

From

$$2q = (a+b+c)^2 - (a^2+b^2+c^2) = p^2 - 3,$$

we get $p > \sqrt{3}$. In addition, from Schur's inequality of degree four, we have

$$abc \geq \frac{(p^2-q)(4q-p^2)}{6p} = \frac{(p^2+3)(p^2-6)}{12p}.$$

Therefore,

$$21 + 18abc - 13(ab+bc+ca) \geq 21 + \frac{3(p^2+3)(p^2-6)}{2p} - \frac{13(p^2-3)}{2}$$

$$= \frac{(p-3)^2(3p^2+5p-6)}{2p} \geq 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.21. *If a,b,c are positive real numbers such that*

$$a^2+b^2+c^2=3,$$

then

$$(2-ab)(2-bc)(2-ca)\ge 1.$$

(Vasile Cîrtoaje, 2005)

First Solution. Let $p=a+b+c$. From

$$3(a^2+b^2+c^2)\ge (a+b+c)^2,$$

we get $p\le 3$. Since

$$\begin{aligned}(2-ab)(2-bc)(2-ca)&=8-4(ab+bc+ca)+2abc(a+b+c)-a^2b^2c^2\\&=8-2(p^2-3)+2abcp-a^2b^2c^2\\&=14-p^2-(p-abc)^2,\end{aligned}$$

we can write the inequality as

$$13-p^2-(p-abc)^2\ge 0.$$

Clearly,

$$3(p-abc)=(a^2+b^2+c^2)(a+b+c)-3abc>0.$$

By Schur's inequality

$$(a+b+c)^3+9abc\ge 4(a+b+c)(ab+bc+ca),$$

we get

$$p^3+9abc\ge 2p(p^2-3),$$

$$abc\ge\frac{p(p^2-6)}{9}.$$

Since

$$0<p-abc\le p-\frac{p(p^2-6)}{9}=\frac{p(15-p^2)}{9},$$

it suffices to prove that

$$13-p^2-\frac{p^2(15-p^2)^2}{81}\ge 0.$$

Setting

$$p=3\sqrt{x},\quad 0<x\le 1,$$

this inequality becomes

$$13-34x+30x^2-9x^3\ge 0.$$

It is true because

$$\begin{aligned}13-34x+30x^2-9x^3 &= (1-x)(13-21x+9x^2)\\ &= (1-x)[1+3(1-x)(4-3x)] \ge 0.\end{aligned}$$

The equality holds for $a=b=c=1$.

Second Solution. We use the mixing variables technique. Assume that $a \le 1$ and show that

$$(2-ab)(2-bc)(2-ca) \ge (2-x^2)(2-ax)^2 \ge 1,$$

where

$$x = \sqrt{\frac{b^2+c^2}{2}} = \sqrt{\frac{3-a^2}{2}}, \quad x < \sqrt{\frac{3}{2}}.$$

Since

$$2-bc \ge 2-\frac{1}{2}(b^2+c^2) \ge 2-\frac{3}{2} > 0$$

and, similarly, $2-ca>0$, $2-ab>0$, we can prove the left inequality by multiplying the inequalities

$$2-bc \ge 2-x^2$$

and

$$(2-ca)(2-ab) \ge (2-ax)^2.$$

The last inequality is true because

$$\begin{aligned}(2-ca)(2-ab)-(2-ax)^2 &= 2a(2x-b-c)-a^2(x^2-bc)\\ &= \frac{2a(b-c)^2}{2x+b+c} - \frac{a^2(b-c)^2}{2}\\ &= \frac{a(b-c)^2[4-a(2x+b+c)]}{2(2x+b+c)}\end{aligned}$$

and

$$\begin{aligned}4-a(2x+b+c) \ge 4(1-ax) &= 2(2-a\sqrt{6-2a^2})\\ &= \frac{4(1-a^2)(2-a^2)}{2+a\sqrt{6-2a^2}} \ge 0.\end{aligned}$$

The right inequality, $(2-x^2)(2-ax)^2 \ge 1$, is equivalent to

$$(1+a^2)(2-ax)^2 \ge 2.$$

Since $2(1+a^2) \ge (1+a)^2$ and $2-ax \ge 2-x > 0$, it suffices to show that

$$(1+a)(2-ax) \ge 2.$$

Indeed,

$$(1+a)(2-ax)-2=a(2-x-ax)=\frac{a(a^4+2a^3-2a^2-6a+5)}{2(2+x+ax)}$$
$$=\frac{a(a-1)^2(a^2+4a+5)}{2(2+x+ax)}\geq 0.$$

□

P 3.22. *Let a,b,c be positive real numbers such that*

$$abc=1.$$

Prove that

$$\left(\frac{a+b+c}{3}\right)^5\geq\frac{a^2+b^2+c^2}{3}.$$

First Solution. Write the inequality in the homogeneous form

$$(a+b+c)^5\geq 81abc(a^2+b^2+c^2).$$

Using to the known inequality

$$(ab+bc+ca)^2\geq 3abc(a+b+c),$$

it suffices to show that

$$(a+b+c)^6\geq 27(ab+bc+ca)^2(a^2+b^2+c^2).$$

Setting $p=a+b+c$ and $q=ab+bc+ca$, we have

$$(a+b+c)^6-27(ab+bc+ca)^2(a^2+b^2+c^2)=p^6-27q^2(p^2-2q)$$
$$=(p^2-3q)^2(p^2+6q)\geq 0.$$

The equality occurs for $a=b=c=1$.

Second Solution. Use the mixing variables method. We show that

$$E(a,b,c)\geq E(a,x,x)\geq 0,$$

where

$$E(a,b,c)=(a+b+c)^5-81abc(a^2+b^2+c^2),\quad x=\frac{b+c}{2}.$$

Indeed, we have

$$\frac{E(a,b,c)-E(a,x,x)}{81}=a^3(x^2-bc)+a[2x^4-bc(b^2+c^2)]$$
$$=\frac{1}{4}a^3(b-c)^2+\frac{1}{8}a(b-c)^4\geq 0$$

and

$$\begin{aligned}E(a,x,x) &= (a+b+c)^5 - \frac{81}{8}a(b+c)^2[2a^2+(b+c)^2] \\ &= \frac{1}{8}(2a-b-c)^2[2a^3+12a^2(b+c)-9a(b+c)^2+8(b+c)^3] \ge 0,\end{aligned}$$

since

$$\begin{aligned}&2a^3+12a^2(b+c)-9a(b+c)^2+8(b+c)^3] > \\ &> 6a^2(b+c)-12a(b+c)^2+6(b+c)^3 \\ &= 6(b+c)(a-b-c)^2 \ge 0.\end{aligned}$$

□

P 3.23. *If a,b,c are positive real numbers such that*

$$abc = 1,$$

then

$$a^3+b^3+c^3+a^{-3}+b^{-3}+c^{-3}+21 \ge 3(a+b+c)(a^{-1}+b^{-1}+c^{-1}).$$

Solution. Since

$$a^3+b^3+c^3+a^{-3}+b^{-3}+c^{-3}+3 = \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)$$

and

$$(a+b+c)(a^{-1}+b^{-1}+c^{-1}) = \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+3,$$

we can write the desired inequality in the homogeneous form

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right)+9 \ge 3\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)+3\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right),$$

or

$$\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}-3\right)\left(\frac{b}{a}+\frac{c}{b}+\frac{a}{c}-3\right) \ge 0.$$

This is true because, by the AM-GM inequality, we have

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \ge 3, \quad \frac{b}{a}+\frac{c}{b}+\frac{a}{c} \ge 3.$$

The equality holds for $a=b=c=1$.

□

P 3.24. *If a, b, c are positive real numbers such that*

$$abc = 1,$$

then

$$a^2 + b^2 + c^2 - ab - bc - ca \geq \frac{9}{4}(a + b + c - 3).$$

Solution. Write the inequality in the form

$$3(4x^2 - 3x + 3) \geq 4(ab + bc + ca),$$

where

$$x = \frac{a + b + c}{3}.$$

The third degree Schur's inequality states that

$$(a + b + c)^3 + 9abc \geq 4(a + b + c)(ab + bc + ca),$$

which is equivalent to

$$4(ab + bc + ca) \leq \frac{3(3x^3 + 1)}{x}.$$

Therefore, it suffices to show that

$$3(4x^2 - 3x + 3) \geq \frac{3(3x^3 + 1)}{x}.$$

This inequality reduces to

$$(x - 1)^3 \geq 0,$$

which is true because

$$x \geq \sqrt[3]{abc} = 1.$$

The equality holds for $a = b = c = 1$.

□

P 3.25. *If a, b, c are positive real numbers such that*

$$abc = 1,$$

then

$$a^2 + b^2 + c^2 + a + b + c \geq 2(ab + bc + ca).$$

Solution. Let

$$p = a + b + c, \qquad q = ab + bc + ca.$$

By virtue of the AM-GM inequality, we have

$$p \ge 3\sqrt[3]{abc} = 3,$$

and by Schur's inequality

$$p^3 + 9abc \ge 4pq,$$

we get

$$4q \le \frac{p^3 + 9}{p}.$$

Therefore,

$$\begin{aligned} a^2 + b^2 + c^2 + a + b + c - 2(ab + bc + ca) &= p^2 + p - 4q \\ &\ge p^2 + p - \frac{p^3 + 9}{p} \\ &= \frac{(p-3)(p+3)}{p} \ge 0. \end{aligned}$$

The equality holds for $a = b = c = 1$.

□

P 3.26. *If a, b, c are positive real numbers such that*

$$abc = 1,$$

then

$$a^2 + b^2 + c^2 + 15(ab + bc + ca) \ge 16(a + b + c).$$

Solution. Write the inequality as $F(a, b, c) \ge 0$, where

$$F(a, b, c) = a^2 + b^2 + c^2 + 15\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 16(a + b + c).$$

Assume that $a \ge b \ge c$ and denote

$$t = \sqrt{bc}, \quad 0 < t \le 1, \quad at^2 = 1.$$

We will show that

$$F(a, b, c) \ge F(a, t, t) \ge 0.$$

Since

$$F(a, b, c) - F(a, t, t) = b^2 + c^2 - 2t^2 + 15\left(\frac{1}{b} + \frac{1}{c} - \frac{2}{t}\right) - 16(b + c - 2t)$$

$$= (b-c)^2 + 15\left(\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{c}}\right)^2 - 16(\sqrt{b} - \sqrt{c}\,)^2$$

$$= (\sqrt{b} - \sqrt{c}\,)^2\left[(\sqrt{b} + \sqrt{c}\,)^2 + \frac{15}{bc} - 16\right] \ge (\sqrt{b} - \sqrt{c}\,)^2\left(4\sqrt{bc} + \frac{15}{bc} - 16\right),$$

it suffices to show that

$$4t + \frac{15}{t^2} - 16 \ge 0.$$

Indeed,

$$4t + \frac{15}{t^2} - 16 > t + \frac{15}{t} - 16 = \frac{(1-t)(15-t)}{t} \ge 0.$$

The inequality $F(a,t,t) \ge 0$ is equivalent to

$$(t-1)^2(17t^4 + 2t^3 - 13t^2 + 2t + 1) \ge 0.$$

We have

$$17t^4 + 2t^3 - 13t^2 + 2t + 1 = (2t-1)^4 + t(t^3 + 34t^2 - 37t + 10)$$

$$= (2t-1)^4 + \frac{t}{4}[t(2t-1)^2 + 140t^2 - 149t + 40] > 0$$

since $D = 149^2 - 4 \cdot 140 \cdot 40 = -199$. The equality holds for $a = b = c = 1$.

□

P 3.27. *If a, b, c are positive real numbers such that*

$$abc = 1,$$

then

$$\frac{2}{a+b+c} + \frac{1}{3} \ge \frac{3}{ab+bc+ca}.$$

Solution. Let

$$x = \frac{ab+bc+ca}{3}.$$

By virtue of the AM-GM inequality, we have

$$x \ge \sqrt[3]{ab \cdot bc \cdot ca} = 1.$$

The third degree Schur's inequality applied to ab, bc, ca, states that

$$(ab+bc+ca)^3 + 9a^2b^2c^2 \ge 4abc(a+b+c)(ab+bc+ca),$$

which is equivalent to

$$\frac{3}{a+b+c} \ge \frac{4x}{3x^3+1}.$$

Therefore,

$$3\left(\frac{2}{a+b+c}+\frac{1}{3}-\frac{3}{ab+bc+ca}\right)\geq\frac{8x}{3x^3+1}+1-\frac{3}{x}$$

$$=\frac{3x^4-9x^3+8x^2+x-3}{x(3x^3+1)}=\frac{(x-1)(3x^3-6x^2+2x+3)}{x(3x^3+1)}.$$

Since $x\geq 1$, we need to show that

$$3x^3-6x^2+2x+3\geq 0.$$

For $x\geq 2$, we have

$$3x^3-6x^2+2x+3>3x^3-6x^2=3x^2(x-2)\geq 0,$$

and for $1\leq x<2$, we have

$$3x^3-6x^2+2x+3=3x(x-1)^2+3-x>0.$$

The equality holds for $a=b=c=1$.

□

P 3.28. *If a,b,c are positive real numbers such that*

$$abc=1,$$

then

$$ab+bc+ca+\frac{6}{a+b+c}\geq 5.$$

(Vasile Cîrtoaje, 2005)

First Solution. Denoting

$$x=\frac{ab+bc+ca}{3},$$

the inequality can be written as

$$(a+b+c)(3x-5)+6\geq 0.$$

In virtue of the AM-GM inequality, we get $x\geq 1$. Since the inequality holds for $x\geq 5/3$, consider next that $1\leq x<5/3$. Applying the third degree Schur's inequality to the numbers ab, bc and ca, we have

$$(ab+bc+ca)^3+9a^2b^2c^2\geq 4abc(a+b+c)(ab+bc+ca),$$

which is equivalent to

$$a+b+c\leq\frac{3(3x^3+1)}{4x}.$$

Since $3x-5<0$, it suffices to prove that

$$\frac{3(3x^3+1)(3x-5)}{4x}+6\geq 0.$$

This inequality is equivalent to

$$9x^4-15x^3+11x-5\geq 0,$$

$$(x-1)(9x^3-6x^2-6x+5)\geq 0.$$

Since

$$9x^3-6x^2-6x+5>9x^3-6x^2-6x+3=3(x-1)(3x^2+x-1)\geq 0,$$

the conclusion follows. The equality holds for $a=b=c=1$.

Second Solution (by *Vo Quoc Ba Can*). Among

$$a-1,\quad b-1,\quad c-1,$$

there are two with the same sign. Due to symmetry, assume that $(b-1)(c-1)\geq 0$; that is,

$$b+c\leq 1+bc.$$

Then,

$$\frac{6}{a+b+c}\geq\frac{6}{a+1+bc}=\frac{6a}{a^2+a+1}.$$

On the other hand, using the AM-GM inequality yields

$$ab+bc+ca=a(b+c)+bc\geq 2a\sqrt{bc}+bc=2\sqrt{a}+\frac{1}{a}.$$

Therefore, it suffices to prove that

$$2\sqrt{a}+\frac{1}{a}+\frac{6a}{a^2+a+1}\geq 5.$$

Setting $\sqrt{a}=x$, this inequality becomes as follows:

$$2x+\frac{1}{x^2}+\frac{6x^2}{x^4+x^2+1}\geq 5,$$

$$2x+\frac{1}{x^2}-3\geq 2-\frac{6x^2}{x^4+x^2+1},$$

$$\frac{(x-1)^2(2x+1)}{x^2}\geq\frac{2(x^2-1)^2}{x^4+x^2+1},$$

$$(x-1)^2(2x^5-x^4-2x^3-x^2+2x+1)\geq 0,$$

$$(x-1)^2[x(x-1)^2(2x^2+3x+2)+1]\geq 0.$$

□

P 3.29. *If a, b, c are positive real numbers such that*

$$abc = 1,$$

then

$$\sqrt[3]{(1+a)(1+b)(1+c)} \geq \sqrt[4]{4(1+a+b+c)}.$$

(Pham Huu Duc, 2008)

Solution. Since

$$(1+a)(1+b)(1+c) = (1+a+b+c) + (1+ab+bc+ca)$$
$$\geq 2\sqrt{(1+a+b+c)(1+ab+bc+ca)},$$

it suffices to prove that

$$(1+ab+bc+ca)^2 \geq 4(1+a+b+c),$$

which is equivalent to

$$(1+q)^2 \geq 4(1+p),$$

where

$$p = a+b+c, \quad q = ab+bc+ca.$$

Setting $x = bc$, $y = ca$, $z = ab$ in Schur's inequality

$$(x+y+z)^3 + 9xyz \geq 4(x+y+z)(xy+yz+zx),$$

we get

$$q^3 + 9 \geq 4pq.$$

Since

$$(1+q)^2 - 4(1+p) \geq (1+q)^2 - 4 - \frac{q^3+9}{q}$$
$$= \frac{(q-3)(2q+3)}{q},$$

it suffices to show that $q \geq 3$. Indeed, by the AM-GM inequality, we have

$$q = ab+bc+ca \geq 3\sqrt[3]{a^2b^2c^2} = 3.$$

The equality holds for $a = b = c = 1$

□

P 3.30. *If a, b, c are positive real numbers, then*

$$a^6 + b^6 + c^6 - 3a^2b^2c^2 \geq 18(a^2 - bc)(b^2 - ca)(c^2 - ab).$$

(Vasile Cîrtoaje, 2007)

Solution. Due to homogeneity, we may assume that $abc = 1$, when the inequality can be written as

$$a^6 + b^6 + c^6 - 3 \geq 18(a^3 + b^3 + c^3 - a^3b^3 - b^3c^3 - c^3a^3).$$

Substituting a^3, b^3, c^3 by a, b, c, respectively, we need to show that $abc = 1$ implies $F(a, b, c) \geq 0$, where

$$F(a, b, c) = a^2 + b^2 + c^2 - 3 - 18(a + b + c - ab - bc - ca).$$

To do this, we use the mixing variables method. Without loss of generality, assume that $a \geq 1$. We claim that

$$F(a, b, c) \geq F(a, \sqrt{bc}, \sqrt{bc}) \geq 0.$$

We have

$$\begin{aligned} F(a, b, c) - F(a, \sqrt{bc}, \sqrt{bc}) &= (b - c)^2 - 18(\sqrt{b} - \sqrt{c})^2 + 18a(\sqrt{b} - \sqrt{c})^2 \\ &= (b - c)^2 + 18(a - 1)(\sqrt{b} - \sqrt{c})^2 \geq 0. \end{aligned}$$

Also, putting $\sqrt{bc} = t$, we have

$$\begin{aligned} F(a, \sqrt{bc}, \sqrt{bc}) = F(\frac{1}{t^2}, t, t) &= \frac{1}{t^4} + 20t^2 - 3 - \frac{18}{t^2} - 36t + \frac{36}{t} \\ &= \frac{(t-1)^2(2t-1)^2(t+1)(5t+1)}{t^4} \geq 0. \end{aligned}$$

The equality holds for $a = b = c$, and for $a/2 = b = c$ (or any cyclic permutation). □

P 3.31. *If a, b, c are positive real numbers such that*

$$a + b + c = 3,$$

then

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq a^2 + b^2 + c^2.$$

(Vasile Cîrtoaje, 2006)

First Solution. Since

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca},$$

it suffices to prove that

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \geq a^2 + b^2 + c^2,$$

which is equivalent to

$$abc(a^2+b^2+c^2) \le 3.$$

Let $x = (ab+bc+ca)/3$. From the known inequality

$$(ab+bc+ca)^2 \ge 3abc(a+b+c),$$

we get

$$abc \le x^2.$$

On the other hand, we have

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 9-6x.$$

Then,

$$abc(a^2+b^2+c^2)-3 \le x^2(9-6x)-3 = -3(x-1)^2(2x+1) \le 0.$$

The equality holds for $a=b=c=1$.

Second Solution. Since $a+b+c=3$, we can write the inequality as

$$\sum\left(\frac{1}{a^2} - a^2 + 4a - 4\right) \ge 0,$$

which is equivalent to

$$\sum \frac{(1-a)^2(1+2a-a^2)}{a^2} \ge 0.$$

Without loss of generality, assume that $a = \max\{a,b,c\}$. We have two cases to consider.

Case 1: $a \le 1+\sqrt{2}$. Since $a,b,c \le 1+\sqrt{2}$, we have

$$1+2a-a^2 \ge 0, \quad 1+2b-b^2 \ge 0, \quad 1+2c-c^2 \ge 0.$$

Thus, the conclusion follows.

Case 2: $a > 1+\sqrt{2}$. Since $b+c = 3-a < 2-\sqrt{2} < \frac{2}{3}$, we have

$$bc \le \frac{1}{4}(b+c)^2 < \frac{1}{9},$$

and hence

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} > \frac{1}{b^2}+\frac{1}{c^2} \ge \frac{2}{bc} > 18 > (a+b+c)^2 > a^2+b^2+c^2.$$

□

P 3.32. *If a, b, c are positive real numbers such that*

$$ab + bc + ca = 3,$$

then

$$a^3 + b^3 + c^3 + 7abc \ge 10.$$

(Vasile Cîrtoaje, 2005)

Solution. Let

$$x = \frac{a+b+c}{3}.$$

By the well-known inequality

$$(a+b+c)^2 \ge 3(ab+bc+ca),$$

we get $x \ge 1$. Since

$$\begin{aligned} a^3 + b^3 + c^3 &= 3abc + (a+b+c)^3 - 3(a+b+c)(ab+bc+ca) \\ &= 3abc + 27x^3 - 27x, \end{aligned}$$

we can write the inequality as

$$10abc + 27x^3 - 27x - 10 \ge 0.$$

For $x \ge \frac{4}{3}$, this inequality is true since

$$27x^3 - 27x - 10 = 27x(x^2-1) - 10 \ge 36(\frac{16}{9} - 1) - 10 = 18.$$

For $1 \le x \le \frac{4}{3}$, we use Schur's inequality

$$(a+b+c)^3 + 9abc \ge 4(a+b+c)(ab+bc+ca),$$

which is equivalent to

$$abc + 3x^3 - 4x \ge 0.$$

Therefore,

$$\begin{aligned} 10abc + 27x^3 - 27x - 10 &\ge 10(-3x^3 + 4x) + 27x^3 - 27x - 10 \\ &= (x-1)[4 - 3x + 3(2 - x^2)] \ge 0. \end{aligned}$$

The equality holds for $a = b = c = 1$.

□

P 3.33. *If a, b, c are nonnegative real numbers such that*

$$a^3 + b^3 + c^3 = 3,$$

then

$$a^4b^4 + b^4c^4 + c^4a^4 \leq 3.$$

(Vasile Cîrtoaje, 2003)

Solution. By virtue of the AM-GM inequality, we have

$$ab \leq \frac{a^3 + b^3 + 1}{3} = \frac{4 - c^3}{3}.$$

Then, we have

$$a^4a^4 \leq \frac{4a^3a^3 - a^3b^3c^3}{3}.$$

Similarly,

$$b^4c^4 \leq \frac{4b^3c^3 - a^3b^3c^3}{3}, \quad c^4a^4 \leq \frac{4c^3a^3 - a^3b^3c^3}{3}.$$

Summing these inequalities, we obtain

$$a^4b^4 + b^4c^4 + c^4a^4 \leq \frac{4(a^3b^3 + b^3c^3 + c^3a^3)}{3} - a^3b^3c^3.$$

Using the substitutions $x = a^3$, $y = b^3$, $z = c^3$, it suffices to prove that

$$4(xy + yz + zx) \leq 3xyz + 9,$$

where x, y, z are nonnegative real numbers satisfying $x + y + z = 3$. This follows immediately from Schur's inequality

$$4(x + y + z)(xy + yz + zx) \leq 9xyz + (x + y + z)^3.$$

The equality holds for $a = b = c = 1$.

Remark 1. We may write the inequality in the homogeneous form

$$\left(\frac{a^4b^4 + b^4c^4 + c^4a^4}{3}\right)^3 \leq \left(\frac{a^3 + b^3 + c^3}{3}\right)^8.$$

From this, we get the reverse statement.

- *If a, b, c are nonnegative real numbers such that $a^4b^4 + b^4c^4 + c^4a^4 = 3$, then*

$$a^3 + b^3 + c^3 \geq 3.$$

Remark 2. The inequality in P 3.33 is a particular case of the following more general statement (*Vasile Cîrtoaje*, 2003).

• *Let a, b, c be nonnegative real numbers such that $a+b+c=3$. If $0<k\le k_0$, where*

$$k_0=\frac{\ln 3}{\ln 9-\ln 4}\approx 1.355,$$

then

$$a^kb^k+b^kc^k+c^ka^k\le 3.$$

□

P 3.34. *If a, b, c are nonnegative real numbers, then*

$$(a+1)^2(b+1)^2(c+1)^2\ge 4(a+b+c)(ab+bc+ca)+28abc.$$

(Vasile Cîrtoaje, 2011)

Solution. By the AM-GM inequality, we have

$$(a+1)(b+1)(c+1)=(abc+1)+(a+b+c)+(ab+bc+ca)$$

$$\ge 2\sqrt{abc}+2\sqrt{(a+b+c)(ab+bc+ca)}.$$

Thus, it suffices to prove that

$$\left[\sqrt{abc}+\sqrt{(a+b+c)(ab+bc+ca)}\right]^2\ge (a+b+c)(ab+bc+ca)+7abc,$$

which can be written as

$$\sqrt{abc(a+b+c)(ab+bc+ca)}\ge 3abc.$$

This is true if

$$\sqrt{(a+b+c)(ab+bc+ca)}\ge 3\sqrt{abc}.$$

Indeed, we have

$$(a+b+c)(ab+bc+ca)-9abc=a(b-c)^2+b(c-a)^2+c(a-b)^2\ge 0.$$

The equality holds for $a=b=c=1$.

□

P 3.35. *If a, b, c are positive real numbers such that*

$$a+b+c=3,$$

then

$$1+8abc\ge 9\min\{a,b,c\}.$$

(Vasile Cîrtoaje, 2007)

Solution. Without loss of generality, assume that

$$a = \min\{a, b, c\}, \quad a \le 1,$$

when the inequality becomes

$$1 + 8abc \ge 9a.$$

From $(a-b)(a-c) \ge 0$, we get

$$bc \ge a(b+c) - a^2 = a(3-a) - a^2 = a(3-2a^2).$$

Therefore,

$$1 + 8abc - 9a \ge 1 + 8a^2(3-2a^2) - 9a = (1-a)(1-4a)^2 \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $(a, b, c) = \left(\frac{1}{4}, \frac{1}{4}, \frac{5}{2}\right)$ (or any cyclic permutation).

□

P 3.36. *If a, b, c are positive real numbers such that*

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 + 4abc \ge 5\min\{a, b, c\}.$$

(Vasile Cîrtoaje, 2007)

Solution. Without loss of generality, assume that

$$a = \min\{a, b, c\}, \quad a \le 1.$$

The inequality can be written as

$$1 + 4abc \ge 5a.$$

From $(a^2 - b^2)(a^2 - c^2) \ge 0$, we get

$$bc \ge a\sqrt{b^2 + c^2 - a^2} = a\sqrt{3 - 2a^2}.$$

Therefore, it suffices to prove that

$$4a^2\sqrt{3 - 2a^2} \ge 5a - 1.$$

We consider two cases.

Case 1: $0 < a \le 1/3$. Since

$$\sqrt{3-2a^2} \ge \frac{5}{3} > \frac{25}{16},$$

it is enough to show that

$$\frac{25}{4}a^2 \ge 5a-1.$$

This inequality is equivalent to $(5a-2)^2 \ge 0$.

Case 2: $1/3 < a \le 1$. Since

$$\sqrt{3-2a^2} \ge 2-a,$$

which is equivalent to the obvious inequality

$$(1-a)(3a-1) \ge 0,$$

it suffices to show that

$$4a^2(2-a) \ge 5a-1.$$

Indeed, we have

$$4a^2(2-a)-5a+1 = (1-a)(2a-1)^2 \ge 0.$$

The proof is completed. The equality holds for $a = b = c = 1$.

□

P 3.37. *If a, b, c are positive real numbers such that*

$$a+b+c = abc,$$

then

$$(1-a)(1-b)(1-c)+(\sqrt{3}-1)^3 \ge 0.$$

Solution. Without loss of generality, assume that

$$a \ge b \ge c.$$

The product $(1-a)(1-b)(1-c)$ is negative for either $a > 1 > b \ge c$ or $a \ge b \ge c > 1$. Since $a > 1 > b \ge c$ involves the contradiction

$$0 = a+b+c-abc > a(1-bc) > 0,$$

it suffices to consider only the case $a \ge b \ge c > 1$. Setting

$$x = a-1, \quad y = b-1, \quad z = c-1,$$

we need to show that

$$xyz \le (\sqrt{3}-1)^3$$

for all $x, y, z > 0$ such that

$$xy + yz + zx + xyz = 2.$$

Let

$$t = \sqrt[3]{xyz}, \quad t > 0.$$

By the AM-GM inequality, we have

$$2 = xy + yz + zx + xyz \ge 3\sqrt[3]{x^2y^2z^2} + xyz = 3t^2 + t^3,$$

hence

$$t^3 + 3t^2 - 2 \le 0,$$
$$(t+1)(t^2 + 2t - 2) \le 0,$$
$$t^2 + 2t - 2 \le 0,$$
$$t \le \sqrt{3} - 1,$$
$$xyz \le (\sqrt{3}-1)^3.$$

The equality holds for $a = b = c = \sqrt{3}$.

□

P 3.38. *If a, b, c are nonnegative real numbers such that*

$$a + b + c = 2,$$

then

$$(a^2 + bc)(b^2 + ca)(c^2 + ab) \le 1.$$

(Vasile Cîrtoaje, 2005)

Solution. Without loss of generality, assume that

$$a \ge b \ge c.$$

Since

$$a^2 + bc \le (a + \frac{c}{2})^2$$

and

$$(b^2 + ca)(c^2 + ab) \le \frac{1}{4}(b^2 + ca + c^2 + ab)^2,$$

it suffices to show that

$$(2a + c)(b^2 + c^2 + ab + ac) \le 4.$$

Let

$$E(a,b,c) = (2a+c)(b^2+c^2+ab+ac).$$

We will show that

$$E(a,b,c) \le E(a,b+c,0) \le 4.$$

Indeed,

$$E(a,b,c) - E(a,b+c,0) = c(b^2+c^2+ac-3ab) \le 0$$

and

$$\begin{aligned} E(a,b+c,0) - 4 &= 2a(a+b+c)(b+c) - 4 \\ &= 4a(2-a) - 4 = -4(a-1)^2 \le 0. \end{aligned}$$

The equality occurs for $a = b = 1$ and $c = 0$ (or any cyclic permutation).

□

P 3.39. *If a,b,c are nonnegative real numbers, then*

$$(8a^2+bc)(8b^2+ca)(8c^2+ab) \le (a+b+c)^6.$$

Solution. We use the mixing variables technique. Without loss of generality, assume that $a \le b \le c$. Let

$$x = \frac{b+c}{2}, \quad x \ge a,$$

and

$$E(a,b,c) = (8a^2+bc)(8b^2+ca)(8c^2+ab) - (a+b+c)^6.$$

We will prove that

$$E(a,b,c) \le E(a,x,x) \le 0.$$

The left inequality is equivalent to

$$(8a^2+x^2)(8x^2+ax)^2 \ge (8a^2+bc)(8b^2+ca)(8c^2+ab),$$

which follows by multiplying the inequalities

$$8a^2+x^2 \ge 8a^2+bc$$

and

$$(8x^2+ax)^2 \ge (8b^2+ca)(8c^2+ab).$$

The first inequality is obvious, and the last inequality is equivalent to

$$64(x^4-b^2c^2) + a^2(x^2-bc) - 8a(b^3+c^3-2x^3) \ge 0.$$

Since

$$b^3+c^3-2x^3 = \frac{3(b+c)(b-c)^2}{4} = 6x(x^2-bc) \ge 0,$$

we need to show that

$$64(x^2+bc)+a^2-48ax \geq 0.$$

This is true, since

$$64(x^2+bc)+a^2-48ax \geq 48x(x-a) \geq 0.$$

The right inequality, $E(a,x,x) \leq 0$, is equivalent to

$$(8a^2+x^2)(8x^2+ax)^2-(a+2x)^6 \leq 0,$$

$$176x^5-273ax^4+32a^2x^3+52a^3x^2+12a^4x+a^5 \geq 0,$$

$$(x-a)^2(176x^3+79ax^2+14a^2x+a^3) \geq 0.$$

The equality holds for $a=b=c$, and for $a=0$ and $b=c$ (or any cyclic permutation).

□

P 3.40. *If a,b,c are positive real numbers such that*

$$a^2b^2+b^2c^2+c^2a^2=3,$$

then

$$a+b+c \geq abc+2.$$

(Vasile Cîrtoaje, 2006)

Solution. Without loss of generality, assume that

$$a \geq b \geq c.$$

From $a^2b^2+b^2c^2+c^2a^2=3$, it follows that

$$1 \leq ab < \sqrt{3}.$$

We have

$$\begin{aligned} a+b+c-abc-2 &= a+b-2-(ab-1)c \\ &\geq 2\sqrt{ab}-2-(ab-1)c \\ &= (\sqrt{ab}-1)\left[2-(\sqrt{ab}+1)c\right]. \end{aligned}$$

So, we need to prove that

$$2 \geq (\sqrt{ab}+1)c.$$

Since

$$\sqrt{ab}+1 \leq 2\sqrt{ab},$$

it suffices to show that $2\sqrt{ab} \le 2\sqrt{ab}$; that is,

$$1 \ge abc^2.$$

Indeed, we have

$$c^2 = \frac{3-a^2b^2}{a^2+b^2} \le \frac{3-a^2b^2}{2ab},$$

hence

$$1-abc^2 \ge 1-\frac{3-a^2b^2}{2} = \frac{a^2b^2-1}{2} \ge 0.$$

The equality holds for $a = b = c = 1$.

□

P 3.41. *Let a,b,c be nonnegative real numbers such that*

$$a+b+c = 5.$$

Prove that

$$(a^2+3)(b^2+3)(c^2+3) \ge 192.$$

First Solution. Without loss of generality, assume that

$$a = \min\{a,b,c\}, \quad a \le \frac{5}{3}.$$

By virtue of the Cauchy-Schwarz inequality, we have

$$(b^2+3)(c^2+3) = (b^2+3)(3+c^2) \ge 3(b+c)^2 = 3(5-a)^2.$$

Therefore, it suffices to sow that

$$(a^2+3)(5-a)^2 \ge 64$$

for $0 \le a \le \frac{5}{3}$. Indeed,

$$(a^2+3)(5-a)^2 - 64 = (a-1)^2(a^2-8a+11) \ge 0,$$

since

$$a^2-8a+11 = \left(\frac{5}{3}-a\right)\left(\frac{19}{3}-a\right)+\frac{4}{9} > 0.$$

The equality holds for $a = 3$ and $b = c = 1$ (or any cyclic permutation).

Second Solution. Without loss of generality, assume that

$$a = \max\{a,b,c\}.$$

First, we show that

$$(b^2+3)(c^2+3) \geq (x^2+3)^2,$$

where

$$x = \frac{b+c}{2}, \quad 0 \leq x \leq \frac{5}{3}.$$

This inequality is equivalent to

$$(b-c)^2(6-bc-x^2) \geq 0,$$

which is true because

$$6-bc-x^2 \geq 2(3-x^2) > 0.$$

Thus, it suffices to prove that

$$(a^2+3)(x^2+3)^2 \geq 192,$$

which is equivalent to

$$[(5-2x)^2+3](x^2+3)^2 \geq 192,$$

$$(x^2-5x+7)(x^2+3)^2 \geq 48,$$

$$(x-1)^2(x^4-3x^3+6x^2-15x+15) \geq 0.$$

This inequality is true since

$$x^4-3x^3+6x^2-15x+15 = x^2\left(x-\frac{3}{2}\right)^2+15\left(\frac{x}{2}-1\right)^2 > 0.$$

□

P 3.42. *If a,b,c are nonnegative real numbers, then*

$$a^2+b^2+c^2+abc+2 \geq a+b+c+ab+bc+ca.$$

(Michael Rozenberg, 2012)

Solution. Among the numbers

$$1-a, \quad 1-b, \quad 1-c,$$

there are always two with the same sign; let us say $(1-b)(1-c) \geq 0$, that is

$$bc \geq b+c-1.$$

Thus, it suffices to show that

$$a^2+b^2+c^2+a(b+c-1)+2 \geq a+b+c+ab+bc+ca,$$

which is equivalent to

$$a^2-2a+b^2+c^2-bc-(b+c)+2\geq 0.$$

Since

$$b^2+c^2-bc\geq \frac{1}{4}(b+c)^2,$$

it suffices to show that

$$a^2-2a+\frac{1}{4}(b+c)^2-(b+c)+2\geq 0,$$

which can be written in the obvious form

$$(a-1)^2+\left(\frac{b+c}{2}-1\right)^2\geq 0.$$

The equality holds for $a=b=c=1$.

□

P 3.43. *If a,b,c are nonnegative real numbers, then*

$$\sum a^3(b+c)(a-b)(a-c)\geq 3(a-b)^2(b-c)^2(c-a)^2.$$

Solution. Without loss of generality, assume that

$$a=\min\{a,b,c\}.$$

Since

$$a^3(b+c)(a-b)(a-c)\geq 0$$

and

$$\begin{aligned}
b^3(c+a)(b-c)(b-a)+c^3(a+b)(c-a)(c-b)&=\\
=(b-c)[bc(b^3-c^3)+(b-c)(b^3+c^3)a-(b^3-c^3)a^2]&\\
=(b-c)^2[(b^2+bc+c^2)(bc-a^2)+(b^3+c^3)a]&\\
\geq (b-c)^2(b^2+bc+c^2)(bc-a^2),&
\end{aligned}$$

it suffices to show that

$$(b^2+bc+c^2)(bc-a^2)\geq 3(a-b)^2(c-a)^2.$$

Since

$$bc-a^2=(a-b)(a-c)+a(b+c-2a)\geq (a-b)(a-c),$$

it suffices to show that

$$b^2+bc+c^2\geq 3(a-b)(a-c),$$

which is equivalent to the obvious inequality

$$(b-c)^2 + 3a(b+c-a) \geq 0.$$

The equality holds for $a=b=c$, for $a=0$ and $b=c$ (or any cyclic permutation), and for $b=c=0$ (or any cyclic permutation).

□

P 3.44. *Find the greatest real number k such that*

$$a+b+c+4abc \geq k(ab+bc+ca)$$

for all $a,b,c \in [0,1]$.

Solution. Setting $a=b=c=1$, we get $k \leq 7/3$, but setting $a=0$ and $b=c=1$, we get $k \leq 2$. So, we claim that $k=2$ is the greatest real number k. To prove this, we only need to show that

$$a+b+c+4abc \geq 2(ab+bc+ca)$$

for all $a,b,c \in [0,1]$. Write the inequality as

$$a(1+4bc-2b-2c)+b+c-2bc \geq 0.$$

Since

$$b+c-2bc = b(1-c)+c(1-b) \geq 0,$$

the inequality is clearly true for $1+4bc-2b-2c \geq 0$. Consider further that $1+4bc-2b-2c < 0$, when it suffices to show that

$$(1+4bc-2b-2c)+b+c-2bc \geq 0.$$

This is equivalent to the obvious inequality

$$bc+(1-b)(1-c) \geq 0.$$

Thus, the proof is completed. If $k=2$, then the equality holds for $a=b=c=0$, and also for $a=0$ and $b=c=1$ (or any cyclic permutation).

Remark. From the proof above it follows that the following stronger inequality holds for all $a,b,c \in [0,1]$:

$$a+b+c+3abc \geq 2(ab+bc+ca),$$

with equality for $a=b=c=0$, for $a=b=c=1$, and also for $a=0$ and $b=c=1$ (or any cyclic permutation).

□

P 3.45. *If $a, b, c \geq \frac{2}{3}$ such that*

$$a + b + c = 3,$$

then

$$a^2b^2 + b^2c^2 + c^2a^2 \geq ab + bc + ca.$$

Solution. We use the mixing variables method. Assume that $a = \max\{a, b, c\}$ and denote

$$x = \frac{b+c}{2}, \quad 2/3 \leq x \leq 1.$$

We will show that

$$E(a, b, c) \geq E(a, x, x) \geq 0,$$

where

$$E(a, b, c) = a^2b^2 + b^2c^2 + c^2a^2 - ab - bc - ca.$$

We have

$$\begin{aligned} E(a, b, c) - E(a, x, x) &= a^2(b^2 + c^2 - 2x^2) - (x^4 - b^2c^2) + (x^2 - bc) \\ &= (x^2 - bc)(2a^2 - x^2 - bc + 1) \\ &= \frac{1}{4}(b-c)^2[a^2 + (a^2 - bc) + (1 - x^2)] \geq 0 \end{aligned}$$

and

$$E(a, x, x) = 2a^2x^2 + x^4 - 2ax - x^2.$$

Since $a + 2x = 3$, we get

$$\begin{aligned} 9E(a, x, x) &= 18a^2x^2 + 9x^4 - (2ax + x^2)(a + 2x)^2 \\ &= x(5x^3 - 12ax^2 + 9a^2x - 2a^3) \\ &= x(x-a)^2(5x - 2a) \\ &= 3x(x-a)^2(3x - 2) \geq 0. \end{aligned}$$

The equality holds for $a = b = c = 1$, and also for $a = 5/3$ and $b = c = 2/3$ (or any cyclic permutation).

□

P 3.46. *If a, b, c are positive real numbers such that*

$$a \leq 1 \leq b \leq c, \quad a + b + c = 3,$$

then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq a^2 + b^2 + c^2.$$

Solution. Let

$$F(a,b,c)=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-a^2-b^2-c^2.$$

We will show that

$$F(a,b,c)\ge F(a,1,b+c-1)\ge 0.$$

The left inequality is true since

$$\begin{aligned}
&F(a,b,c)-F(a,1,b+c-1)=\\
&\quad=\left(\frac{1}{b}+\frac{1}{c}-1-\frac{1}{b+c-1}\right)+1+(b+c-1)^2-b^2-c^2\\
&\quad=(b+c)\left(\frac{1}{bc}-\frac{1}{b+c-1}\right)+2(b-1)(c-1)\\
&\quad=(b-1)(c-1)\left[2-\frac{b+c}{bc(b+c-1)}\right]
\end{aligned}$$

and

$$\begin{aligned}
2bc(b+c-1)-b-c&=(2bc-1)(b+c)-2bc\\
&\ge 2(2bc-1)\sqrt{bc}-2bc\\
&=2\sqrt{bc}\,(\sqrt{bc}-1)(2\sqrt{bc}+1)\ge 0.
\end{aligned}$$

The right inequality $F(a,1,b+c-1)\ge 0$ is equivalent to $F(a,1,x)\ge 0$, where $x>0$ and $x+a=2$. We have

$$F(a,1,x)=\frac{1}{a}+\frac{1}{x}-a^2-x^2=\frac{(x+a)^4}{8ax}-a^2-x^2=\frac{(x-a)^4}{8ax}\ge 0.$$

The equality holds for $a=b=c=1$.

□

P 3.47. *If a,b,c are positive real numbers such that*

$$a\le 1\le b\le c,\qquad a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

then

$$a^2+b^2+c^2\le\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

(Vasile Cîrtoaje, 2008)

Solution. Write the inequality as

$$b^2-\frac{1}{b^2}\le(a^2+c^2)\left(\frac{1}{a^2c^2}-1\right).$$

From $a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$, we have

$$b-\frac{1}{b}=(a+c)\left(\frac{1}{ac}-1\right)\geq 0.$$

Thus, the desired inequality holds if

$$(a+c)(b+\frac{1}{b})\leq (a^2+c^2)(\frac{1}{ac}+1).$$

On the other hand, from $(b-c)(1-\frac{1}{bc})\leq 0$, we get

$$b+\frac{1}{b}\geq c+\frac{1}{c}.$$

Then, it suffices to prove that

$$(a+c)(c+\frac{1}{c})\leq (a^2+c^2)(\frac{1}{ac}+1),$$

which is equivalent to the obvious inequality

$$c(1-a^2)(a-c)\leq 0.$$

The proof is completed. The equality holds for $b=1$ and $ac=1$.

□

P 3.48. *If a,b,c are positive real numbers such that*

$$a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

then

$$(abc-1)\left(a^n+b^n+c^n-\frac{1}{a^n}-\frac{1}{b^n}-\frac{1}{c^n}\right)\leq 0$$

for any integer $n\geq 2$.

(Vasile Cîrtoaje, 2007)

Solution. Since the statement remains unchanged by substituting a, b, c with $1/a$, $1/b$, $1/c$, respectively, it suffices to prove that

$$a^n+b^n+c^n-\frac{1}{a^n}-\frac{1}{b^n}-\frac{1}{c^n}\leq 0$$

for

$$abc\geq 1, \quad a+b+c=1/a+1/b+1/c.$$

It is easy to check that $a+b+c=1/a+1/b+1/c$ is equivalent to

$$(ab-1)(bc-1)(ca-1)=a^2b^2c^2-1,$$

and the desired inequality is equivalent to

$$(a^nb^n-1)(b^nc^n-1)(c^na^n-1)\ge a^{2n}b^{2n}c^{2n}-1.$$

Setting

$$x=bc,\quad y=ca,\quad z=ab,$$

we need to show that

$$(x-1)(y-1)(z-1)=xyz-1\ge 0$$

involves

$$(x^n-1)(y^n-1)(z^n-1)\ge x^ny^nz^n-1.$$

This inequality holds if

$$(x^{n-1}+x^{n-2}+\cdots+1)(y^{n-1}+y^{n-2}+\cdots+1)(z^{n-1}+z^{n-2}+\cdots+1)\ge$$

$$\ge x^{n-1}y^{n-1}z^{n-1}+x^{n-2}y^{n-2}z^{n-2}+\cdots+1.$$

Since the last inequality is clearly true, the proof is completed. The equality occurs for $a=bc=1$ (or any cyclic permutation).

□

P 3.49. *Let a,b,c be positive real numbers, and let*

$$E(a,b,c)=a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b).$$

Prove that

(a) $$(a+b+c)E(a,b,c)\ge ab(a-b)^2+bc(b-c)^2+ca(c-a)^2;$$

(b) $$2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)E(a,b,c)\ge (a-b)^2+(b-c)^2+(c-a)^2.$$

Solution. (a) Using Schur's inequality of degree four

$$\sum a^2(a-b)(a-c)\ge 0,$$

we have

$$\begin{aligned}(a+b+c)E(a,b,c) &= \sum a^2(a-b)(a-c) + \sum a(b+c)(a-b)(a-c)\\ &\ge \sum a(b+c)(a-b)(a-c)\\ &= \sum ab(a-b)(a-c) + \sum ac(a-b)(a-c)\\ &= \sum ab(a-b)(a-c) + \sum ba(b-c)(b-a)\\ &= \sum ab(a-b)^2 \ge 0.\end{aligned}$$

The equality holds for $a=b=c$. If a,b,c are nonnegative real numbers, then the equality also holds for $a=0$ and $b=c$ (or any cyclic permutation).

(b) Since

$$\begin{aligned}&(ab+bc+ca)E(a,b,c) =\\ &= abc\sum(a-b)(a-c) + \sum(a^2b+a^2c)(a-b)(a-c)\\ &= \frac{1}{2}abc\sum(a-b)^2 + \sum[a^2b(a-b)(a-c) + b^2a(b-c)(b-a)]\\ &= \frac{1}{2}abc\sum(a-b)^2 + \sum ab(a-b)^2(a+b-c),\end{aligned}$$

the required inequality is equivalent to

$$\sum ab(a-b)^2(a+b-c) \ge 0.$$

Without loss of generality, assume that $a \ge b \ge c$. Then,

$$\begin{aligned}\sum ab(a-b)^2(a+b-c) &\ge bc(b-c)^2(b+c-a) + ac(a-c)^2(a+c-b)\\ &\ge bc(b-c)^2(b+c-a) + ac(b-c)^2(a+c-b)\\ &= c(b-c)^2[(a-b)^2 + c(a+b)] \ge 0.\end{aligned}$$

The equality holds for $a=b=c$.

□

P 3.50. *Let $a \ge b \ge c$ be nonnegative real numbers. Schur's inequalities of third and fourth degree state that*

(a) $\qquad a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \ge 0;$

(b) $\qquad a^2(a-b)(a-c) + b^2(b-c)(b-a) + c^2(c-a)(c-b) \ge 0.$

Prove that (a) is sharper than (b) if

$$\sqrt{b} + \sqrt{c} \le \sqrt{a},$$

and (b) is sharper than (a) if

$$\sqrt{b}+\sqrt{c}\ge\sqrt{a}.$$

(Vasile Cîrtoaje, 2005)

Solution. Let

$$p=a+b+c,\quad q=ab+bc+ca.$$

If we rewrite Schur's inequalities as

$$abc\ge f(p,q)$$

and

$$abc\ge g(p,q),$$

respectively, then (a) is sharper than (b) if $f(p,q)\ge g(p,q)$, while (b) is sharper than (a) if $g(p,q)\ge f(p,q)$. Therefore, we need to show that

$$(\sqrt{b}+\sqrt{c}-\sqrt{a}\ )[g(p,q)-f(p,q)]\ge 0.$$

From the known relation

$$4q-p^2=(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{b}+\sqrt{c}-\sqrt{a})(\sqrt{c}+\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}-\sqrt{c}),$$

it follows that $4q-p^2$ and $\sqrt{b}+\sqrt{c}-\sqrt{a}$ has the same sign. Therefore, it suffices to prove that

$$(4q-p^2)[g(p,q)-f(p,q)]\ge 0.$$

In order to find $f(p,q)$, write the inequality in (a) as follows

$$a^3+b^3+c^3+3abc\ge ab(a+b)+bc(b+c)+ca(c+a)$$
$$(a+b+c)^3+9abc\ge 4(a+b+c)(ab+bc+ca),$$

from which

$$f(p,q)=\frac{p(4q-p^2)}{9}.$$

Analogously, write the inequality in (b) as follows

$$a^4+b^4+c^4+abc(a+b+c)\ge ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2),$$

$$a^4+b^4+c^4+2abc(a+b+c)\ge(ab+bc+ca)(a^2+b^2+c^2).$$

Since

$$a^2+b^2+c^2=p^2-2q$$

and

$$\begin{aligned}a^4+b^4+c^4&=(a^2+b^2+c^2)^2-2(a^2b^2+b^2c^2+c^2a^2)\\&=(p^2-2q)^2-2q^2+4abcp,\end{aligned}$$

we get

$$g(p,q)=\frac{(p^2-q)(4q-p^2)}{6p}.$$

Therefore, we have

$$g(p,q)-f(p,q)=\frac{(p^2-3q)(4q-p^2)}{18p},$$

hence

$$(4q-p^2)[g(p,q)-f(p,q)]=\frac{(p^2-3q)(4q-p^2)^2}{18p}\geq 0.$$

Remark. If a,b,c are the lengths of the sides of a triangle, then Schur's inequality of degree four is always stronger than Schur's inequality of degree three.

□

P 3.51. *If a,b,c are nonnegative real numbers such that*

$$(a+b)(b+c)(c+a)=8,$$

then

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\geq ab+bc+ca.$$

(Vasile Cîrtoaje, 2010)

First Solution. Assume that $a\geq b\geq c$, and write the inequality in the equivalent homogeneous forms

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})\sqrt{(a+b)(b+c)(c+a)}\geq 2\sqrt{2}(ab+bc+ca),$$

$$\sum\sqrt{a(b+c)}[\sqrt{(a+b)(a+c)}-\sqrt{2a(b+c)}]\geq 0,$$

$$\sum\frac{(a-b)(a-c)\sqrt{a(b+c)}}{\sqrt{(a+b)(a+c)}+\sqrt{2a(b+c)}}\geq 0.$$

Since $(c-a)(c-b)\geq 0$, it suffices to prove that

$$\frac{(a-b)(a-c)\sqrt{a(b+c)}}{\sqrt{(a+b)(a+c)}+\sqrt{2a(b+c)}}+\frac{(b-c)(b-a)\sqrt{b(c+a)}}{\sqrt{(b+c)(b+a)}+\sqrt{2b(c+a)}}\geq 0,$$

which is true if

$$\frac{(a-c)\sqrt{a(b+c)}}{\sqrt{(a+b)(a+c)}+\sqrt{2a(b+c)}}\geq\frac{(b-c)\sqrt{b(c+a)}}{\sqrt{(b+c)(b+a)}+\sqrt{2b(c+a)}}.$$

Since $\sqrt{a}\geq\sqrt{b}$,

$$\sqrt{(a+b)(a+c)}\geq\sqrt{2a(b+c)}$$

and

$$\sqrt{(b+c)(b+a)} \le \sqrt{2b(a+c)},$$

it suffices to show that

$$\frac{(a-c)\sqrt{b+c}}{\sqrt{(a+b)(a+c)}} \ge \frac{(b-c)\sqrt{c+a}}{\sqrt{(b+c)(b+a)}}.$$

This is equivalent to the obvious inequality

$$c(a-b) \ge 0.$$

The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=\sqrt[3]{4}$ (or any cyclic permutation).

Second Solution. Let

$$p = a+b+c, \quad q = ab+bc+ca.$$

By squaring, the inequality becomes

$$p + 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}) \ge q^2.$$

Since

$$\sqrt{ab} \ge \frac{2ab}{a+b} = \frac{ab(b+c)(c+a)}{4} = \frac{ab(q+c^2)}{4},$$

$$\sqrt{bc} \ge \frac{bc(q+c^2)}{4}, \quad \sqrt{ca} \ge \frac{ca(q+c^2)}{4},$$

we have

$$2(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}) \ge \frac{q(ab+bc+ca)+abc(a+b+c)}{2} = \frac{q^2+abcp}{2}.$$

Using this result, it suffices to show that

$$p + \frac{q^2+abcp}{2} \ge q^2,$$

which is equivalent to

$$p(2+abc) \ge q^2.$$

Having in view the hypothesis $pq - abc = 8$, we can write this inequality in the homogeneous forms

$$p\left(\frac{pq-abc}{4}+abc\right) \ge q^2,$$

$$q(p^2-4q)+3abcp \ge 0.$$

For the non-trivial case where $p^2-4q \le 0$, using Schur's inequality

$$p^3+3abc \ge 4pq$$

gives

$$3(p^2q-4q^2+3abcp) \ge 3p^2q-12q^2+(4pq-p^3)p = (p^2-3q)(4q-p^2) \ge 0.$$

□

P 3.52. *If* $a,b,c \in [1,\ 4+3\sqrt{2}]$, *then*

$$9(ab+bc+ca)(a^2+b^2+c^2) \ge (a+b+c)^4.$$

(Vasile Cîrtoaje, 2005)

Solution. Let

$$A = a^2+b^2+c^2, \qquad B = ab+bc+ca.$$

Since

$$9(ab+bc+ca)(a^2+b^2+c^2)-(a+b+c)^4 = 9AB-(A+2B)^2$$

$$= (A-B)(4B-A)$$

and

$$2(A-B) = (a-b)^2+(b-c)^2+(c-a)^2 \ge 0,$$

we need to show that $4B-A \ge 0$; that is, to show that $E(a,b,c) \le 0$, where

$$E(a,b,c) = a^2+b^2+c^2-4(ab+bc+ca).$$

We claim that $E(a,b,c)$ is maximal for $a,b,c \in \{1,\ w\}$, where $w = 4+3\sqrt{2}$. For the sake of contradiction, assume that there exists a triple (a,b,c) with $a \in (1,\ w)$ such that

$$E(a,b,c) \ge \max\{E(1,b,c), E(w,b,c)\}.$$

From

$$E(a,b,c)-E(1,b,c) = (a-1)(a+1-4b-4c) \ge 0,$$

we get

$$a-4(b+c) \ge -1,$$

and from

$$E(a,b,c)-E(w,b,c) = (a-w)(a+w-4b-4c) \ge 0,$$

we get

$$a-4(b+c) \le -w.$$

These results involve $w \le 1$, which is false. Therefore, since $E(a,b,c)$ is symmetric, we have

$$\begin{aligned} E(a,b,c) &\le \max\{E(1,1,1), E(1,1,w), E(1,w,w), E(w,w,w)\} \\ &= \max\{-9, w^2-8w-2, 1-2w^2-8w, -9w^2\} \\ &= \max\{-9, 0, -99-72\sqrt{2}, -306-216\sqrt{2}\} = 0. \end{aligned}$$

This completes the proof. The equality holds for $a=b=c$, and also for $a=b=1$ and $c=4+3\sqrt{2}$ (or any cyclic permutation).

□

P 3.53. *If a,b,c are nonnegative real numbers such that*

$$a+b+c+abc=4,$$

then

(a) $$a^2+b^2+c^2+12 \ge 5(ab+bc+ca);$$

(b) $$3(a^2+b^2+c^2)+13(ab+bc+ca) \ge 48.$$

Solution. Let

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc.$$

(a) We need to show that

$$p^2+12 \ge 7q$$

for $p+r=4$. By Schur's inequality of degree three, we have $p^3+9r \ge 4pq$. Therefore, we get

$$\begin{aligned}4p(p^2+12-7q) &\ge 4p^3+48p-7(p^3+9r)\\ &= -3(p^3-37p+84)\\ &= 3(p-3)(4-p)(7+p).\end{aligned}$$

Since $4-p=r\ge 0$, we only need to show that $p \ge 3$. By virtue of the AM-GM inequality, we get

$$p^3 \ge 27r,$$

$$p^3 \ge 27(4-p),$$

$$(p-3)(p^2+3p+36) \ge 0,$$

$$p \ge 3.$$

The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=2$ (or any cyclic permutation).

(b) We need to show that

$$3p^2+7q \ge 48$$

for $p+r=4$. Using the known inequality $pq \ge 9r$, we get

$$\begin{aligned}p(3p^2+7q-48) &\ge 3(p^3+21r-16p)\\ &= 3(p^3-37p+84)\\ &= 3(p-3)(4-p)(7+p) \ge 0.\end{aligned}$$

The equality holds for $a=b=c=1$.

□

P 3.54. *Let a,b,c be the lengths of the sides of a triangle. If*

$$a+b+c=3,$$

then

$$a^2b^2+b^2c^2+c^2a^2 \ge ab+bc+ca.$$

Solution. Write the inequality as follows:

$$9(a^2b^2+b^2c^2+c^2a^2) \ge (ab+bc+ca)(a+b+c)^2;$$

$$3[3(a^2b^2+b^2c^2+c^2a^2)-(ab+bc+ca)^2] \ge$$

$$\ge (ab+bc+ca)[(a+b+c)^2-3(ab+bc+ca)];$$

$$6[a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2] \ge$$

$$\ge (ab+bc+ca)[(b-c)^2+(c-a)^2+(a-b)^2] \ge 0;$$

$$\sum S_a(b-c)^2 \ge 0,$$

where

$$S_a = 6a^2-ab-bc-ca.$$

Without loss of generality, assume that $a \ge b \ge c$. It suffices to show that

$$S_b(a-c)^2+S_c(a-b)^2 \ge 0.$$

Since

$$(a-c)^2 \ge (a-b)^2,$$

$$S_b = 6b^2-bc-a(b+c) \ge 6b^2-bc-(b+c)^2 > 0$$

and

$$\begin{aligned} S_b+S_c &= 6(b^2+c^2)-2bc-2a(b+c) \ge 6(b^2+c^2)-2bc-2(b+c)^2 \\ &= 4(b-c)^2+2bc > 0, \end{aligned}$$

we get

$$S_b(a-c)^2+S_c(a-b)^2 \ge (S_b+S_c)(a-b)^2 \ge 0.$$

The equality holds for an equilateral triangle.

□

P 3.55. *Let a,b,c be the lengths of the sides of a triangle. If*

$$a^2+b^2+c^2=3,$$

then

$$ab+bc+ca \ge 1+2abc.$$

(Vasile Cîrtoaje, 2005)

Solution. Write the inequality as

$$3(ab+bc+ca)-a^2-b^2-c^2 \geq 6abc,$$

where the both sides are homogeneous. From

$$3(a^2+b^2+c^2) \geq (a+b+c)^2,$$

we get

$$a+b+c \leq 3.$$

Therefore, it suffices to prove the homogeneous inequality

$$(a+b+c)[3(ab+bc+ca)-a^2-b^2-c^2] \geq 18abc.$$

This is equivalent to

$$2ab(a+b)+2bc(b+c)+2ca(c+a) \geq a^3+b^3+c^3+9abc.$$

Using the known substitution

$$a=y+z, \quad b=z+x, \quad c=x+y, \qquad x,y,z \geq 0,$$

the inequality can be written as

$$x^3+y^3+z^3+3xyz \geq xy(x+y)+yz(y+z)+zx(z+x),$$

which is just the third degree Schur's inequality. The equality holds for an equilateral triangle.

□

P 3.56. *Let a,b,c be the lengths of the sides of a triangle. If*

$$a+b+c=3,$$

then

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{41}{6} \geq 3(a^2+b^2+c^2).$$

(Vasile Cîrtoaje, 2010)

Solution *(by Vo Quoc Ba Can)*. Using the substitution

$$a=\frac{y+z}{2}, \quad b=\frac{z+x}{2}, \quad c=\frac{x+y}{2},$$

where $x,y,z \geq 0$ such that $x+y+z=3$, the inequality becomes as follow

$$\frac{1}{y+z}+\frac{1}{z+x}+\frac{1}{x+y}+\frac{41}{12} \geq \frac{3}{8}[(y+z)^2+(z+x)^2+(x+y)^2],$$

$$\sum \frac{x+y+z}{y+z} + \frac{41}{4} \ge \frac{9}{4}\left(\sum x^2 + \sum xy\right),$$

$$\sum \frac{x}{y+z} + 3 + \frac{41}{4} \ge \frac{9}{4}\left(9 - \sum xy\right),$$

$$\sum \frac{x}{y+z} \ge 7 - \frac{9}{4}\sum xy.$$

Let us denote $t = xy + yz + zx$. Since

$$\sum \frac{x}{y+z} = \frac{1}{t}\sum \frac{x(xy+yz+zx)}{y+z} \ge \frac{1}{t}\sum \frac{x(xy+zx)}{y+z}$$
$$= \frac{1}{t}\sum x^2 = \frac{9-2t}{t},$$

it suffices to show that

$$\frac{9-2t}{t} \ge 7 - \frac{9}{4}t,$$

which is equivalent to

$$(t-2)^2 \ge 0.$$

The equality holds for a degenerate triangle having $a = 3/2$, $b = 1$, $c = 1/2$ (or any permutation thereof).

□

P 3.57. *Let $a \ge b \ge c$ such that*

$$a + b + c = p, \qquad ab + bc + ca = q,$$

where p and q are fixed real numbers satisfying $p^2 \ge 3q$.

(a) If a, b, c are nonnegative real numbers, then the product $r = abc$ is minimal only when $a = b$ or $c = 0$, and is maximal only when $b = c$;

(b) If a, b, c are the lengths of the sides of a triangle (non-degenerate or degenerate), then the product $r = abc$ is minimal only when $a = b \ge c$, and is maximal only when $b = c \ge \frac{a}{2}$ or $b + c = a$.

(Vasile Cîrtoaje, 2005)

Solution. (a) Following Third Solution of P 2.53, we have $c \in [c_1, c_2]$, where

$$c_1 = \begin{cases} \dfrac{p - 2\sqrt{p^2 - 3q}}{3}, & 3q \le p^2 \le 4q \\ 0, & p^2 \ge 4q \end{cases},$$

$$c_2 = \frac{p - \sqrt{p^2 - 3q}}{3}.$$

On the other hand, the function $r(c)$ is strictly increasing on $[c_1, c_2]$. Since c attains its minimum c_1 only when $a = b$ (if $p^2 \le 4q$) or $c = 0$ (if $p^2 \ge 4q$), r is minimal only when $a = b$ or $c = 0$. Since c attains its maximum c_2 only when $b = c$, r is maximal only when $b = c$.

(b) Using the known substitution

$$a = y + z, \quad b = z + x, \quad c = x + y,$$

where $0 \le x \le y \le z$, from

$$a + b + c = 2(x + y + z),$$
$$ab + bc + ca = (x + y + z)^2 + xy + yz + zx,$$
$$abc = (x + y + z)(xy + yz + zx) - xyz,$$

it follows that

$$x + y + z = \frac{p}{2}, \quad xy + yz + zx = q - \frac{p^2}{4},$$

hence

$$abc = \frac{p}{2}\left(q - \frac{p^2}{4}\right) - xyz.$$

Therefore, the product abc is minimal when xyz is maximal; that is, according to (a), only when $x = y$, which is equivalent to $a = b \ge c$. Also, the product abc is maximal when xyz is minimal; that is, according to (a), only when $y = z$ or $x = 0$, which is equivalent to $b = c \ge a/2$ or $b + c = a$.

Remark 1. Using the result in (a), we can prove by the contradiction method (as in Remark 1 from P 2.53) the following generalization:

- *If $a_1, a_2, \ldots, a_n$ are nonnegative numbers such that $a_1 \ge a_2 \ge \cdots \ge a_n$ and*

$$a_1 + a_2 + \cdots + a_n = p, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = p_1,$$

where p and p_1 are fixed real numbers satisfying $p^2 \le np_1$, then the product

$$a_1 a_2 \cdots a_n$$

is minimal for $a_1 = \cdots = a_{n-1} \ge a_n$ or $a_n = 0$, and maximal for $a_1 \ge a_2 = \cdots = a_n$.

More general, according to Remark 3 from P 2.53, the following statement is valid:

- *If $a_1, a_2, \ldots, a_n \in [0, M]$ are real numbers such that $a_1 \ge a_2 \ge \cdots \ge a_n$ and*

$$a_1 + a_2 + \cdots + a_n = p, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = p_1,$$

where p and p_1 are fixed real numbers satisfying $p^2 \le np_1$, then the product

$$r = a_1 a_2 \cdots a_n$$

is minimal for $a_1 = \cdots = a_{n-1} \ge a_n$ *or* $a_n = 0$*, and maximal for* $a_1 \ge a_2 = \cdots = a_n$ *or* $a_1 = M$.

Remark 2. The statements for real variables in Remark 2 and Remark 3 from P 2.53 are also valid for nonnegative variables $a, b, c \in [m, M]$ and $a_1, a_2, \ldots, a_n \in [m, M]$, where $0 \le m < M$.

- *If* $0 \le m < M$ *and* $a, b, c \in [m, M]$ *such that* $a \ge b \ge c$ *and*

$$a + b + c = p, \quad ab + bc + ca = q,$$

where p *and* q *are fixed real numbers satisfying* $p^2 \ge 3q$*, then the product*

$$r = abc$$

is minimal when $a = b \ge c$ *or* $c = m$*, and maximal when* $a \ge b = c$ *or* $a = M$.

- *If* $0 \le m < M$ *and* $a_1, a_2, \ldots, a_n \in [m, M]$ *are real numbers such that* $a_1 \ge a_2 \ge \cdots \ge a_n$ *and*

$$a_1 + a_2 + \cdots + a_n = p, \quad a_1^2 + a_2^2 + \cdots + a_n^2 = p_1,$$

where p *and* p_1 *are fixed real numbers satisfying* $p^2 \le np_1$*, then the product*

$$r = a_1 a_2 \cdots a_n$$

is minimal for $a_1 = \cdots = a_{n-1} \ge a_n$ *or* $a_n = m$*, and maximal for* $a_1 \ge a_2 = \cdots = a_n$ *or* $a_1 = M$.

□

P 3.58. *Let* $a \ge b \ge c > 0$ *be positive real numbers such that*

$$a + b + c = p, \quad abc = r,$$

where p *and* r *are fixed positive numbers satisfying* $p^3 \ge 27r$*. Prove that*

$$q = ab + bc + ca$$

is minimal only when $b = c$*, and is maximal only when* $a = b$.

(Vasile Cîrtoaje, 2005)

Solution. Since $p^3 = 27r$ involves $a = b = c = p/3$, consider further that $p^3 > 27r$. As in P 2.53, we can show that

$$c \in [c_1, c_2], \quad b \in [b_1, b_2], \quad a \in [a_1, a_2], \quad 0 < c_1 < c_2 = b_1 < b_2 = a_1 < a_2,$$

where c_1 is the smallest positive root of the equation

$$x^3 - 2px^2 + p^2x - 4r = 0,$$

c_2 and b_1 are the smallest positive root of the equation

$$2x^3 - px^2 + r = 0,$$

b_2 and a_1 are the largest positive root of the equation

$$2x^3 - px^2 + r = 0,$$

a_2 is the largest positive root of the equation

$$x^3 - 2px^2 + p^2x - 4r = 0.$$

In addition, if $a = b$, then $c = c_1$, $b = b_2$, $a = a_1$, and if $b = c$, then $c = c_2$, $b = b_1$, $a = a_2$.

First solution. From

$$q = b(a + c) + ac = b(p - b) + \frac{r}{b},$$

we get

$$q(b) = pb - b^2 + \frac{r}{b}.$$

Since

$$q'(b) = p - 2b - \frac{r}{b^2} = \frac{-(b - a)(b - c)}{b} \geq 0,$$

$q(b)$ is increasing on $[b_1, b_2]$, hence $q(b)$ is minimal only for $b = b_1$, when $b = c$, and is maximal only for $b = b_2$, when $b = a$.

Second solution. From

$$q = a(b + c) + bc = a(p - a) + \frac{r}{a},$$

we get

$$q(a) = pa - a^2 + \frac{r}{a}.$$

Since

$$q'(a) = p - 2a - \frac{r}{a^2} = \frac{-(a - b)(a - c)}{a} \leq 0,$$

$q(a)$ is decreasing on $[a_1, a_2]$, hence $q(a)$ is minimal only for $a = a_2$, when $b = c$, and is maximal only for $a = a_1$, when $a = b$.

Third solution. From

$$q = c(a + b) + ab = c(p - c) + \frac{r}{c},$$

we get

$$q(c) = pc - c^2 + \frac{r}{c}.$$

Since

$$q'(c) = p - 2c - \frac{r}{c^2} = \frac{-(c-a)(c-b)}{c} \le 0,$$

$q(c)$ is decreasing on $[c_1, c_2]$, hence $q(c)$ is minimal only for $c = c_2$, when $c = b$, and is maximal only for $c = c_1$, when $a = b$.

Remark 1. Since

$$q = r\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = \frac{1}{2}(p^2 - a^2 - b^2 - c^2),$$

we can extend the statement of P 3.58 as follows:

- *If $a \ge b \ge c$ are positive real numbers such that*

$$a + b + c = p, \quad abc = r > 0,$$

where p and r are fixed positive numbers satisfying $p^3 \ge 27r$, then the sums

$$q = ab + bc + ca, \qquad q_1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \qquad p_1 = -(a^2 + b^2 + c^2)$$

are minimal only when $b = c$, and are maximal only when $a = b$.

Remark 2. Replacing a, b, c in P 3.58 with $1/a$, $1/b$, $1/c$, respectively, we get the following statement:

- *If $0 < a \le b \le c$ are positive real numbers such that*

$$ab + bc + ca = q, \quad abc = r,$$

where q and r are fixed positive numbers satisfying $q^3 \ge 27r^2$, then the sum

$$p = a + b + c$$

is minimal only when $b = c$, and is maximal only when $a = b$.

The statement remains valid by replacing q with $q_1 = \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$.

Remark 3. We can prove by the contradiction method (as in Remark 1 from P 2.53) the following generalization:

- *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 \ge a_2 \ge \cdots \ge a_n$ and*

$$a_1 + a_2 + \cdots + a_n = np, \quad a_1 a_2 \cdots a_n = r > 0,$$

where p and r are fixed positive numbers satisfying $p^n \ge r$, then the sums

$$q_1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}, \qquad p_1 = -(a_1^2 + a_2^2 + \cdots + a_n^2)$$

are minimal and maximal when $n-1$ of $a_1, a_2, \ldots, a_n$ are equal, more precisely, they are minimal when $a_1 \geq a_2 = \cdots = a_n$, and are maximal when $a_1 = \cdots = a_{n-1} \geq a_n$.

Replacing $a_1, a_2, \ldots, a_n$ with $1/a_1$, $1/a_2$, $\ldots$, $1/a_n$, respectively, we get:

- *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that $a_1 \leq a_2 \leq \cdots \leq a_n$ and*

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = \frac{n}{p}, \quad a_1 a_2 \cdots a_n = r > 0,$$

where p and r are fixed positive numbers satisfying $p^n \leq r$, then the sums

$$p = a_1 + a_2 + \cdots + a_n, \qquad p_2 = -\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2}\right)$$

are minimal and maximal when $n-1$ of $a_1, a_2, \ldots, a_n$ are equal, more precisely, it is minimal when $a_1 \leq a_2 = \cdots = a_n$, and is maximal when $a_1 = \cdots = a_{n-1} \leq a_n$.

Remark 4. Another extension of P 3.58 is the following:

- *If $0 < m < M$ and $a, b, c \in [m, M]$ such that $a \geq b \geq c$ and*

$$a + b + c = p, \quad abc = r,$$

where p and r are fixed positive numbers satisfying $p^3 \geq 27r$, then the sums

$$q = ab + bc + ca, \qquad q_1 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}, \qquad p_1 = -(a^2 + b^2 + c^2)$$

are minimal when $b = c$ or $a = M$, and are maximal when $a = b$ or $c = m$.

We have $m \leq c_1$ or $c_1 \leq m \leq c_2$, and $M \geq a_2$ or $a_1 \leq M \leq a_2$. Thus, we have

$$c \in [c_1', c_2], \qquad c_1' = \max\{c_1, m\},$$

and

$$a \in [a_1, a_2'], \qquad a_2' = \min\{a_2, M\}.$$

According to Second Solution, the sum $q = ab + bc + ca$ is minimal for $a = a_2'$, when either $b = c$ or $a = M$. Similarly, according to Third Solution, the sum $q = ab + bc + ca$ is maximal for $c = c_1'$, when either $a = b$ or $c = m$.

We can generalize this result as follows (Vasile Cîrtoaje, 2017):

- *If $0 < m < M$ and $a_1, a_2, \ldots, a_n \in [m, M]$ such that $a_1 \geq a_2 \geq \cdots \geq a_n$ and*

$$a_1 + a_2 + \cdots + a_n = np, \quad a_1 a_2 \cdots a_n = r,$$

where p and r are fixed real numbers satisfying $p^n \geq r$, then the sums

$$q_1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}, \qquad p_1 = -(a_1^2 + a_2^2 + \cdots + a_n^2)$$

are minimal for $a_1 \ge a_2 = \cdots = a_n$ or $a_1 = M$, and are maximal for $a_1 = \cdots = a_{n-1} \ge a_n$ or $a_n = m$.

Replacing $a_1, a_2, \ldots, a_n$ with $1/a_1$, $1/a_2$, $\ldots$, $1/a_n$, respectively, we get:

- *If $0 < m < M$ and $a_1, a_2, \ldots, a_n \in [m, M]$ such that $a_1 \le a_2 \le \cdots \le a_n$ and*

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} = \frac{n}{p}, \quad a_1 a_2 \cdots a_n = r,$$

where p and r are fixed real numbers satisfying $p^n \le r$, then the sums

$$p = a_1 + a_2 + \cdots + a_n, \qquad p_2 = -\left(\frac{1}{a_1^2} + \frac{1}{a_2^2} + \cdots + \frac{1}{a_n^2}\right)$$

are minimal for $a_1 \le a_2 = \cdots = a_n$ or $a_1 = m$, and are maximal for $a_1 = \cdots = a_{n-1} \le a_n$ or $a_n = M$.

□

P 3.59. *If a, b, c are positive real numbers such that*

$$a + b + c = 3,$$

then

$$\frac{9}{abc} + 16 \ge \frac{75}{ab + bc + ca}.$$

(Vasile Cîrtoaje, 2005)

Solution. Let

$$q = ab + bc + ca.$$

For fixed q, the product abc is maximal when two of a, b, c are equal - see P 3.57-(a). Therefore, it suffices to prove the inequality for

$$a = b, \quad c = 3 - 2a, \quad a < 3/2.$$

We have

$$\begin{aligned}
\frac{9}{abc} + 16 - \frac{75}{ab + bc + ca} &= \frac{9}{a^2 c} + 16 - \frac{75}{a(a + 2c)} \\
&= \frac{9}{a^2(3 - 2a)} + 16 - \frac{25}{a(2 - a)} \\
&= \frac{2(16a^4 - 56a^3 + 73a^2 - 42a + 9)}{a^2(3 - 2a)(2 - a)} \\
&= \frac{2(a - 1)^2(4a - 3)^2}{a^2(3 - 2a)(2 - a)} \ge 0,
\end{aligned}$$

as desired. The equality holds for $(a, b, c) = (1, 1, 1)$, and also for $(a, b, c) = \left(\frac{3}{4}, \frac{3}{4}, \frac{3}{2}\right)$ or any cyclic permutation.

□

P 3.60. *If a,b,c are positive real numbers such that*

$$a+b+c=3,$$

then

$$8\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+9\geq 10(a^2+b^2+c^2).$$

(Vasile Cîrtoaje, 2006)

First Solution. Putting

$$q=ab+bc+ca,$$

we can write the inequality as

$$\frac{8q}{abc}+20q\geq 81.$$

By P 3.57-(a), the product abc is maximal for fixed q when two of a,b,c are equal. Therefore, it suffices to prove the inequality for

$$a=b,\quad c=3-2a,\quad a<3/2.$$

We have

$$\begin{aligned}\frac{8q}{abc}+20q-81&=\frac{24a(2-a)}{a^2(3-2a)}+60a(2-a)-81\\&=\frac{3a(40a^4-140a^3+174a^2-89a+16)}{a^2(3-2a)}\\&=\frac{3a(2a-1)^2(10a^2-25a+16)}{a^2(3-2a)}.\end{aligned}$$

Since

$$10a^2-25a+16=10\left(a-\frac{5}{4}\right)^2+\frac{3}{8}>0,$$

the proof is completed. The equality holds for $(a,b,c)=\left(\frac{1}{2},\frac{1}{2},2\right)$ or any cyclic permutation.

Second Solution (by *Vo Quoc Ba Can*). It is easy to check that the equality holds when two of a,b,c are $1/2$. Then, let us define

$$f(x)=\frac{8}{x}-10x^2-\alpha x-\beta,$$

such that $(2x-1)^2$ divides $f(x)$. From $f(1/2)=0$, we get $\alpha+2\beta=27$. Therefore,

$$f(x)=\frac{8}{x}-10x^2-(27-2\beta)x-\beta=\frac{(1-2x)h(x)}{x},$$

where

$$h(x) = 5x^2 - (\beta - 16)x + 8.$$

From $h(1/2) = 0$, we get $\beta = 69/2$, then $\alpha = 27 - 2\beta = -42$; therefore,

$$f(x) = \frac{8}{x} - 10x^2 + 42x - \frac{69}{2} = \frac{(1-2x)^2(16-5x)}{2x}.$$

From

$$f(a) + f(b) + f(c) = 8\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 10(a^2 + b^2 + c^2) + \frac{45}{2},$$

it follows that the desired inequality is equivalent to

$$f(a) + f(b) + f(c) \ge \frac{27}{2},$$

$$f(b) + f(c) \ge \frac{27}{2} - f(a).$$

Assume that

$$a = \max\{a, b, c\}, \quad a \ge 1.$$

Since

$$\frac{27}{2} - f(a) = \frac{2(a-2)^2(5a-1)}{a},$$

we can rewrite the inequality as

$$\frac{(1-2b)^2(16-5b)}{b} + \frac{(1-2c)^2(16-5c)}{c} \ge \frac{4(a-2)^2(5a-1)}{a}.$$

Since

$$16 - 5b > 0, \quad 16 - 5c > 0,$$

the Cauchy-Schwarz inequality yields

$$\frac{(1-2b)^2(16-5b)}{b} + \frac{(1-2c)^2(16-5c)}{c} \ge \frac{(1-2b+1-2c)^2}{\dfrac{b}{16-5b} + \dfrac{c}{16-5c}}$$
$$= \frac{4(a-2)^2}{\dfrac{b}{16-5b} + \dfrac{c}{16-5c}}.$$

Therefore, it suffices to prove that

$$\frac{1}{\dfrac{b}{16-5b} + \dfrac{c}{16-5c}} \ge \frac{5a-1}{a},$$

which is equivalent to

$$\frac{a}{5a-1} \ge \frac{b}{16-5b} + \frac{c}{16-5c}.$$

Indeed,

$$\frac{a}{5a-1}-\frac{b}{16-5b}-\frac{c}{16-5c} \geq \frac{a}{5a-1}-\frac{b+c}{16-5a}$$
$$= \frac{3}{(5a-1)(16-5a)} > 0.$$

Remark. Using the second method, we can prove the following more general inequality.

- *If $x_1, x_2, \ldots, x_n$ are positive real numbers such that*

$$x_1 + x_2 + \cdots + x_n = n,$$

then

$$(n+1)^2\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right) \geq n(n^2-3n-6)+4(n+2)(x_1^2+x_2^2+\cdots+x_n^2),$$

with equality for $x_1 = (n+1)/2$ and $x_2 = \cdots = x_n = 1/2$ (or any cyclic permutation).

□

P 3.61. *If a, b, c are positive real numbers such that*

$$a+b+c=3,$$

then

$$7(a^2+b^2+c^2)+8(a^2b^2+b^2c^2+c^2a^2)+4a^2b^2c^2 \geq 49.$$

Solution. Let

$$q = ab+bc+ca.$$

Since

$$a^2+b^2+c^2 = 9-2q,$$
$$a^2b^2+b^2c^2+c^2a^2 = q^2-6abc,$$

we can rewrite the inequality as

$$7(9-2q)+8(q^2-6abc)+4a^2b^2c^2 \geq 49,$$

$$2(6-abc)^2+4q^2-7q-65 \geq 0.$$

Since

$$abc \leq \left(\frac{a+b+c}{3}\right)^3 = 1,$$

we have $6-abc>0$. By P 3.57-(a), the product abc is maximal for fixed q when two of a,b,c are equal. Therefore, it suffices to prove the inequality for $a=b$, when $c=3-2a$. Since

$$q=a^2+2ac=a^2+2a(3-2a)=3a(2-a), \qquad abc=a^2c=a^2(3-2a),$$

we have

$$\begin{aligned}2(6-abc)^2+4q^2-7q-65&=8a^6-24a^5+54a^4-96a^3+83a^2-42a+7\\&=(a-1)^2(2a-1)^2(2a^2+7\geq 0.\end{aligned}$$

The equality holds for $a=b=c=1$, and for $(a,b,c)=\left(\frac{1}{2},\frac{1}{2},2\right)$ or any cyclic permutation.

□

P 3.62. *If a,b,c are nonnegative real numbers, then*

$$(a^3+b^3+c^3+abc)^2\geq 2(a^2+b^2)(b^2+c^2)(c^2+a^2).$$

(Aleksandar Bulj, 2011)

First Solution. Let

$$p=a+b+c, \quad q=ab+bc+ca, \quad r=abc.$$

Using the identities

$$a^3+b^3+c^3=3r+p^3-3pq$$

and

$$\begin{aligned}(a^2+b^2)(b^2+c^2)(c^2+a^2)&=(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)-a^2b^2c^2\\&=(p^2-2q)(q^2-2pr)-r^2,\end{aligned}$$

we can write the required inequality as $f_6(a,b,c)\geq 0$, where

$$f_6(a,b,c)=18r^2+4p(3p^2-8q)r+p^6-6p^4q+7p^2q^2+4q^3.$$

Since

$$3p^2-8q=3(p^2-3q)+q\geq 0,$$

for fixed p and q, f_6 is an increasing function of r. Therefore, it suffices to prove the inequality $f_6(a,b,c)\geq 0$ for the case when r is minimal; that is, when one of a,b,c is zero or two of a,b,c are equal (see P 3.57). For $a=0$ and for $b=c$, the original inequality becomes

$$(b-c)^2(b^4+2b^3c+b^2c^2+2bc^3+c^4)\geq 0$$

and

$$a(a-b)^2(a^3+2a^2b+ab^2+4b^3) \geq 0,$$

respectively. The equality holds for $a=b=c$, and also for $a=0$ and $b=c$ (or any cyclic permutation).

Second Solution. Without loss of generality, assume that $a=\min\{a,b,c\}$. From

$$a(a-b)(a-c) \geq 0,$$

we get

$$a^3+abc \geq a^2(b+c),$$

and hence

$$a^3+b^3+c^3+abc \geq a^2(b+c)+b^3+c^3=(b+c)(a^2+b^2+c^2-bc).$$

On the other hand,

$$2(a^2+b^2)(c^2+a^2) \leq \frac{1}{2}(2a^2+b^2+c^2)^2.$$

Therefore, it suffices to prove that

$$(b+c)^2(a^2+b^2+c^2-bc)^2 \geq \frac{1}{2}(b^2+c^2)(2a^2+b^2+c^2)^2.$$

We can obtain this inequality by multiplying the obvious inequality

$$a^2+b^2+c^2-bc \geq \frac{1}{2}(2a^2+b^2+c^2)$$

and

$$(b+c)^2(a^2+b^2+c^2-bc) \geq (b^2+c^2)(2a^2+b^2+c^2).$$

The last inequality is equivalent to

$$(b-c)^2(bc-a^2) \geq 0,$$

which is also true.

Remark. Using the first method, we can prove the following stronger inequality (*Vasile Cîrtoaje*, 2011).

- *If a,b,c are nonnegative real numbers then*

$$(a^3+b^3+c^3+abc)^2 \geq 2(a^2+b^2)(b^2+c^2)(c^2+a^2)+7(a-b)^2(b-c)^2(c-a)^2.$$

Since

$$(a-b)^2(b-c)^2(c-a)^2=-27r^2+2(9pq-2p^3)r+p^2q^2-4q^3,$$

we can write this inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = 207r^2 + 2(20p^3 - 79pq)r + p^6 - 6p^4q + 32q^3.$$

We will show that for fixed p and q, f_6 is an increasing function of r. From

$$f_6 = 27r^2 + 2pqr + 20g(r) + p^6 - 6p^4q + 32q^3,$$

where

$$g(r) = 9r^2 + 2(p^3 - 4pq)r,$$

it suffices to show that $g(r)$ is increasing. Indeed, by the third degree Schur's inequality, we have

$$g'(r) = 2(9r + p^3 - 4pq) \ge 0.$$

Therefore, it suffices to prove the inequality $f_6(a,b,c) \ge 0$ for the case when r is minimal; that is, when $a = 0$ or $b = c$ (see P 3.57). In these cases, the original inequality becomes

$$(b-c)^4(b^2 + 4bc + c^2) \ge 0$$

and

$$a(a-b)^2(a^3 + 2a^2b + ab^2 + 4b^3) \ge 0,$$

respectively.

□

P 3.63. *If a,b,c are positive real numbers, then*

$$(a+b+c-3)(ab+bc+ca-3) \ge 3(abc-1)(a+b+c-ab-bc-ca).$$

(Vasile Cîrtoaje, 2011)

Solution. Setting

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc,$$

the inequality becomes

$$(p-3)(q-3) \ge 3(r-1)(p-q),$$

or

$$3r(q-p) + pq - 6q + 9 \ge 0.$$

First Solution. For fixed p and q, the linear function $f(r) = 3r(q-p) + pq - 6q + 9$ is minimal when r is either minimal or maximal. Thus, according to P 3.57-(a), we need only to prove that $f(r) \ge 0$ for $a = 0$ and for $b = c$.

For $a = 0$, the inequality becomes

$$(b+c)bc - 6bc + 9 \geq 0.$$

Putting $x = \sqrt{bc}$, we have

$$(b+c)bc - 6bc + 9 \geq 2x^3 - 6x^2 + 9 = 2(x+1)(x-2)^2 + 1 > 0.$$

For $b = c$, since $p = a + 2b$, $q = 2ab + b^2$ and $r = ab^2$, we need to show that

$$3ab^2(2ab + b^2 - a - 2b) + (a + 2b - 6)(2ab + b^2) + 9 \geq 0;$$

that is,

$$Aa^2 + Ba + C \geq 0,$$

where

$$A = b(6b^2 - 3b + 2), \quad B = b(3b^3 - 6b^2 + 5b - 12), \quad C = 2b^3 - 6b^2 + 9.$$

We have $A > 0$ and

$$C = 2(b^3 - 3b^2 + 4) + 1 = 2(b+1)(b-2)^2 + 1 > 0.$$

Consider two cases.

Case 1: $b \geq 12/5$. Since

$$B = 3b^2(b-2) + b(5b-12) > 0,$$

we have $Aa^2 + Ba + C > 0$.

Case 2: $0 < b < 12/5$. Since

$$Aa^2 + Ba + C = (Aa^2 + C) + Ba \geq a(2\sqrt{AC} + B),$$

it suffices to show that $4AC \geq B^2$, which is equivalent to

$$4b(6b^2 - 3b + 2)(2b^3 - 6b^2 + 9) \geq b^2(3b^3 - 6b^2 + 5b - 12)^2,$$

$$b(b-1)^4(8 + 4b - b^3) \geq 0.$$

This inequality is true since

$$8 + 4b - b^3 = 8 + 4b - 3b^2 + b^2(3-b) > 8 + 4b - 3b^2 > 0.$$

The equality holds for $a = b = c = 1$.

Second Solution. Consider the following two cases.

Case 1: $p \geq q$. We have

$$3r(q-p) + pq - 6q + 9 = (q - 3r)(p - q) + (q-3)^2 \geq 0$$

since

$$q-3r \ge q-\frac{q^2}{p} \ge 0.$$

Case 2: $p \le q$. For $p \ge 6$, we have

$$3r(q-p)+pq-6q+9 = 3r(q-p)+q(p-6)+9 > 0.$$

Consider further that $p < 6$. From $p^2 \ge 3q \ge 3p$, we get $p \ge 3$. From Schur's inequality

$$p^3+9r \ge 4pq,$$

we get

$$p^3+p^2r \ge 4pq,$$

hence

$$pr \ge 4q-p^2.$$

Using this result, we have

$$\begin{aligned} p[3r(q-p)+pq-6q+9)] &\ge 3(4q-p^2)(q-p)+p(pq-6q+9) \\ &= 12q^2-2p(p+9)q+3p(p^2+3) \\ &= 12\left(q-\frac{p^2+9p}{12}\right)^2+\frac{p(12-p)(p-3)^2}{12} \ge 0. \end{aligned}$$

□

P 3.64. *Let a,b,c be nonnegative real numbers such that*

$$ab+bc+ca=3.$$

Prove that

$$4(a^3+b^3+c^3)+7abc+125 \ge 48(a+b+c).$$

(Vasile Cîrtoaje, 2011)

Solution. Since

$$a^3+b^3+c^3 = 3abc+(a+b+c)^3-9(a+b+c),$$

we can write the inequality as

$$19abc+4(a+b+c)^3-84(a+b+c)+125 \ge 0.$$

As it is known, for fixed $a+b+c$, the product abc is minimal when $a=0$ or $b=c$ (see P 3.57). Therefore it suffices to consider these cases.

Case 1: $a = 0$. We need to show that $bc = 3$ yields

$$4(b^3 + c^3) + 125 \ge 48(b + c).$$

Since

$$b^3 + c^3 = (b+c)^3 - 3bc(b+c) = (b+c)^3 - 9(b+c)$$

and

$$b + c \ge 2\sqrt{bc} = 2\sqrt{3},$$

we have

$$4(b^3 + c^3) + 125 - 48(b+c) = 4(b+c)^3 - 84(b+c) + 125 > 0.$$

Case 2: $b = c$. We need to show that $2ab + b^2 = 3$ yields

$$4(a^3 + 2b^3) + 7ab^2 + 125 \ge 48(a + 2b).$$

This inequality is equivalent to

$$8b^6 - 114b^4 + 250b^3 - 171b^2 + 27 \ge 0,$$

$$(b-1)^2(2b-3)^2(2b^2 + 10b + 3) \ge 0.$$

The equality holds for $a = b = c = 1$, and also for $a = 1/4$ and $b = c = 3/2$ (or any cyclic permutation).

□

P 3.65. *If $a, b, c \in [0, 1]$, then*

(a) $$a\sqrt{a} + b\sqrt{b} + c\sqrt{c} + 4abc \ge 2(ab + bc + ca);$$

(b) $$a\sqrt{a} + b\sqrt{b} + c\sqrt{c} \ge \frac{3}{2}(ab + bc + ca - abc);$$

(c) $$3(a\sqrt{a} + b\sqrt{b} + c\sqrt{c}) + \frac{500}{81}abc \ge 5(ab + bc + ca).$$

(Vasile Cîrtoaje, 2012)

Solution. Consider the inequality

$$a\sqrt{a} + b\sqrt{b} + c\sqrt{c} + kabc \ge m(ab + bc + ca), \quad k, m > 0.$$

This inequality is equivalent to

$$x^3 + y^3 + z^3 + kx^2y^2z^2 \ge m(x^2y^2 + y^2z^2 + z^2x^2),$$

where

$$x, y, z \in [0, 1].$$

In addition, using the notation

$$p = x + y + z, \quad q = xy + yz + zx,$$

we can rewrite the inequality as

$$kx^2y^2z^2 + (2mp+3)xyz + p^3 - 3pq - mq^2 \ge 0.$$

Consider $x \le y \le z$. According to Remark 1 from P 3.57, for fixed p and q, the product xyz is minimal when $x = 0$ or $y = z$. Therefore, it suffices to consider these cases.

(a) We need to show that

$$x^3 + y^3 + z^3 + 4x^2y^2z^2 \ge 2(x^2y^2 + y^2z^2 + z^2x^2).$$

For $x = 0$, the required inequality becomes $y^3 + z^3 \ge 2y^2z^2$. Indeed, we have

$$y^3 + z^3 - 2y^2z^2 \ge y^4 + z^4 - 2y^2z^2 = (y^2 - z^2)^2 \ge 0.$$

For $y = z$, write the inequality as $f(x) \ge 0$, $x \in [0,1]$, where

$$f(x) = x^3 - 4x^2y^2(1-y^2) + 2y^3(1-y), \qquad y \in [0,1].$$

If $y \in \{0,1\}$, we have $f(x) = x^3 \ge 0$. Consider further that $y \in (0,1)$. From

$$f'(x) = x[3x - 8y^2(1-y^2)],$$

it follows that $f(x)$is decreasing on $[0, x_1]$ and increasing on $[x_1, 1]$, where

$$x_1 = \frac{8y^2(1-y^2)}{3}.$$

Since

$$y^2(1-y^2) \le \frac{1}{4},$$

we have

$$x_1 \in \left(0, \frac{2}{3}\right].$$

Thus, it remains to show that $f(x_1) \ge 0$, which is equivalent to

$$128y^3(1-y)^2(1+y)^3 \le 27.$$

Since $y^2(1-y^2) \le 1/4$, it suffices to show that

$$32y(1-y)(1+y)^2 \le 27.$$

Using the AM-GM inequality, we get

$$32y(1-y)(1+y)^2 = 1024 \cdot \frac{y}{2}(1-y)\left(\frac{1+y}{4}\right)^2$$
$$\leq 1024 \left[\frac{\frac{y}{2}+(1-y)+2\cdot\frac{1+y}{4}}{4}\right]^4 = \frac{81}{4} < 27.$$

The equality holds for $a = 0$ and $b = c = 1$ (or any cyclic permutation), and also for $a = b = c = 0$.

(b) We need to show that

$$2(x^3+y^3+z^3)+3x^2y^2z^2 \geq 3(x^2y^2+y^2z^2+z^2x^2).$$

For $x = 0$, the required inequality becomes $2(y^3+z^3) \geq 3y^2z^2$. Indeed, we have

$$2(y^3+z^3)-3y^2z^2 \geq 2(y^4+z^4)-4y^2z^2 = 2(y^2-z^2)^2 \geq 0.$$

For $y = z$, write the inequality as $f(x) \geq 0$, $x \in [0,1]$, where

$$f(x) = 2x^3-3x^2y^2(2-y^2)+y^3(4-3y), \quad y \in [0,1].$$

If $y = 0$, we have $f(x) = x^3 \geq 0$. Consider further that $y \in (0,1]$. From

$$f'(x) = 6x[x-y^2(2-y^2)],$$

it follows that $f(x)$is decreasing on $[0, x_1]$ and increasing on $[x_1, 1]$, where

$$x_1 = y^2(2-y^2), \quad x_1 \in (0,1].$$

Therefore, we only need to show that $f(x_1) \geq 0$, which is equivalent to

$$y^3(2-y^2)^3 \leq 4-3y.$$

Indeed, since

$$y^2(2-y^2) \leq 1,$$

we have

$$y^3(2-y^2)^3-(4-3y) \leq y(2-y^2)^2-(4-3y)$$
$$= (y-1)^2(y^3+2y^2-y-4) \leq 0.$$

The equality occurs for $a = b = c = 1$, and also for $a = b = c = 0$.

(c) We need to show that

$$3(x^3+y^3+z^3)+\frac{500}{81}x^2y^2z^2 \geq 5(x^2y^2+y^2z^2+z^2x^2),$$

For $x=0$, the required inequality becomes $3(y^3+z^3)\ge 5y^2z^2$. Indeed, we have

$$3(y^3+z^3)-5y^2z^2 \ge 3(y^4+z^4)-6y^2z^2 = 3(y^2-z^2)^2 \ge 0.$$

For $y=z$, write the inequality as $f(x)\ge 0$, $x\in[0,1]$, where

$$f(x)=3x^3-10x^2y^2\left(1-\frac{50}{81}y^2\right)+y^3(6-5y), \quad y\in[0,1].$$

If $y=0$, we have $f(x)=3x^3\ge 0$. For $y\in(0,1]$, from

$$f'(x)=x\left[9x-20y^2\left(1-\frac{50}{81}y^2\right)\right],$$

it follows that $f(x)$ is decreasing on $[0,x_1]$ and increasing on $[x_1,1]$, where

$$x_1=\frac{20}{9}y^2\left(1-\frac{50}{81}y^2\right), \quad x_1\in(0,1).$$

Therefore, it remains to show that $f(x_1)\ge 0$, which is equivalent to

$$\frac{4000}{243}y^3\left(1-\frac{50}{81}y^2\right)^3 \le 6-5y.$$

Substituting

$$y=\frac{9t}{10}, \quad 0<t\le\frac{10}{9},$$

this inequality can be written as

$$t^3(2-t^2)^3\le 4-3t,$$

Indeed, since

$$t^2(2-t^2)\le 1,$$

we have

$$\begin{aligned} t^3(2-t^2)^3-(4-3t) &\le t(2-t^2)^2-(4-3t) \\ &= (t-1)^2(t^3+2t^2-t-4)\le 0. \end{aligned}$$

The equality holds for $a=b=c=0$, and also for $a=b=c=81/100$.

□

P 3.66. *If*

$$a,b,c\ge\frac{13-4\sqrt{10}}{3}\approx 0.117$$

such that $a+b+c=9$, then

$$\sqrt{a}+\sqrt{b}+\sqrt{c}\ge\sqrt{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2018)

Solution. Denote

$$k = \frac{13 - 4\sqrt{10}}{3}$$

and

$$\sqrt{a} = x, \quad \sqrt{b} = y, \quad \sqrt{c} = z.$$

We need to show that

$$x + y + z \ge \sqrt{(xy + yz + zx)^2 - 2xyz(x + y + z)}$$

for $x \ge y \ge z \ge \sqrt{k}$ and $(x + y + z)^2 - 2(xy + yz + zx) = 9$. According to Remark 2 from P 3.57, for fixed $x + y + z$ and $xy + yz + zx$, the product xyz is minimal for $x = y \ge z$ or $z = \sqrt{k}$. So, we only need to consider these cases. This means to consider the cases $a = b \ge c \ge k$ and $a \ge b \ge c = k$.

Case 1: $a = b \ge c \ge k$. We need to show that $2a + c = 9$ involves

$$2\sqrt{a} + \sqrt{c} \ge \sqrt{a^2 + 2ac},$$

that is

$$2\sqrt{2(9 - c)} + 2\sqrt{c} \ge \sqrt{3(9 - c)(3 + c)}.$$

By squaring, the inequality becomes

$$8\sqrt{2c(9 - c)} \ge -3c^2 + 22c + 9.$$

This is true if

$$128c(9 - c) \ge (-3c^2 + 22c + 9)^2,$$

which is equivalent to

$$3c^4 - 44c^3 + 186c^2 - 252c + 27 \le 0,$$

$$(c - 3)^2(3c^2 - 26c + 3) \le 0.$$

Since $3c^2 - 26c + 3 \le 0$ for $c \in [k, 3]$, the inequality is true.

Case 2: $a \ge b \ge c = k$. We need to show that

$$\sqrt{a} + \sqrt{b} + \sqrt{k} \ge \sqrt{ab + k(a + b)},$$

that is

$$\sqrt{a + b + 2\sqrt{ab}} + \sqrt{k} \ge \sqrt{ab + k(a + b)},$$

$$\sqrt{9 - k + 2\sqrt{ab}} + \sqrt{k} \ge \sqrt{ab + k(9 - k)}.$$

From

$$(a - c)(b - c) \ge 0,$$

we get

$$ab \ge c(a + b) - c^2 = c(9 - 2c) = k(9 - 2k) > 1.$$

Substituting $\sqrt{ab} = x$, we need to show that $f(x) \ge 0$ for

$$1 < x \le \frac{a+b}{2} = \frac{9-k}{2} \approx 4.4415,$$

where

$$f(x) = \sqrt{2x+9-k} + \sqrt{k} - \sqrt{x^2+k(9-k)},$$

with

$$f'(x)\frac{1}{\sqrt{2x+9-k}} - \frac{x}{\sqrt{x^2+k(9-k)}}.$$

We show that $f'(x) < 0$, therefore f is decreasing. This is true if

$$x^2(2x+9-k) > x^2 + k(9-k),$$

that is

$$2x^3 + (8-k)x^2 - k(9-k) > 0.$$

Indeed,

$$2x^3 + (8-k)x^2 - k(9-k) > x^2 + (8-k)x^2 - k(9-k) = (9-k)(x^2-k) > 0.$$

So, we have

$$f(x) \ge f\left(\frac{9-k}{2}\right) = 0.$$

The inequality is an equality for $a = b = c = 3$, and also for $a = b = \dfrac{7+2\sqrt{10}}{3}$ and $c = \dfrac{13-4\sqrt{10}}{3}$ (or any cyclic permutation).

□

P 3.67. *Let a, b, c be the lengths of the sides of a triangle. If*

$$a^2 + b^2 + c^2 = 3,$$

then

$$a + b + c \ge 2 + abc.$$

(Vasile Cîrtoaje, 2005)

Solution. Let

$$p = a + b + c, \qquad q = ab + bc + ca.$$

We need to show that

$$p^2 - 2q = 3$$

involves

$$p \ge 2 + abc.$$

First Solution. According to P 3.57-(b), for fixed p and q, the product abc is maximal when $c/2 \le a = b \le c$ or $a + b = c$. Therefore, it suffices to consider only these two cases.

Case 1: $c/2 \le a = b \le c$. From

$$3 = 2b^2 + c^2 \ge 3b^2,$$

we get $b \le 1$. In addition, from

$$3 = 2b^2 + c^2 \le 2b^2 + 4b^2 = 6b^2,$$

it follows that $2b^2 \ge 1$. Therefore, we need to prove that

$$2b^2 + c^2 = 3, \quad b \le 1, \quad 2b^2 \ge 1$$

involve

$$2b + c \ge 2 + b^2c.$$

Since

$$2b + c - 2 - b^2c = (1 - b)(c + bc - 2),$$

it suffices to show that $c(1 + b) \ge 2$. This is true, since

$$\begin{aligned} c^2(1+b)^2 - 4 &= (3 - 2b^2)(1+b)^2 - 4 \\ &= -1 + 6b + b^2 - 4b^3 - 2b^4 \\ &= (1 - b)(-1 + 5b + 6b^2 + 2b^3) \ge 0. \end{aligned}$$

Case 2: $a + b = c$. From $a^2 + b^2 + c^2 = 3$, we get

$$2ab = 2c^2 - 3, \quad c^2 \ge 3/2.$$

In addition, from $4ab \le c^2$, we get $c^2 \le 2$, and hence

$$3/2 \le c^2 \le 2.$$

Since

$$a + b + c - 2 - abc = 2c - 2 - c(c^2 - \frac{3}{2}) = \frac{-2c^3 + 7c - 4}{2},$$

we need to show that

$$2c^3 - 7c + 4 \le 0.$$

From

$$(c^2 - 2)(2c^2 - 3) \le 0,$$

we get

$$2c^4 - 7c^2 \le -6.$$

Therefore,

$$c(2c^3 - 7c + 4) \le -6 + 4c < 0.$$

This completes the proof. The equality holds for $a=b=c=1$.

Second Solution. Since

$$pq \ge 9abc,$$

it suffices to show that $p^2-2q=3$ implies

$$9p \ge 18+pq;$$

that is,

$$18p \ge 36+p(p^2-3),$$

$$p^3-21p+36 \le 0,$$

$$(p-3)(p^2+3p-12) \le 0.$$

Since

$$p^2 \le 3(a^2+b^2+c^2)=9,$$

we need to show that

$$p^2+3p-12 \ge 0.$$

From

$$2(ab+bc+ca)-a^2-b^2-c^2 =$$

$$=(\sqrt{a}+\sqrt{b}+\sqrt{c})(-\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c}) \ge 0,$$

we get

$$p^2 \ge 2(a^2+b^2+c^2)=6,$$

hence

$$p^2+3p-12 \ge 6+3\sqrt{6}-12 > 0.$$

□

P 3.68. *Let $f_n(a,b,c)$ be a symmetric homogeneous polynomial of degree $n \le 5$. Prove that*

(a) the inequality $f_n(a,b,c) \ge 0$ holds for all nonnegative real numbers a,b,c if and only if $f_n(a,1,1) \ge 0$ and $f_n(0,b,c) \ge 0$ for all $a,b,c \ge 0$;

(b) the inequality $f_n(a,b,c) \ge 0$ holds for all lengths a,b,c of the sides of a non-degenerate or degenerate triangle if and only if $f_n(x,1,1) \ge 0$ for $0 \le x \le 2$, and $f_n(y+z,y,z) \ge 0$ for all $y,z \ge 0$.

(Vasile Cîrtoaje, 2005)

Solution. Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

Any symmetric homogeneous polynomial $f_n(a, b, c)$ of degree $n \le 5$ can be written as

$$f_n(a, b, c) = A_n(p, q)r + B_n(p, q),$$

where $A_n(p, q)$ and $B_n(p, q)$ are polynomial functions. For fixed p and q, the linear function

$$g_n(r) = A_n(p, q)r + B_n(p, q)$$

is minimal when r is either minimal or maximal.

(a) By P 3.57-(a), for fixed p and q, the product r is minimal and maximal when two of a, b, c are equal or one of a, b, c is 0. Due to symmetry and homogeneity, the conclusion follows.

(b) By P 3.57-(b), for fixed p and q, the product r is minimal and maximal when two of a, b, c are equal or one of a,b,c is the sum of the others. Due to symmetry and homogeneity, the conclusion follows.

Remark. Similarly, we can prove the following statement, which does not involve the homogeneity property.

- *Let $f_n(a, b, c)$ be a symmetric polynomial function of degree $n \le 5$. The inequality $f_n(a, b, c) \ge 0$ holds for all nonnegative real numbers a, b, c if and only if it holds for $a = 0$ and for $b = c$.*

□

P 3.69. *If a, b, c are nonnegative real numbers such that*

$$a + b + c = 3,$$

then

$$4(a^4 + b^4 + c^4) + 45 \ge 19(a^2 + b^2 + c^2).$$

(Vasile Cîrtoaje, 2009)

First Solution. We use the mixing variables method. Write the inequality as

$$F(a, b, c) \ge 0,$$

where

$$F(a, b, c) = 4(a^4 + b^4 + c^4) + 45 - 19(a^2 + b^2 + c^2).$$

Due to symmetry, we may assume that $a \le b \le c$. Let us denote

$$x = \frac{b + c}{2}, \quad 1 \le x \le 3/2.$$

We will show that

$$F(a,b,c) \ge F(a,x,x) \ge 0.$$

We have

$$F(a,b,c) - F(a,x,x) = 4(b^4 + b^4 - 2x^4) - 19(b^2 + c^2 - 2x^2)$$

$$= 4[(b^2 + c^2)^2 - 4x^4] + 8(x^4 - b^2c^2) - 19(b^2 + c^2 - 2x^2)$$

$$= (b^2 + c^2 - 2x^2)[4(b^2 + c^2 + 2x^2) - 19] + 8(x^2 - bc)(x^2 + bc).$$

Since

$$b^2 + c^2 - 2x^2 = 2(x^2 - bc) = \frac{1}{2}(b-c)^2,$$

we get

$$F(a,b,c) - F(a,x,x) = \frac{1}{2}(b-c)^2[4(b^2 + c^2 + 2x^2) - 19 + 4(x^2 + bc)]$$

$$= \frac{1}{2}(b-c)^2[4(x^2 - bc) + 24x^2 - 19] \ge 0.$$

Also,

$$F(a,x,x) = F(3-2x,x,x) = 6(x-1)^2(3-2x)(11-6x) \ge 0.$$

This completes the proof. The equality holds for $a = b = c = 1$, and for $a = 0$ and $b = c = 3/2$ (or any cyclic permutation).

Second Solution. Write the inequality in the homogeneous form $f_4(a,b,c) \ge 0$, where

$$f_4(a,b,c) = 36(a^4 + b^4 + c^4) + 5(a+b+c)^4 - 19(a^2 + b^2 + c^2)(a+b+c)^2.$$

According to P 3.68-(a), it suffices to prove that $f_4(a,1,1) \ge 0$ and $f_4(0,b,c) \ge 0$ for all $a,b,c \ge 0$. We have

$$f_4(a,1,1) = 2a(11a^3 - 18a^2 + 3a + 4) = 2a(a-1)^2(11a + 4) \ge 0,$$

$$f_4(0,b,c) = 2(b-c)^2(11b^2 + 11c^2 + 13bc).$$

Remark. Similarly, we can prove the following more general statement (*Vasile Cîrtoaje* and *Le Huu Dien Khue*, 2008).

- *Let α, β, γ be real numbers such that*

$$1 + \alpha + \beta = 2\gamma.$$

The inequality

$$\sum a^4 + \alpha \sum a^2b^2 + \beta abc \sum a \ge \gamma \sum ab(a^2 + b^2)$$

holds for all $a,b,c \ge 0$ if and only if

$$\alpha \ge (\gamma - 1)\max\{2, \gamma + 1\}.$$

□

P 3.70. *Let a,b,c be nonnegative real numbers. If $k \le 2$, then*

$$\sum a(a-b)(a-c)(a-kb)(a-kc) \ge 0.$$

(Vasile Cîrtoaje, 2008)

Solution. Let us denote

$$f_5(a,b,c) = \sum a(a-b)(a-c)(a-kb)(a-kc).$$

By P 3.68-(a), it suffices to show that $f_5(a,1,1) \ge 0$ and $f_5(0,b,c) \ge 0$ for all $a,b,c \ge 0$. Indeed, we have

$$f_5(a,1,1) = a(a-1)^2(a-k)^2 \ge 0$$

and

$$f_5(0,b,c) = (b+c)(b-c)^2(b^2-kbc+c^2) \ge 0.$$

The equality holds for $a=b=c$, for $a=0$ and $b=c$ (or any cyclic permutation), and for $a/k=b=c$, $k>0$ (or any cyclic permutation).

□

P 3.71. *Let a,b,c be nonnegative real numbers. If $k \in \mathbf{R}$, then*

$$\sum (b+c)(a-b)(a-c)(a-kb)(a-kc) \ge 0.$$

(Vasile Cîrtoaje, 2008)

Solution. Let

$$f_5(a,b,c) = \sum (b+c)(a-b)(a-c)(a-kb)(a-kc).$$

By P 3.68-(a), it suffices to show that $f_5(a,1,1) \ge 0$ and $f_5(0,b,c) \ge 0$ for all $a,b,c \ge 0$. Indeed, we have

$$f_5(a,1,1) = 2(a-1)^2(a-k)^2 \ge 0$$

and

$$f_5(0,b,c) = k^2(b+c)b^2c^2 + bc(b+c)(b-c)^2 \ge 0.$$

The equality holds for $a=b=c$, for $b=c=0$ (or any cyclic permutation), and for $a/k=b=c$, $k>0$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *Let a,b,c be nonnegative real numbers. If $m \ge 0$ and $m(k-2) \le 1$, then*

$$\sum (ma+b+c)(a-b)(a-c)(a-kb)(a-kc) \ge 0.$$

□

P 3.72. *If a, b, c are nonnegative real numbers, then*

$$\sum a(a-2b)(a-2c)(a-5b)(a-5c) \geq 0.$$

(Vasile Cîrtoaje, 2008)

Solution. Let

$$f_5(a,b,c) = \sum a(a-2b)(a-2c)(a-5b)(a-5c).$$

By P 3.68-(a), it suffices to show that $f_5(a,1,1) \geq 0$ and $f_5(0,b,c) \geq 0$ for all $a,b,c \geq 0$. Indeed, we have

$$f_5(a,1,1) = a^3(a-7)^2 + 20a^3 - 60a^2 + 44a + 8$$
$$\geq 20a^3 - 60a^2 + 44a + 8 \geq 0,$$

since

$$20a^3 - 60a^2 + 44a + 8 > 20a^2(a-3) \geq 0$$

for $a \geq 3$, and

$$20a^3 - 60a^2 + 44a + 8 = 5(2a-3)^2 + 8 - a \geq 8 - a \geq 0$$

for $a \leq 8$. Also,

$$f_5(0,b,c) = (b+c)(b^2 - 4bc + c^2)^2 \geq 0.$$

The equality holds for $a = 0$ and $b^2 - 4bc + c^2 = 0$ (or any cyclic permutation).

□

P 3.73. *If a, b, c are the lengths of the side of a triangle, then*

$$a^4 + b^4 + c^4 + 9abc(a+b+c) \leq 10(a^2b^2 + b^2c^2 + c^2a^2).$$

First Solution. Let

$$f_4(a,b,c) = 10(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4 - 9abc(a+b+c).$$

By P 3.68-(b), it suffices to show that $f_4(x,1,1) \geq 0$ for $0 \leq x \leq 2$ and $f_4(y+z,y,z) \geq 0$ for $y,z \geq 0$. We have

$$f_4(x,1,1) = 8 - 18x + 11x^2 - x^4 = (2-x)(4+x)(1-x)^2 \geq 0$$

and

$$f_4(y+z,y,z) = 8(y^2+z^2)^2 - 2yz(y^2+z^2) - 28y^2z^2$$
$$= 2(y-z)^2(4y^2 + 4z^2 + 7yz) \geq 0.$$

The equality holds for an equilateral triangle and for a degenerate triangle with $a/2 = b = c$ (or any cyclic permutation).

Second Solution. We use the sum-of-squares method (SOS method). Write the inequality as follows

$$9(\sum b^2c^2 - abc\sum a) - (\sum a^4 - \sum b^2c^2) \geq 0,$$

$$9\sum a^2(b-c)^2 - \sum(b^2-c^2)^2 \geq 0,$$

$$\sum(b-c)^2(3a-b-c)(3a+b+c) \geq 0.$$

Without loss of generality, assume that $a \geq b \geq c$. Since

$$(b-c)^2(3a-b-c)(3a+b+c) \geq 0,$$

it suffices to show that

$$(c-a)^2(3b-c-a)(3b+c+a) + (a-b)^2(3c-a-b)(3c+a+b) \geq 0.$$

Since

$$3b-c-a \geq 2b-a \geq b+c-a \geq 0$$

and $(c-a)^2 \geq (a-b)^2)$, it is enough to prove that

$$(3b-c-a)(3b+c+a) + (3c-a-b)(3c+a+b) \geq 0.$$

We have

$$(3b+c+a) - (3c+a+b) = 2(b-c) \geq 0,$$

hence

$$\begin{aligned}(3b-c-a)(3b+c+a) + (3c-a-b)(3c+a+b) &\geq \\ &\geq (3b-c-a)(3c+a+b) + (3c-a-b)(3c+a+b) \\ &= 2(b+c-a)(3c+a+b) \geq 0.\end{aligned}$$

□

P 3.74. *If a, b, c are the lengths of the sides of a triangle, then*

$$3(a^4+b^4+c^4) + 7abc(a+b+c) \leq 5\sum ab(a^2+b^2).$$

Solution. Let

$$f_4(a,b,c) = 5\sum ab(a^2+b^2) - 3(a^4+b^4+c^4) - 7abc(a+b+c).$$

By P 3.68-(b), it suffices to show that $f_4(x,1,1) \ge 0$ for $0 \le x \le 2$ and $f_4(y+z,y,z) \ge 0$ for $y,z \ge 0$. Indeed,

$$f_4(x,1,1) = 4 - 4x - 7x^2 + 10x^3 - 3x^4 = (2-x)(2+3x)(1-x)^2 \ge 0$$

and

$$\begin{aligned} f_4(y+z,y,z) &= 4(y^2+z^2)^2 + 4yz(y^2+z^2) - 24y^2z^2 \\ &= 4(y-z)^2(y^2+z^2+3yz) \ge 0. \end{aligned}$$

The equality holds for an equilateral triangle and for a degenerate triangle with $a/2 = b = c$ (or any cyclic permutation).

□

P 3.75. *If a,b,c are the lengths of the sides of a triangle, then*

$$\frac{b^2+c^2-6bc}{a} + \frac{c^2+a^2-6ca}{b} + \frac{a^2+b^2-6ab}{c} + 4(a+b+c) \le 0.$$

(Vasile Cîrtoaje, 2005)

First Solution. Write the inequality as $f_4(a,b,c) \ge 0$, where

$$f_4(a,b,c) = \sum bc(6bc - b^2 - c^2) - 4abc(a+b+c).$$

By P 3.68-(b), it suffices to show that $f_4(x,1,1) \ge 0$ for $0 \le x \le 2$ and $f_4(y+z,y,z) \ge 0$ for $y,z \ge 0$. Since

$$f_4(x,1,1) = 2(2 - 5x + 4x^2 - x^3) = 2(1-x)^2(2-x) \ge 0$$

and

$$\begin{aligned} f_4(y+z,y,z) &= 4(y^2+z^2)^2 - 2yz(y^2+z^2) - 12y^2z^2 \\ &= 2(y-z)^2(2y^2+3yz+2z^2) \ge 0, \end{aligned}$$

the proof is completed. The equality holds for an equilateral triangle and for a degenerate triangle with $a/2 = b = c$ (or any cyclic permutation).

Second Solution. We use the SOS method. Write the inequality as follows:

$$\sum bc(b^2+c^2-6bc) + 4abc\sum a \le 0,$$

$$\sum bc(b^2+c^2-2bc) - 4(\sum b^2c^2 - abc\sum a) \le 0,$$

$$\sum bc(b-c)^2 - 2\sum a^2(b-c)^2 \le 0,$$
$$\sum (b-c)^2(2a^2-bc) \ge 0.$$

Without loss of generality, assume that $a \ge b \ge c$. Since $(b-c)^2(2a^2-bc) \ge 0$, it suffices to prove that

$$(c-a)^2(2b^2-ca) + (a-b)^2(2c^2-ab) \ge 0.$$

Since
$$2b^2 - ca \ge 2b^2 - c(b+c) = (b-c)(2b+c) \ge 0$$
and $(c-a)^2 \ge (a-b)^2$, it is enough to show that

$$(2b^2-ca) + (2c^2-ab) \ge 0.$$

Indeed,

$$(2b^2-ca) + (2c^2-ab) = (b-c)^2 + (b+c)(b+c-a) \ge 0.$$

□

P 3.76. *Let $f_6(a,b,c)$ be a sixth degree symmetric homogeneous polynomial written in the form*
$$f_6(a,b,c) = Ar^2 + B(p,q)r + C(p,q), \quad A \le 0,$$
where
$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc.$$
Prove that

(a) the inequality $f_6(a,b,c) \ge 0$ holds for all nonnegative real numbers a,b,c if and only if $f_6(a,1,1) \ge 0$ and $f_6(0,b,c) \ge 0$ for all $a,b,c \ge 0$;

(b) the inequality $f_6(a,b,c) \ge 0$ holds for all for all lengths a,b,c of the sides of a non-degenerate or degenerate triangle if and only if $f_6(x,1,1) \ge 0$ for $0 \le x \le 2$, and $f_6(y+z,y,z) \ge 0$ for all $y,z \ge 0$.

(Vasile Cîrtoaje, 2006)

Solution. For fixed p and q, the function f defined by

$$f(r) = Ar^2 + B(p,q)r + C(p,q)$$

is a quadratic concave function of r. Therefore, $f(r)$ is minimal when r is minimal or maximal. According to P 3.57, the conclusion follows. As we have shown in the proof of P 2.75, A is called the *highest coefficient* of $f_6(a,b,c)$.

Remark 1. We can extend the part (a) of P 3.76 as follows:

(a1) *For $A \le 0$, the inequality $f_6(a,b,c) \ge 0$ holds for all nonnegative real numbers a,b,c satisfying $p^2 \le 4q$ if and only if $f_6(a,1,1) \ge 0$ for all $0 \le a \le 4$;.*

(a2) *For $A \le 0$, the inequality $f_6(a,b,c) \ge 0$ holds for all nonnegative real numbers a,b,c satisfying $p^2 > 4q$ if and only if $f_6(a,1,1) \ge 0$ for all $a > 4$ and $f_6(0,b,c) \ge 0$ for all $b,c \ge 0$.*

Notice that the restriction $0 \le a \le 4$ in (a1) follows by setting $b = c = 1$ in $p^2 \le 4q$. In addition, the condition $f_6(0,b,c) \ge 0$ is not necessary in (a1) since $a = 0$ and $p^2 \le 4q$ involve $b = c$; therefore, the condition $f_6(0,b,c) \ge 0$ is equivalent to $f_6(0,1,1) \ge 0$, which follows from $f_6(a,1,1) \ge 0$ for all $0 \le a \le 4$.

Also, the restriction $a > 4$ in (a2) follows by setting $b = c = 1$ in $p^2 > 4q$.

Remark 2. The statement in P 3.76 and its extension in Remark 1 are also valid in the more general case when $f_6(a,b,c)$ is a symmetric homogeneous function of the form

$$f_6(a,b,c) = Ar^2 + B(p,q)r + C(p,q),$$

where $B(p,q)$ and $C(p,q)$ are rational functions.

□

P 3.77. *If a,b,c are nonnegative real numbers, then*

$$\sum a(b+c)(a-b)(a-c)(a-2b)(a-2c) \ge (a-b)^2(b-c)^2(c-a)^2.$$

(Vasile Cîrtoaje, 2008)

Solution. Let

$$p = a+b+c, \quad q = ab+bc+ca, \quad r = abc,$$

and

$$f_6(a,b,c) = f(a,b,c) - (a-b)^2(b-c)^2(c-a)^2,$$

where

$$f(a,b,c) = \sum a(b+c)(a-b)(a-c)(a-2b)(a-2c).$$

Since

$$\sum a(b+c)(a-b)(a-c)(a-2b)(a-2c) =$$

$$= \sum a(p-a)(a^2+2bc-q)(a^2+6bc-2q),$$

$f(a,b,c)$ has the same highest coefficient A_0 as

$$P_1(a,b,c) = -\sum a^2(a^2+2bc)(a^2+6bc);$$

that is, according to Remark 2 from P 2.75,

$$A_0 = P_1(1,1,1) = -3(1+2)(1+6) = -63.$$

Then, $f_6(a,b,c)$ has the highest coefficient

$$A = A_0 + 27 = -36.$$

Since $A < 0$, according to P 3.76-(a), it suffices to prove that $f_6(a,1,1) \ge 0$ and $f_6(0,b,c) \ge 0$ for all $a,b,c \ge 0$. Indeed, we have

$$f_6(a,1,1) = 2a(a-1)^2(a-2)^2 \ge 0$$

and

$$f_6(0,b,c) = bc(b-c)^4 \ge 0.$$

The equality holds for $a = b = c$, for $a = 0$ and $b = c$ (or any cyclic permutation), and for $a/2 = b = c$ (or any cyclic permutation).

□

P 3.78. *Let a,b,c be nonnegative real numbers.*

(a) If $2 \le k \le 6$, then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) + \frac{4(k-2)(a-b)^2(b-c)^2(c-a)^2}{a+b+c} \ge 0;$$

(b) If $k \ge 6$, then

$$\sum a(a-b)(a-c)(a-kb)(a-kc) + \frac{(k+2)^2(a-b)^2(b-c)^2(c-a)^2}{4(a+b+c)} \ge 0.$$

(Vasile Cîrtoaje, 2009)

Solution. a) We need to prove that $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = (a+b+c)\sum a(a-b)(a-c)(a-kb)(a-kc)$$

$$+4(k-2)(a-b)^2(b-c)^2(c-a)^2.$$

Since $f_6(a,b,c)$ has the same highest coefficient as

$$4(k-2)(a-b)^2(b-c)^2(c-a)^2$$

and $(a-b)^2(b-c)^2(c-a)^2$ has the highest coefficient -27, it follows that $f_6(a,b,c)$ has the highest coefficient

$$A = -108(k-2).$$

Since $A \le 0$ for $k \ge 2$, according to P 3.76-(a), it suffices to prove that $f_6(a,1,1) \ge 0$ and $f_6(0,b,c) \ge 0$ for all $a,b,c \ge 0$. Indeed, we have

$$f_6(a,1,1) = a(a+2)(a-1)^2(a-k)^2 \ge 0$$

and

$$f_6(0,b,c) = (b-c)^6 + (6-k)bc(b-c)^4 \geq 0.$$

The equality holds for $a = b = c$, for $a = 0$ and $b = c$ (or any cyclic permutation, and for and for $a/k = b = c$ (or any cyclic permutation).

(b) We need to prove that $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = 4(a+b+c)\sum a(a-b)(a-c)(a-kb)(a-kc)$$

$$+(k+2)^2(a-b)^2(b-c)^2(c-a)^2.$$

Since $f_6(a,b,c)$ has the same highest coefficient as

$$(k+2)^2(a-b)^2(b-c)^2(c-a)^2$$

and $(a-b)^2(b-c)^2(c-a)^2$ has the highest coefficient -27, it follows that $f_6(a,b,c)$ has the highest coefficient

$$A = -27(k+2)^2 < 0.$$

According to P 3.76-(a), it suffices to prove that $f_6(a,1,1) \geq 0$ and $f_6(0,b,c) \geq 0$ for all $a,b,c \geq 0$. Indeed, we have

$$f_6(a,1,1) = 4a(a+2)(a-1)^2(a-k)^2 \geq 0$$

and

$$f_6(0,b,c) = (b-c)^2[2(b^2+c^2)-(k-2)bc]^2 \geq 0.$$

The equality holds for $a = b = c$, for $a = 0$ and $b = c$ (or any cyclic permutation, and for and for $a/k = b = c$ (or any cyclic permutation), and for $a = 0$ and $\frac{b}{c}+\frac{c}{b} = \frac{k-2}{2}$ (or any cyclic permutation).

□

P 3.79. *If a,b,c are nonnegative real numbers, then*

$$(3a^2+2ab+3b^2)(3b^2+2bc+3c^2)(3c^2+2ca+3a^2) \geq 8(a^2+3bc)(b^2+3ca)(c^2+3ab).$$

Solution. Let

$$p = a+b+c, \quad q = ab+bc+ca,$$

$$f(a,b,c) = (3a^2+2ab+3b^2)(3b^2+2bc+3c^2)(3c^2+2ca+3a^2),$$

$$g(a,b,c) = (a^2+3bc)(b^2+3ca)(c^2+3ab).$$

We need to prove that $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = f(a,b,c) - 8g(a,b,c).$$

By Remark 2 from P 2.75, $f(a,b,c)$ has the highest coefficient

$$A_1 = (2-3)^3 = -1,$$

and $g(a,b,c)$ has the highest coefficient

$$A_2 = g(1,1,1) = 64.$$

Therefore, $f_6(a,b,c)$ has the highest coefficient

$$A = A_1 - 8A_2 = -1 - 512 = -513.$$

Then, by P 3.76-(a), it suffices to prove that $f_6(a,1,1) \geq 0$ and $f_6(0,b,c) \geq 0$ for all nonnegative real a,b,c. Indeed,

$$\begin{aligned} f_6(a,1,1) &= 8(3a^2+2a+3)^2 - 8(a^2+3)(3a+1)^2 \\ &= 48(a+1)(a-1)^2 \geq 0, \end{aligned}$$

$$f_6(0,b,c) = 3b^2c^2(9b^2 - 2bc + 9c^2) \geq 0.$$

The equality holds for $a = b = c$.

□

P 3.80. *Let a,b,c be nonnegative real numbers such that*

$$a+b+c = 2.$$

If

$$\frac{-2}{3} \leq k \leq \frac{11}{8},$$

then

$$(a^2+kab+b^2)(b^2+kbc+c^2)(c^2+kca+a^2) \leq k+2.$$

(Vasile Cîrtoaje, 2011)

Solution. Let

$$f(a,b,c) = (a^2+kab+b^2)(b^2+kbc+c^2)(c^2+kca+a^2).$$

We need to prove that $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = (k+2)(a+b+c)^6 - 64f(a,b,c).$$

According to Remark 2 from P 2.75, $f(a,b,c)$ has the highest coefficient

$$A_1 = (k-1)^3.$$

Thus, $f_6(a,b,c)$ has the highest coefficient

$$A = -64A_1 = 64(1-k)^3.$$

Also, we have

$$f_6(a,1,1) = (k+2)a[(a-1)^2 + 11 - 8k][a^3 + 14a^2 + (8k+12)a + 16]$$

and

$$\begin{aligned} f_6(0,b,c) &= (k+2)(b+c)^6 - 64b^2c^2(b^2 + kbc + c^2) \\ &= (b-c)^2\left[(k+2)(b^2+c^2)^2 + 8(k+2)bc(b^2+c^2) + 4(7k-2)b^2c^2\right]. \end{aligned}$$

Case 1: $1 \le k \le \dfrac{11}{8}$. Since $A \le 0$, according to P 3.76-(a), it suffices to prove that $f_6(a,1,1) \ge 0$ and $f_6(0,b,c) \ge 0$ for all $a,b,c \ge 0$. Clearly, these conditions are satisfied. The equality holds for $a = 0$ and $b = c = 1$ (or any cyclic permutation). If $k = 11/8$, then the equality holds also for $a = b = c = 2/3$.

Case 2: $\dfrac{-2}{3} \le k < 1$. Since $A > 0$, we will use the *highest coefficient cancellation method*. We will prove the sharper inequality $g_6(a,b,c) \ge 0$, where

$$g_6(a,b,c) = f_6(a,b,c) - 64(1-k)^3a^2b^2c^2.$$

Since $g_6(a,b,c)$ has the highest coefficient equal to zero, it suffices to show that $g_6(a,1,1) \ge 0$ and $g_6(0,b,c) \ge 0$ for all $a,b,c \ge 0$ (see P 3.76). The inequality $g_6(a,1,1) \ge 0$ is true if

$$(k+2)[(a-1)^2 + 11 - 8k][a^3 + 14a^2 + (8k+12)a + 16] \ge 64(1-k)^3a.$$

It suffices to show that

$$(k+2)(11-8k)(6a^2 + (8k+12)a + 16] \ge 64(1-k)^3a.$$

Moreover, since $11 - 8k > 8(1-k)$, we only need to show that

$$(k+2)[3a^2 + (4k+6)a + 8] \ge 4(1-k)^2a.$$

Since

$$\begin{aligned} 3a^2 + (4k+6)a + 8 &\ge 3(2a-1) + (4k+6)a + 8 \\ &= (4k+12)a + 5 > 4(k+3)a, \end{aligned}$$

it is enough to show that

$$(k+2)(k+3) \ge (1-k)^2.$$

Indeed,

$$(k+2)(k+3)-(1-k)^2 = 7k+5 = 7\left(k+\frac{2}{3}\right)+\frac{1}{3} > 0.$$

The inequality $g_6(0,b,c) \geq 0$ is also true since

$$\begin{aligned} g_6(0,b,c) &\geq (b-c)^2[4(k+2)b^2c^2 + 16(k+2)b^2c^2 + 4(7k-2)b^2c^2] \\ &= 16(3k+2)b^2c^2(b-c)^2 \geq 0. \end{aligned}$$

Thus, the proof is completed. The equality holds for $a = 0$ and $b = c = 1$ (or any cyclic permutation). If $k = 11/8$, then the equality holds also for $a = b = c = 2/3$.

□

P 3.81. *Let a, b, c be nonnegative real numbers such that*

$$a + b + c = 2.$$

Prove that

$$(2a^2 + bc)(2b^2 + ca)(2c^2 + ab) \leq 4.$$

Solution. Write the inequality in the homogeneous form $f_6(a,b,c) \geq 0$, where

$$f_6(a,b,c) = (a+b+c)^6 - 16(2a^2+bc)(2b^2+ca)(2c^2+ab).$$

Since $f_6(a,b,c)$ has the same highest coefficient A as $P_2(a,b,c)$, where

$$P_2(a,b,c) = -16(2a^2+bc)(2b^2+ca)(2c^2+ab),$$

according to Remark 2 from P 2.75, we have

$$A = P_2(1,1,1) = -432.$$

Since $A < 0$, according to P 3.76-(a), it suffices to prove that $f_6(a,1,1) \geq 0$ and $f_6(0,b,c) \geq 0$ for all $a, b, c \geq 0$. We have

$$\begin{aligned} f_6(a,1,1) &= a(a+2)^2(a^3+8a^2-8a+32) \\ &= a(a+2)^2[a^3+4a^2+28+4(a-1)^2] \geq 0, \end{aligned}$$

$$f_6(0,b,c) = (b+c)^6 - 64b^3c^3 \geq 0.$$

The equality holds for $a = 0$ and $b = c = 1$ (or any cyclic permutation).

□

P 3.82. *Let a,b,c be nonnegative real numbers, no two of which are zero. Then,*

$$\sum(a-b)(a-c)(a-2b)(a-2c) \ge \frac{5(a-b)^2(b-c)^2(c-a)^2}{ab+bc+ca}.$$

(Vasile Cîrtoaje, 2010)

Solution. Denote

$$p = a+b+c, \quad q = ab+bc+ca,$$

and write the inequality as $f_6(a,b,c) \ge 0$, where

$$f_6(a,b,c) = q\sum(a-b)(a-c)(a-2b)(a-2c) - 5(a-b)^2(b-c)^2(c-a)^2.$$

Clearly, $f_6(a,b,c)$ has the highest coefficient

$$A = (-5)(-27) = 135.$$

Since $A > 0$, we will use the *highest coefficient cancellation method*. We have

$$f_6(a,1,1) = (2a+1)(a-1)^2(a-2)^2, \quad f_6(0,b,c) = bc[(b+c)^2 - 6bc]^2.$$

Consider two cases: $p^2 \le 4q$ and $p^2 > 4q$.

Case 1: $p^2 \le 4q$. Since

$$f_6(1,1,1) = 0, \quad f_6(2,1,1) = 0,$$

we define the symmetric homogeneous polynomial of degree three

$$P(a,b,c) = abc + Bp^3 + Cpq$$

such that $P(1,1,1) = 0$ and $P(2,1,1) = 0$. We get $B = 1/18$ and $C = -5/18$, hence

$$P(a,b,c) = abc + \frac{1}{18}p^3 - \frac{5}{18}pq.$$

Consider now the sharper inequality $g_6(a,b,c) \ge 0$, where

$$g_6(a,b,c) = f_6(a,b,c) - 135P^2(a,b,c).$$

Clearly, $g_6(a,b,c)$ has the highest coefficient $A = 0$. Then, according to Remark 1 from the proof of P 3.76, it suffices to prove that $g_6(a,1,1) \ge 0$ for $0 \le a \le 4$. We have

$$P(a,1,1) = \frac{1}{18}(a-1)^2(a-2),$$

hence

$$\begin{aligned} g_6(a,1,1) &= f_6(a,1,1) - 135P^2(a,1,1) \\ &= \frac{1}{12}(a-1)^2(a-2)^2(7+34a-5a^2) \ge 0. \end{aligned}$$

Case 2: $p^2 > 4q$. Define the symmetric homogeneous function

$$R(a,b,c) = abc + Cpq - (9C+1)\frac{q^2}{3p},$$

which satisfies

$$R(1,1,1) = 0,$$
$$R(a,1,1) = \frac{(a-1)^2[3C(2a+1)-1]}{3(a+2)},$$
$$R(0,b,c) = \frac{bc[3C(b+c)^2-(9C+1)bc]}{3(b+c)}.$$

Consider further the sharper inequality $g_6(a,b,c) \geq 0$, where

$$g_6(a,b,c) = f_6(a,b,c) - 135R^2(a,b,c).$$

Since $g_6(a,b,c)$ has the highest coefficient $A = 0$, according to Remark 2 from the proof of P 3.76, it suffices to prove that $g_6(a,1,1) \geq 0$ for $a > 4$, and $g_6(0,b,c) \geq 0$ for all $b,c \geq 0$. Since $f_6(0,b,c) = bc[(b+c)^2 - 6bc]^2$, the inequality $g_6(0,b,c) \geq 0$ holds for all $b,c \geq 0$ only if

$$\frac{9C+1}{3C} = 6;$$

that is, $C = 1/9$. For this value of C, we have

$$R(a,1,1) = \frac{2(a-1)^3}{9(a+2)}, \quad R(0,b,c) = \frac{bc[(b+c)^2-6bc]}{9(b+c)},$$

hence

$$g_6(a,1,1) = f_6(a,1,1) - 135R^2(a,1,1) = \frac{(a-1)^2}{3(a+2)^2}g(a),$$

where

$$g(a) = 3(2a+1)(a^2-4)^2 - 20(a-1)^4,$$

and

$$\begin{aligned} g_6(0,b,c) &= f_6(0,b,c) - 135R^2(0,b,c) \\ &= \frac{bc(3b^2+bc+3c^2)[(b+c)^2-6bc]^2}{3(b+c)^2} \geq 0. \end{aligned}$$

To complete the proof, we need to show that $g(a) \geq 0$ for $a > 4$. This is true since $a^2 - 4 > (a-1)^2$ and $3(2a+1) > 20$.

The equality holds for $a = b = c$, for $a/2 = b = c$ (or any cyclic permutation), and for $a = 0$ and $b/c + c/b = 4$ (or any cyclic permutation).

□

P 3.83. *If a, b, c are positive real numbers such that*

$$abc = 1,$$

then

$$ab + bc + ca + \frac{50}{a+b+c+5} \geq \frac{37}{4}.$$

(Michael Rozenberg, 2013)

Solution. For $abc = 1$ and fixed $a+b+c$, the sum $ab+bc+ca$ is minimal when two of a, b, c are equal (see P 3.58). Thus,it suffices to prove the desired inequality for $a = b$; that is, to show that $a^2c = 1$ involves

$$a^2 + 2ac + \frac{50}{2a+c+5} \geq \frac{37}{4}.$$

This is equivalent to

$$a^2 + \frac{2}{a} + \frac{50a^2}{2a^3+5a^2+1} \geq \frac{37}{4},$$

which can be written in the obvious form

$$(a-1)^2(2a-1)^2(2a^2+11a+8) \geq 0.$$

The equality holds for $a = b = c = 1$, and for $a = b = 1/2$ and $c = 4$ (or any cyclic permutation).

□

P 3.84. *If a, b, c are positive real numbers, then*

$$(a+b+c-3)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-3\right) + abc + \frac{1}{abc} \geq 2.$$

(Vasile Cîrtoaje, 2004)

Solution. Since the inequality does not exchange by substituting a, b, c with $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, respectively, we may consider only the case $abc \geq 1$. Using the notation

$$p = a+b+c, \quad r = abc, \quad r \geq 1,$$

we can write the inequality as

$$(p-3)\left(\frac{ab+bc+ca}{r} - 3\right) + r + \frac{1}{r} \geq 2.$$

By P 3.58, for fixed p and r, the sum $q = ab+bc+ca$ is minimal when two of a, b, c are equal. Since $p \geq 3\sqrt[3]{r} \geq 3$ (by the AM-GM inequality), it suffices to prove the desired inequality for $b = c$, when it becomes as follows

$$a\left(b^2 + \frac{2}{b} - 3\right) + \frac{1}{a}\left(\frac{1}{b^2} + 2b - 3\right) \geq 6\left(b + \frac{1}{b} - 2\right),$$

$$(b-1)^2\left[\left(ab+\frac{1}{ab}-2\right)+2\left(a+\frac{1}{a}-2\right)\right]\geq 0.$$

Since $ab+\dfrac{1}{ab}\geq 2$ and $a+\dfrac{1}{a}\geq 2$, the conclusion follows. The equality holds for $a=b=1$, or $b=c=1$, or $c=a=1$.

□

P 3.85. *If a,b,c are positive real numbers such that*

$$abc=1,$$

then

(a) $$\frac{3}{7}\left(ab+bc+ca-\frac{2}{3}\right)\geq\sqrt{\frac{2}{3}(a+b+c)-1};$$

(b) $$ab+bc+ca-3\geq\frac{46}{27}(\sqrt{a+b+c-2}-1).$$

(Vasile Cîrtoaje, 2009)

Solution. Let

$$p=a+b+c,\quad p\geq 3.$$

For $abc=1$ and fixed p, the sum $ab+bc+ca$ is minimal when two of a,b,c are equal (see P 3.58). Thus, it suffices to consider the case $a=b$.

(a) For $a=b$, the desired inequality is equivalent to

$$3a^3-2a+6\geq 7\sqrt{\frac{4a^3-3a^2+2}{3}}.$$

By squaring, we get

$$(a-1)^2(3a-1)^2(3a^2+8a+10)\geq 0,$$

which is true. The equality holds for $a=b=c=1$, and also for $(a,b,c)=\left(\dfrac{1}{3},\dfrac{1}{3},9\right)$ or any cyclic permutation.

(b) For $a=b$, the desired inequality becomes

$$27a^3-35a+54\geq 46\sqrt{2a^3-2a^2+1}.$$

By squaring, we get the obvious inequality

$$(a-1)^2(9a-5)^2(9a^2+28a+32)\geq 0.$$

The equality holds for $a=b=c=1$, and also for $(a,b,c)=\left(\dfrac{5}{9},\dfrac{5}{9},\dfrac{81}{25}\right)$ or any cyclic permutation.

□

P 3.86. *Let a,b,c be positive real numbers.*

(a) If $abc=2$, then

$$(a+b+c-3)^2+1\geq\frac{a^2+b^2+c^2}{3};$$

(b) If $abc=\frac{1}{2}$, then

$$a^2+b^2+c^2+3(3-a-b-c)^2\geq 3.$$

(Vasile Cîrtoaje, 2007)

Solution. Let

$$p=a+b+c.$$

(a) Write the inequality as

$$(p-3)^2+1\geq\frac{p^2-2(ab+bc+ca)}{3}.$$

For $abc=2$ and fixed p, the sum $ab+bc+ca$ is minimal when two of a,b,c are equal (see P 3.58). Thus, it suffices to consider the case $a=b$, when the inequality becomes as follows

$$\left(2a+\frac{2}{a^2}-3\right)^2+1\geq\frac{2a^2}{3}+\frac{4}{3a^4},$$

$$5a^6-18a^5+15a^4+12a^3-18a^2+4\geq 0,$$

$$(a-1)^2(5a^4-8a^3-6a^2+8a+4)\geq 0.$$

Since

$$5a^4-8a^3-6a^2+8a+4=4(a-1)^4+a(a^3+8a^3-30a+24),$$

it suffices to prove that $a^3+8a^3-30a+24\geq 0$. Indeed, for $a\geq 1$, we have

$$a^3+8a^3-30a+24=(a-1)^3+11a^2-33a+25$$

$$=(a-1)^3+11(a-\frac{3}{2})^2+\frac{1}{4}>0,$$

and for $a<1$, we have

$$a^3+8a^3-30a+24=a(1-a)^2+2(2-a)(6-5a)>0.$$

The equality holds for $(a,b,c)=(1,1,2)$ or any cyclic permutation.

(b) Write the inequality as

$$p^2-2(ab+bc+ca)+3(3-p)^2\geq 3.$$

For $abc = 1/2$ and fixed p, the sum $ab + bc + ca$ is maximal when two of a, b, c are equal (see P 3.58). Thus, it suffices to consider the case $a = b$, when the inequality becomes in succession

$$2a^2 + \frac{1}{4a^2} + 3\left(3 - 2a - \frac{1}{2a^2}\right)^2 \geq 3,$$

$$14a^6 - 36a^5 + 24a^4 + 6a^3 - 9a^2 + 1 \geq 0,$$

$$(a-1)^2(14a^4 - 8a^3 - 6a^2 + 2a + 1) \geq 0.$$

Since

$$14a^4 - 8a^3 - 6a^2 + 2a + 1 = (a-1)^4 + a(13a^3 - 4a^2 - 12a + 6),$$

it suffices to prove that $13a^3 - 4a^2 - 12a + 6 \geq 0$. Indeed,

$$9(13a^3 - 4a^2 - 12a + 6) = 13(a+1)(3a-2)^2 + 2(a-1)^2 + a^2 > 0.$$

The equality holds for $(a, b, c) = \left(1, 1, \frac{1}{2}\right)$ or any cyclic permutation.

□

P 3.87. *If a, b, c are positive real numbers such that*

$$a + b + c = 3,$$

then

$$4\left(\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c}\right) + 9abc \geq 21.$$

Solution. Let

$$p = a + b + c, \quad q = ab + bc + ca, \quad r = abc.$$

We write the required inequality in the homogeneous form

$$\frac{4p^2q^2}{9r} + 9r \geq \frac{5p^3}{3}.$$

For fixed p and r, it suffices to consider the case when q is minimal; that is, when two of a, b, c are equal (see P 3.58). Due to symmetry and homogeneity, we can set $b = c = 1$, when

$$p = a + 2, \quad q = 2a + 1, \quad r = a,$$

the inequality becomes

$$4(a+2)^2(2a+1)^2 + 81a^2 \geq 15(a+2)^3,$$

which is equivalent to

$$(a-1)^2(a-4)^2 \geq 0.$$

The equality holds for $a = b = c = 1$, and for $a = 2$ and $b = c = 1/2$ (or any cyclic permutation).

□

P 3.88. *If a,b,c are nonnegative real numbers such that*

$$ab+bc+ca=abc+2,$$

then

$$a^2+b^2+c^2+abc\geq 4.$$

(Vasile Cîrtoaje, 2011)

First Solution. Among the numbers $1-a$, $1-b$ and $1-c$ there are always two with the same sign; let us say $(1-b)(1-c)\geq 0$. Thus, we have

$$a(1-b)(1-c)\geq 0,$$

$$a+abc\geq ab+ac,$$

$$a+(ab+bc+ca-2)\geq ab+ac,$$

$$a+bc\geq 2,$$

and hence

$$\begin{aligned}a^2+b^2+c^2+abc-4&\geq a^2+2bc+abc-4\\&=(a+2)(a+bc-2)\geq 0.\end{aligned}$$

The equality holds for $a=b=c=1$, and for $a=0$ and $b=c=\sqrt{2}$ (or any cyclic permutation).

Second Solution. For $a=0$, we need to show that $bc=2$ involves $b^2+c^2\geq 4$. This is true since

$$b^2+c^2\geq 2bc=4.$$

Consider further that a,b,c are positive, and write the required inequality as

$$a^2+b^2+c^2+abc\geq 2(ab+bc+ca-abc),$$

$$3abc\geq 2(ab+bc+ca)-a^2-b^2-c^2.$$

Let

$$p=a+b+c,\quad q=ab+bc+ca,\quad r=abc.$$

We need to show that $q=r+2$ implies $3r\geq 4q-p^2$. For fixed q and r, the sum $p=a+b+c$ is minimal when two of a,b,c are equal (see Remark 2 from P 3.58). Thus, it suffices to consider the case $b=c$, when $p=a+2b$, $q=2ab+b^2$, $r=ab^2$. We need to prove that

$$2ab+b^2=ab^2+2$$

implies

$$3ab^2\geq 4(2ab+b^2)-(a+2b)^2,$$

which is equivalent to

$$a(a+3b^2-4b)\geq 0.$$

For the nontrivial case $b < 4/3$, we have

$$a+3b^2-4b = \frac{2-b^2}{b(2-b)} + 3b^2 - 4b = \frac{(1-b)^2(2+4b-3b^2)}{b(2-b)} \geq 0.$$

□

P 3.89. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

$$(a+b)(b+c)(c+a) \geq (a+bc)(b+ca)(c+ab).$$

Solution. Write the inequality in the homogeneous form $f_6(a, b, c) \geq 0$, where

$$f_6(a,b,c) = p^3(a+b)(b+c)(c+a) - (pa+3bc)(pb+3ca)(pc+3ab), \quad p = a+b+c.$$

Clearly, $f_6(a, b, c)$ has the highest coefficient $A = -27$. Thus, according to P 3.76-(a), it suffices to prove that $f_6(a, 1, 1) \geq 0$ and $f_6(0, b, c) \geq 0$ for $a, b, c \geq 0$. We have

$$f_6(a,1,1) = 2(a+2)^3(a+1)^2 - 4(a^2+2a+3)(2a+1)^2$$
$$= 2(a^5 + a^3 - 4a^2 + 2) = 2(a-1)^2(a^3 + 2a^2 + 4a + 2) \geq 0.$$

Also,

$$f_6(0,b,c) = bc(b+c)^4 - 3b^2c^2(b+c)^2 = bc(b+c)^2(b^2 - bc + c^2) \geq 0.$$

The equality holds for $a = b = c = 1$, and also for $a = b = 0$ and $c = 3$ (or any cyclic permutation).

□

P 3.90. *Let a, b, c be positive numbers such that*

$$a + b + c \leq 3\sqrt[4]{abc}.$$

Prove that

$$a^2 + b^2 + c^2 \leq 3.$$

(Vasile Cîrtoaje, 2018)

Solution. By the AM-GM inequality, we have

$$a + b + c \geq 3\sqrt[3]{abc},$$

and from

$$3\sqrt[3]{abc} \leq a + b + c \leq 3\sqrt[4]{abc},$$

we get $abc \le 1$. Denote

$$x = \sqrt[6]{abc}, \quad x \le 1.$$

Since

$$a^2+b^2+c^2 = (a+b+c)^2 - 2(ab+bc+ca) \le 9\sqrt{abc} - 6\sqrt[3]{a^2b^2c^2} = 9x^3 - 6x^4,$$

we only need to show that

$$9x^3 - 6x^4 \le 3,$$

which is equivalent to

$$(1-x)(1+x+x^2-2x^3) \ge 0.$$

This is true since

$$1+x+x^2-2x^3 = 1+x(1-x^2)+x^2(1-x) > 0.$$

The equality occurs for $a=b=c=1$.

□

P 3.91. *If a,b,c are positive real numbers, then*

$$\left(\frac{b+c}{a}-2-\sqrt{2}\right)^2+\left(\frac{c+a}{b}-2-\sqrt{2}\right)^2+\left(\frac{a+b}{c}-2-\sqrt{2}\right)^2 \ge 6.$$

(Vasile Cîrtoaje, 2012)

Solution. Without loss of generality, we can assume that $a = \max\{a,b,c\}$. Let

$$m = 2+\sqrt{2}, \qquad t = \frac{b+c}{2}.$$

We will show that

$$E(a,b,c) \ge E(a,t,t) \ge 6,$$

where

$$E = \left(\frac{b+c}{a}-m\right)^2+\left(\frac{c+a}{b}-m\right)^2+\left(\frac{a+b}{c}-m\right)^2.$$

Write the left inequality as

$$\left(\frac{c+a}{b}-m\right)^2+\left(\frac{a+b}{c}-m\right)^2 \ge 2\left(\frac{2a}{b+c}-m+1\right)^2. \qquad (*)$$

According to the identity

$$2x^2+2y^2 = (x-y)^2+(x+y)^2,$$

we have

$$2\left(\frac{c+a}{b}-m\right)^2+2\left(\frac{a+b}{c}-m\right)^2=\left(\frac{c+a}{b}-\frac{a+b}{c}\right)^2+\left(\frac{c+a}{b}+\frac{a+b}{c}-2m\right)^2.$$

Thus, we can rewrite the inequality (*) as

$$\left(\frac{c+a}{b}-\frac{a+b}{c}\right)^2\geq 4\left(\frac{2a}{b+c}-m+1\right)^2-\left(\frac{c+a}{b}+\frac{a+b}{c}-2m\right)^2,$$

$$\frac{(a+b+c)^2(b-c)^2}{b^2c^2}+\left(\frac{4a}{b+c}+\frac{c+a}{b}+\frac{a+b}{c}-4m+2\right)\frac{(a+b+c)(b-c)^2}{bc(b+c)}\geq 0.$$

This is true if $f(a)\geq 0$, where

$$f(a)=\frac{(a+b+c)(b+c)}{bc}+\frac{4a}{b+c}+\frac{c+a}{b}+\frac{a+b}{c}-4m+2.$$

Since $f(a)$ is increasing and $a=\max\{a,b,c\}$, it suffices to show that $f\left(\frac{b+c}{2}\right)\geq 0$. Indeed,

$$f\left(\frac{b+c}{2}\right)=\frac{3(b-c)^2}{bc}+6-4\sqrt{2}\geq 6-4\sqrt{2}>0.$$

Write now the right inequality $E(a,t,t)\geq 6$ as

$$\left(\frac{b+c}{a}-m\right)^2+2\left(\frac{2a}{b+c}-m+1\right)^2\geq 6.$$

Setting $\frac{b+c}{a}=x$, this inequality becomes

$$(x-m)^2+2\left(\frac{2}{x}-m+1\right)^2\geq 6,$$

$$\frac{(x-2)^2(x-\sqrt{2})^2}{x^2}\geq 0.$$

The proof is completed. The equality holds for $a=b=c$, and for $\frac{a}{\sqrt{2}}=b=c$ (or any cyclic permutation).

□

P 3.92. *If a,b,c are positive real numbers, then*

$$2(a^3+b^3+c^3)+9(ab+bc+ca)+39\geq 24(a+b+c).$$

(Vasile Cîrtoaje, 2010)

Solution. Let $p=a+b+c$ and $q=ab+bc+ca$. Since $a^3+b^3+c^3=3abc+p^3-3pq$, we can write the inequality as

$$6abc+2p^3+3(3-2p)q+39\geq 24p.$$

By Schur's inequality of degree three, we have

$$9abc\geq 4pq-p^3.$$

Therefore, it suffices to show that

$$\frac{2}{3}(4pq-p^3)+2p^3+3(3-2p)q+39\geq 24p,$$

which is equivalent to

$$4p^3+117\geq 72p+(10p-27)q.$$

Case 1: $10p-27\geq 0$. Since $3q\leq p^2$, we have

$$4p^3+117-72p-(10p-27)q\geq 4p^3+117-72p-\frac{(10p-27)p^2}{3}$$
$$=\frac{1}{3}(p-3)^2(2p+39)\geq 0.$$

Case 2: $10p-27<0$. From $(3p-8)^2\geq 0$, we get

$$9p^2-48p+64\geq 0,$$

$$18q\geq -9\sum a^2+48p-64.$$

Using this inequality and

$$\sum(10a-9)=10p-27<0,$$

we get

$$2\left(2\sum a^3+9q+39-24p\right)\geq 4\sum a^3+\left(-9\sum a^2+48p-64\right)+78-48p$$

$$=\sum\left(4a^3-9a^2+\frac{14}{3}\right)>\sum\left(4a^3-9a^2+\frac{14}{3}\right)+\frac{14}{27}\sum(10a-9)$$

$$=\sum a\left(4a^2-9a+\frac{140}{27}\right)>\sum a\left(4a^2-9a+\frac{81}{16}\right)=\sum a\left(2a-\frac{9}{4}\right)^2\geq 0.$$

The equality holds for $a=b=c=1$.

□

P 3.93. *If a, b, c are nonnegative real numbers such that $a^2+b^2+c^2=3$, then*

$$a^3+b^3+c^3-3 \ge |(a-b)(b-c)(c-a)|.$$

Solution. Assume that $a \le b \le c$, write the inequality in the homogeneous form

$$a^3+b^3+c^3-3\left(\frac{a^2+b^2+c^2}{3}\right)^{3/2} \ge (b-a)(c-b)(c-a),$$

and use the substitution

$$b=a+p, \qquad c=a+q, \qquad 0 \le p \le q.$$

For fixed p and q, we need to show that

$$f(a) \ge pq(q-p),$$

where

$$f(a)=a^3+b^3+c^3-3\left(\frac{a^2+b^2+c^2}{3}\right)^{3/2}.$$

Since $a'=b'=c'=1$, we have

$$\begin{aligned} f'(a) &= 3(a^2+b^2+c^2)-3(a+b+c)\sqrt{\frac{a^2+b^2+c^2}{3}} \\ &= 9\sqrt{\frac{a^2+b^2+c^2}{3}}\left(\sqrt{\frac{a^2+b^2+c^2}{3}}-\frac{a+b+c}{3}\right) \ge 0. \end{aligned}$$

Thus, $f(a)$ is increasing, $f(a) \ge f(0)$, and it suffices to show that $f(0) \ge pq(q-p)$, that is

$$p^3+q^3-3\left(\frac{p^2+q^2}{3}\right)^{3/2} \ge pq(q-p).$$

Consider the non-trivial case $p > 0$. Due to homogeneity, we may assume that $p=1$ and $q \ge 1$, when the inequality becomes as follows:

$$q^3-q^2+q+1 \ge 3\left(\frac{1+q^2}{3}\right)^{3/2},$$

$$3(q^3-q^2+q+1)^2 \ge (q^2+1)^3,$$

$$q^6-3q^5+3q^4-3q^2+3q+1 \ge 0,$$

$$q^3(q-1)^3+q^3-3q^2+3q+1 \ge 0,$$

$$(q^3+1)(q-1)^3+2 \ge 0.$$

The last inequality is clearly true. The equality holds for $a=b=c=1$.

□

P 3.94. *Let a, b, c be nonnegative real numbers such that $a^2+b^2+c^2=3$. Prove that*

$$1-abc \ge \frac{5}{3}\min\{(a-b)^2,(b-c)^2,(c-a)^2\}.$$

(Vasile Cîrtoaje, 2019)

Solution. Assume that $a \ge b \ge c$. For $a=c$, the inequality is a trivial equality. Consider next that $a>c$. There are two cases to consider: $a-b \ge b-c$ and $a-b \le b-c$.

Case 1: $a-b \ge b-c$. Write the inequality in the homogeneous form $f(a) \le 0$, where

$$f(a)=abc+\frac{5}{3}(b-c)^2g^{1/2}(a)-g^{3/2}(a), \qquad g(a)=\frac{a^2+b^2+c^2}{3}.$$

We will show that

$$f(a) \le f(2b-c) \le 0.$$

The left inequality is true if $f'(a) \le 0$. Since

$$g'(a)=\frac{2a}{3}$$

and

$$f'(a)=bc+\frac{5a(b-c)^2}{9g^{1/2}}-ag^{1/2}=bc+\frac{5a(b-c)^2}{9}-a,$$

we need to show that

$$a\left[1-\frac{5(b-c)^2}{9}\right] \ge bc.$$

Since $a \ge b$, this is true if

$$1-\frac{5(b-c)^2}{9} \ge c,$$

that is

$$3(1-c) \ge \frac{5(b-c)^2}{3}.$$

It is enough to show that

$$3(1-c) \ge 2(b-c)^2.$$

From

$$3=a^2+b^2+c^2 \ge 2b^2+c^2,$$

we get

$$b \le \sqrt{\frac{3-c^2}{2}}.$$

Thus, it suffices to show that

$$3(1-c) \ge 2\left(\sqrt{\frac{3-c^2}{2}}-c\right)^2,$$

which is equivalent to

$$c(2\sqrt{6-2c^2}-3-c) \geq 0,$$

$$\frac{3c(1-c)(5+3c)}{2\sqrt{6-2c^2}+3+c} \geq 0.$$

Since f is decreasing and $a \geq 2b-c$, we have $f(a) \leq f(2b-c)$.

The inequality $f(2b-c) \geq 0$ is true if the original inequality holds for $a = 2b-c$. Thus, we need to show that

$$3bc(2b-c)+5(b-c)^2 \leq 3$$

for

$$(2b-c)^2+b^2+c^2 = 3,$$

which involves

$$2(b-c)^2+3b^2 = 3, \qquad b \leq 1.$$

Indeed, we have

$$3bc(2b-c)+5(b-c)^2-3 \leq 3c(2b-c)+5(b-c)^2-3 = 0.$$

Case 2: $a-b \leq b-c$. We have

$$c \leq 2b-a, \qquad 2b-a \geq 0.$$

Write the inequality in the homogeneous form $f(c) \leq 0$, where

$$f(c) = abc+\frac{5}{3}(a-b)^2g^{1/2}(c)-g^{3/2}(c), \qquad g(c) = \frac{a^2+b^2+c^2}{3}.$$

We will show that

$$f(c) \leq f(2b-a) \leq 0.$$

The left inequality is true if $f'(c) \geq 0$. We have

$$g'(c) = \frac{2c}{3}$$

and

$$f'(c) = ab+\frac{5c(a-b)^2}{9g^{1/2}}-cg^{1/2} = ab+\frac{5c(b-c)^2}{9}-c \geq ab-c \geq b-c \geq 0.$$

The inequality $f(2b-a) \geq 0$ is true if the original inequality holds for $c = 2b-a$. Thus, we need to show that

$$3ab(2b-a)+5(a-b)^2 \leq 3$$

for

$$a^2+b^2+(2b-a)^2 = 3,$$

which involves

$$2(a-b)^2+3b^2=3, \qquad b\le 1.$$

Indeed, we have

$$3ab(2b-a)+5(a-b)^2-3\le 3a(2b-a)+5(a-b)^2-3=0.$$

The proof is completed. The equality occurs for $a=b=c=1$, and also for $a=2\sqrt{\frac{3}{5}}$, $b=\sqrt{\frac{3}{5}}$ and $c=0$ (or any permutation).

□

P 3.95. *If a,b,c are nonnegative real numbers, then*

$$a^4+b^4+c^4-a^2b^2-b^2c^2-c^2a^2\ge 2|a^3b+b^3c+c^3a-ab^3-bc^3-ca^3|.$$

Solution. Assume that $a\le b\le c$ and write the inequality as

$$(a^2-b^2)^2+(b^2-c^2)^2+(c^2-a^2)^2\ge 4(a+b+c)(a-b)(b-c)(c-a).$$

Using the substitution $b=a+p$ and $c=a+q$, where $q\ge p\ge 0$, the inequality can be restated as

$$4Aa^2+4Ba+C\ge 0,$$

where

$$A=p^2-pq+q^2, \quad B=p^3+q(p-q)^2,$$

$$C=p^4+2p^3q-p^2q^2-2pq^3+q^4=(p^2+pq-q^2)^2.$$

Since $A\ge 0$, $B\ge 0$ and $C\ge 0$, the inequality is obviously true. For $a\le b\le c$, the equality occurs when $a=b=c$, and also when $a=0$ and $\frac{c}{b}=\frac{1+\sqrt{5}}{2}$.

□

P 3.96. *If a,b,c are nonnegative real numbers, then*

$$a^4+b^4+c^4-abc(a+b+c)\ge 2\sqrt{2}\,|a^3b+b^3c+c^3a-ab^3-bc^3-ca^3|.$$

(Pham Kim Hung, 2006)

Solution. Assume that $a\le b\le c$ and write the inequality as

$$a^2(a^2-bc)+b^2(b^2-ca)+c^2(c^2-ab)\ge 2\sqrt{2}(a+b+c)(a-b)(b-c)(c-a).$$

Using the substitution $b=a+p$ and $c=a+q$, where $q\ge p\ge 0$, the inequality becomes

$$Aa^2+Ba+C\ge 0,$$

where

$$A = 5(p^2 - pq + q^2), \quad B = 4p^3 + (6\sqrt{2} - 1)p^2q - (6\sqrt{2} + 1)pq^2 + 4q^3,$$

$$C = p^4 + q^4 + 2\sqrt{2}pq(p^2 - q^2).$$

Since

$$A \geq 0,$$

$$B \geq \frac{25}{4}p^2q - 10pq^2 + 4q^3 = q\left(\frac{5p}{2} - 2q\right)^2 \geq 0$$

and

$$C = (p^2 + \sqrt{2}pq - q^2)^2 \geq 0,$$

the conclusion follows. For $a \leq b \leq c$, the equality occurs when $a = b = c$, and also when $a = 0$ and $\frac{c}{b} = \frac{\sqrt{2} + \sqrt{6}}{2}$.

□

P 3.97. *If a, b, c are nonnegative real numbers such that $a + b + c = 3$, then*

$$(a^3b + b^3c + c^3a - 3abc)(ab^3 + bc^3 + ca^3 - 3abc) \geq (a^2b^2 + b^2c^2 + c^2a^2 - 3abc)^2.$$

(Vasile Cîrtoaje, 2008)

Solution. Write the inequality in the homogeneous form

$$AB \geq C^2,$$

where

$$A = a^3b + b^3c + c^3a - abc(a + b + c),$$

$$B = ab^3 + bc^3 + ca^3 - abc(a + b + c),$$

$$C = a^2b^2 + b^2c^2 + c^2a^2 - abc(a + b + c).$$

From the Cauchy-Schwarz inequality

$$(c + a + b)(a^3b + b^3c + c^3a) \geq abc(a + b + c)^2,$$

it follows that $A \geq 0$, with equality for $a = b = c$. Thus, the desired inequality $AB \geq C^2$ is true if

$$At^2 - 2Ct + B \geq 0$$

for all real t. This inequality is equivalent to

$$(a^3b + b^3c + c^3a)t^2 - 2(a^2b^2 + b^2c^2 + c^2a^2)t + ab^3 + bc^3 + ca^3 \geq abc(a + b + c)(t - 1)^2,$$

$$ab(at - b)^2 + bc(bt - c)^2 + ca(ct - a)^2 \geq abc(a + b + c)(t - 1)^2.$$

Clearly, the last inequality follows immediately from the Cauchy-Schwarz inequality

$$(c+a+b)[ab(at-b)^2+bc(bt-c)^2+ca(ct-a)^2] \ge abc(a+b+c)^2(t-1)^2.$$

The equality holds for $a=b=c$, and also for $a=0$ or $b=0$ or $c=0$.
Remark. Actually, the following identity holds:

$$AB-C^2 = abc(a+b+c)(a^2+b^2+c^2-ab-bc-ca)^2.$$

□

P 3.98. *If $a,b,c \ge -5$ such that*

$$a+b+c=3,$$

then

$$\frac{1-a}{1+a+a^2}+\frac{1-b}{1+b+b^2}+\frac{1-c}{1+c+c^2} \ge 0.$$

(Vasile Cîrtoaje, 2014)

First Solution. Using the substitution

$$a=x-5, \quad b=y-5, \quad c=z-5,$$

we need to prove that if $x,y,z \ge 0$ such that $x+y+z=18$, then

$$\frac{6-x}{x^2-9x+21}+\frac{6-y}{y^2-9y+21}+\frac{6-z}{z^2-9z+21} \ge 0.$$

Denoting

$$p=\frac{x+y+z}{18},$$

we can write this inequality as $f_5(x,y,z) \ge 0$, where

$$f_5(x,y,z)=\sum(6p-x)(y^2-9yp+21p^2)(z^2-9zp+21p^2)$$

is a symmetric homogeneous polynomial of degree 5. According to Remark from P 3.68, it suffices to prove this inequality for $y=z$ and for $x=0$. Therefore, we only need to prove the original inequality for $b=c$ and for $a=-5$.

Case 1: $b=c=\dfrac{3-a}{2}$. Since

$$\frac{1-b}{1+b+b^2}=\frac{1-c}{1+c+c^2}=\frac{2(a-1)}{a^2-8a+19},$$

we need to show that

$$\frac{1-a}{1+a+a^2}+\frac{4(a-1)}{a^2-8a+19} \ge 0,$$

which is equivalent to

$$(a-1)^2(a+5) \geq 0.$$

Case 2: $a = -5,\ b + c = 8$. We can write the desired inequality as follows:

$$\left(\frac{1}{7} + \frac{1-b}{1+b+b^2}\right) + \left(\frac{1}{7} + \frac{1-c}{1+c+c^2}\right) \geq 0,$$

$$\frac{(b-4)(b-2)}{1+b+b^2} + \frac{(c-4)(c-2)}{1+c+c^2} \geq 0,$$

$$\frac{b-c}{2}\left(\frac{b-2}{1+b+b^2} - \frac{c-2}{1+c+c^2}\right) \geq 0,$$

$$\frac{(b-c)^2[3+2(b+c)-bc]}{2(1+b+b^2)(1+c+c^2)} \geq 0.$$

The last inequality is true since

$$2(b+c) - bc = \left(\frac{b+c}{2}\right)^2 - bc = \left(\frac{b-c}{2}\right)^2 \geq 0.$$

The proof is completed. The equality occurs for $a = b = c = 1$, and also for $a = -5$ and $b = c = 4$ (or any cyclic permutation).

Second Solution. Assume that $a \leq b \leq c$ and denote

$$E(a,b,c) = \frac{1-a}{1+a+a^2} + \frac{1-b}{1+b+b^2} + \frac{1-c}{1+c+c^2}.$$

We will show that

$$E(a,b,c) \geq E(a,t,t) \geq 0,$$

where

$$t = \frac{b+c}{2} = \frac{3-a}{2}.$$

From $-5 \leq a \leq 1$ it follows that

$$t \in [1,4].$$

Write the left inequality as follows:

$$\left(\frac{1-b}{1+b+b^2} - \frac{1-t}{1+t+t^2}\right) + \left(\frac{1-c}{1+c+c^2} - \frac{1-t}{1+t+t^2}\right) \geq 0,$$

$$(b-c)\left[\frac{(b-1)t-b-2}{1+b+b^2} - \frac{(c-1)t-c-2}{1+c+c^2}\right] \geq 0,$$

$$(b-c)^2[(2+b+c-bc)t + 1 + 2(b+c) + bc] \geq 0,$$

$$(b-c)^2[2t^2 + 6t + 1 - bc(t-1)] \geq 0.$$

The last inequality is true since

$$\begin{aligned}2t^2+6t+1-bc(t-1) &\geq 2t^2+6t+1-t^2(t-1)\\ &= t(4-t)(1+t)+2t+1>0.\end{aligned}$$

Also, we have

$$\begin{aligned}E(a,t,t) &= \frac{1-a}{1+a+a^2}+\frac{2(1-t)}{1+t+t^2}\\ &= \frac{2(t-1)}{4t^2-14t+13}+\frac{2(1-t)}{1+t+t^2}\\ &= \frac{6(1-t)^2(4-t)}{(4t^2-14t+13)(1+t+t^2)} \geq 0.\end{aligned}$$

□

P 3.99. *Let $a,b,c \neq \frac{1}{k}$ be nonnegative real numbers such that*

$$a+b+c=3.$$

If $k \geq \frac{4}{3}$, then

$$\frac{1-a}{(1-ka)^2}+\frac{1-b}{(1-kb)^2}+\frac{1-c}{(1-kc)^2} \geq 0.$$

(Vasile Cîrtoaje, 2012)

Solution. Denoting $p=(a+b+c)/3$, we may write the inequality as $f_5(x,y,z) \geq 0$, where

$$f_5(x,y,z)=\sum(p-a)(p-kb)^2(p-kc)^2$$

is a symmetric homogeneous polynomial of degree 5. According to P 3.68, it suffices to prove this inequality for $b=c$ and for $a=0$.

Case 1: $b=c$. Since $a=3-2b$, the original inequality is equivalent to the following sequence of inequalities:

$$\frac{1-a}{(1-ka)^2}+\frac{2(1-b)}{(1-kb)^2} \geq 0,$$

$$\frac{2(b-1)}{(1-3k+2kb)^2}+\frac{2(1-b)}{(1-kb)^2} \geq 0,$$

$$k(b-1)^2[k(3-b)-2] \geq 0.$$

The last inequality holds since

$$k(3-b)-2 \geq \frac{4}{3}(3-b)-2=\frac{2(3-2b)}{3}=\frac{2a}{3} \geq 0.$$

Case 2: $a=0$. Since $b+c=3$, the original inequality becomes as follows:

$$1 \ge \frac{b-1}{(1-kb)^2}+\frac{c-1}{(1-kc)^2},$$

$$(1-kb)^2(1-kc)^2 \ge (b-1)(1-kc)^2+(c-1)(1-kb)^2,$$

$$(k^2bc-3k+1)^2 \ge 1+6k-9k^2+(5k^2-4k)bc,$$

$$k^3b^2c^2+18k-12 \ge (6k^2+3k-4)bc.$$

Since

$$k^3b^2c^2+18k-12 \ge 2\sqrt{k^3(18k-12)}\ bc,$$

it suffices to show that

$$4k^3(18k-12) \ge (6k^2+3k-4)^2,$$

which is equivalent to

$$36k^4-84k^3+39k^2+24k-16 \ge 0,$$

$$(3k-4)(12k^3-12k^2-3k+4) \ge 0.$$

The last inequality holds since

$$12k^3-12k^2-3k+4 > 12k^3-12k^2-4k+4 = 4(k-1)(3k^2-1) > 0.$$

The proof is completed. The equality occurs for $a=b=c=1$. If $k=4/3$, then the equality holds also for $a=0$ and $b=c=3/2$ (or any cyclic permutation).

□

P 3.100. *If a,b,c are positive real numbers such that*

$$abc=1,$$

then

$$3(2a^2+1)(2b^2+1)(2c^2+1) \le (a+b+c)^4.$$

(Vasile Cîrtoaje, 2015)

Solution. Since

$$2a^2+1 = 2a^2+(abc)^{2/3} = a^{2/3}[2a^{4/3}+(bc)^{2/3}],$$

we can write the inequality in the homogeneous form

$$3[2a^{4/3}+(bc)^{2/3}][2b^{4/3}+(ca)^{2/3}][2c^{4/3}+(ab)^{2/3}] \le (a+b+c)^4.$$

Since the condition $abc = 1$ becomes superfluous, we may now assume that $a + b + c = 3$, when the inequality can be written as

$$[2a^{4/3} + (bc)^{2/3}][2b^{4/3} + (ca)^{2/3}][2c^{4/3} + (ab)^{2/3}] \le 27,$$

$$4\sum a^2b^2 + 2(abc)^{2/3}\sum a^2 + 9(abc)^{4/3} \le 27.$$

By the AM-GM inequality, we have $abc \le 1$ and

$$2abc + 1 \ge 3(abc)^{2/3}.$$

Therefore, it suffices to show that

$$4\sum a^2b^2 + \frac{2}{3}(2abc + 1)\sum a^2 + 9abc \le 27,$$

which is equivalent to

$$(9 + 8q)r + 63 + 4q - 12q^2 \ge 0,$$

where

$$q = ab + bc + ca, \quad r = abc, \quad 0 < q \le 3, \ \ 0 < r \le 1.$$

By Schur's inequality

$$(a + b + c)^3 + 9abc \ge 4(a + b + c)(ab + c + ca),$$

we get

$$3r \ge 4q - 9,$$

hence

$$\begin{aligned}(9 + 8q)r + 63 + 4q - 12q^2 &\ge (6 + 9q)r + 63 + 4q - 12q^2\\ &\ge (2 + 3q)(4q - 9) + 63 + 4q - 12q^2\\ &= 15(3 - q) \ge 0.\end{aligned}$$

The equality occurs for $a = b = c = 1$.

□

P 3.101. *If a, b, c are positive real numbers such that*

$$a + b + c = \sqrt{3},$$

then

$$(3\sqrt{3} - 5)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{b}\right) \ge a^2 + b^2 + c^2.$$

(Vasile Cîrtoaje, 2014)

Solution. Assume that

$$a \ge b \ge c, \quad a \ge \frac{\sqrt{3}}{3},$$

and show that

$$F(a,b,c) \ge F(a,x,x) \ge 0,$$

where

$$x = \frac{b+c}{2} = \frac{\sqrt{3}-a}{2}, \qquad F(a,b,c) = (3\sqrt{3}-5)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{b}\right) - a^2 - b^2 - c^2.$$

The left inequality is equivalent to

$$(b-c)^2\left[2(3\sqrt{3}-5) - bc(b+c)\right] \ge 0.$$

This is true since

$$bc(b+c) \le \frac{(b+c)^3}{4} = \frac{(\sqrt{3}-a)^3}{4}$$
$$\le \frac{(\sqrt{3}-\sqrt{3}/3)^3}{4} = \frac{2}{3\sqrt{3}} < 2(3\sqrt{3}-5).$$

The right inequality, $F(a,x,x) \ge 0$, is equivalent to

$$\sqrt{3}\,a^4 - 5a^3 + 3\sqrt{3}\,a^2 + 5(3-2\sqrt{3}\,)a + 6\sqrt{3} - 10 \ge 0,$$

$$(a-1)^2\left[\sqrt{3}\,a^2 - (5-2\sqrt{3}\,)a + 6\sqrt{3} - 10\right] \ge 0.$$

Thus, it suffices to show that

$$\sqrt{3}\,a^2 - (5-2\sqrt{3}\,)a + 6\sqrt{3} - 10 \ge 0.$$

This inequality is strict because

$$D = (5-2\sqrt{3}\,)^2 - 4\sqrt{3}\,(6\sqrt{3}-10) = 20\sqrt{3} - 35 < 0.$$

The equality occurs for $a = 1$ and $b = c = \dfrac{\sqrt{3}-1}{2}$ (or any cyclic permutation).

□

P 3.102. *If* $a,b,c \ge 1$ *such that*

$$a+b+c = 4,$$

then

$$12\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{b}\right) \ge 5(a^2+b^2+c^2).$$

(Vasile Cîrtoaje, 2014)

Solution. Assume that

$$a \ge b \ge c, \quad a \ge \frac{4}{3},$$

and show that

$$F(a,b,c) \ge F(a,x,x) \ge 0,$$

where

$$F(a,b,c) = 12\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{b}\right) - 5(a^2+b^2+c^2)$$

and

$$x = \frac{b+c}{2}.$$

From

$$x \ge \frac{1+1}{2} = 1, \quad x = \frac{4-a}{2} \le \frac{4-4/3}{2} = \frac{4}{3},$$

we get

$$1 \le x \le \frac{4}{3}.$$

The left inequality is equivalent to

$$(b-c)^2[24-5bc(b+c)] \ge 0.$$

This is true since

$$5bc(b+c) \le \frac{5(b+c)^3}{4} = 10x^3 \le \frac{640}{27} < 24.$$

The right inequality, $F(a,x,x) \ge 0$, is equivalent to

$$F(4-2x,x,x) \ge 0,$$

$$15x^4 - 70x^3 + 120x^2 - 89x + 24 \ge 0,$$

$$(x-1)(15x^3 - 55x^2 + 65x - 24) \ge 0,$$

$$(x-1)[1-5(x-1)^2(5-3x)] \ge 0.$$

We only need to show that

$$1 \ge 15(x-1)^2\left(\frac{5}{3}-x\right).$$

Since

$$(x-1)\left(\frac{5}{3}-x\right) \le \frac{1}{4}\left[(x-1)+\left(\frac{5}{3}-x\right)\right]^2 = \frac{1}{9},$$

it suffices to show that

$$1 \ge \frac{15}{9}(x-1),$$

which is equivalent to

$$x \le \frac{8}{5}.$$

The proof is completed. The equality occurs for $a = 2$ and $b = c = 1$ (or any cyclic permutation).

□

P 3.103. *If a, b, c are positive real numbers such that*

$$a + b + c = 3, \qquad c \le \frac{15}{32},$$

then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge a^2 + b^2 + c^2.$$

(Vasile Cîrtoaje, 2018)

Solution. Write the inequality as

$$\frac{a+b}{ab} + 2ab \ge (a+b)^2 + c^2 - \frac{1}{c}.$$

Since

$$\frac{a+b}{ab} + 2ab \ge 2\sqrt{2(a+b)},$$

it suffices to show that

$$2\sqrt{2(a+b)} \ge (a+b)^2 + c^2 - \frac{1}{c},$$

which is equivalent to

$$\sqrt{8(3-c)} + \frac{1}{c} + 6c - 2c^2 - 9 \ge 0.$$

We can get this inequality by summing the inequalities

$$\sqrt{8(3-c)} - \frac{9}{2} \ge 0$$

and

$$\frac{1}{c} + 6c - 2c^2 - \frac{9}{2} \ge 0.$$

The first inequality reduces to $c \le \dfrac{15}{32}$, and the second inequality to

$$(2-c)(1-2c)^2 \ge 0.$$

The inequality is an equality for $a = b = c = 1$.

□

P 3.104. *If* $a \ge b \ge c \ge 0$ *and* $ab + bc + ca = 3$ *, then*

(a) $b + c \le 2;$

(b) $b^2 + bc + c^2 \le 3.$

(Vasile Cîrtoaje, 2018)

Solution. From

$$3a^2 \ge ab + bc + ca = 3,$$

it follows that $a \ge 1$, and from $(a-b)(a-c) \ge 0$, we get

$$a^2 + bc \ge a(b+c),$$

hence

$$3 = a(b+c) + bc \ge 2a(b+c) - a^2,$$

$$b + c \le \frac{a^2+3}{2a}.$$

(a) It suffices to show that

$$\frac{a^2+3}{2a} \le 2,$$

that is

$$(a-1)(a-3) \le 0.$$

Since this is true for $a \le 3$, it remains to consider the case $a \ge 3$, when

$$b + c = \frac{3-bc}{a} \le \frac{3}{a} \le 1 < 2.$$

The equality occurs for $a = b = c = 1$.

(b) From

$$b^2 + bc + c^2 - 3 = (b+c)^2 - bc = (b+c)^2 + a(b+c) - 6 \le \frac{(a^2+3)^2}{4a^2} + \frac{a^2+3}{2} - 6$$

$$= \frac{3(a^4 - 4a^2 + 3)}{4a^2} = \frac{3(a^2-1)(a^2-3)}{4a^2},$$

it follows that $b^2 + bc + c^2 \le 3$ for $a^2 \le 3$. Consider further the case $a^2 \ge 3$, when

$$b^2 + bc + c^2 \le (b+c)^2 = \left(\frac{3-bc}{a}\right)^2 \le \frac{9}{a^2} \le 3.$$

The equality occurs for $a = b = c = 1$, and also for $a = b = \sqrt{3}$ and $c = 0$.

□

P 3.105. *If $a, b, c \in \left[0, 1+\frac{1}{\sqrt{2}}\right]$ and $a^2+b^2+c^2=3$, then*

$$a+b+c \geq abc+2.$$

(Vasile Cîrtoaje, 2019)

Solution. Assume that $a \geq b \geq c$ and denote $x = b+c$. From

$$b+c \leq \sqrt{2(b^2+c^2)} = \sqrt{6-2a^2}$$

and

$$b+c \geq \sqrt{b^2+c^2} = \sqrt{3-a^2}.$$

we get

$$\sqrt{3-a^2} \leq x \leq \sqrt{6-2a^2}, \quad 1 \leq a \leq 1+\frac{1}{\sqrt{2}}.$$

Since

$$2bc = (b+c)^2 - b^2 - c^2 = (b+c)^2 + a^2 - 3,$$

the required inequality can be written as

$$2a + 2(b+c) \geq a[(b+c)^2 + a^2 - 3] + 4,$$

that is $f(x) \leq 0$, where

$$f(x) = ax^2 - 2x + a^3 - 5a + 4.$$

Since f is a quadratic convex function, it suffices to show that $f(\sqrt{3-a^2}) \leq 0$ and $f(\sqrt{6-2a^2}) \leq 0$. We have

$$f\left(\sqrt{3-a^2}\right) = 2(2-a-\sqrt{3-a^2}) = \frac{2(2a^2-4a+1)}{2-a+\sqrt{3-a^2}} \leq 0.$$

Also,

$$f\left(\sqrt{6-2a^2}\right) = 4+a-a^3-2\sqrt{6-2a^2} = \frac{a^6-2a^4-8a^3+9a^2+8a-8}{4+a-a^3+2\sqrt{6-2a^2}}$$

$$= \frac{(a+1)(a-1)^2(a^3+a^2-8)}{4+a-a^3+2\sqrt{6-2a^2}}.$$

Because

$$a^3+a^2-8 \leq \left(1+\frac{1}{\sqrt{2}}\right)^3 + \left(1+\frac{1}{\sqrt{2}}\right)^2 - 8 = \frac{11\sqrt{2}-16}{4} < 0,$$

we get $f\left(\sqrt{6-2a^2}\right) \leq 0$.

The inequality is an equality for $a=b=c=1$, and also for

$$a = 1+\frac{1}{\sqrt{2}}, \quad b = 1-\frac{1}{\sqrt{2}}, \quad c = 0$$

(or any permutation).

□

P 3.106. *Let $a,b,c \ge \frac{1}{6}$ be real numbers such that $a^2+b^2+c^2=3$. Then,*

$$a+b+c \ge abc+2.$$

(Vasile Cîrtoaje, 2019)

Solution. Assume that $a=\min\{a,b,c\}$, hence $\frac{1}{6} \le a \le 1$, and denote $s=a+b+c$ and $x=b+c$. From

$$\left(a-\frac{1}{6}\right)\left(b-\frac{1}{6}\right)+\left(b-\frac{1}{6}\right)\left(c-\frac{1}{6}\right)+\left(c-\frac{1}{6}\right)\left(a-\frac{1}{6}\right) \ge 0,$$

we get

$$ab+bc+ca+\frac{1}{12} \ge \frac{1}{3}(a+b+c),$$

$$(a+b+c)^2-(a^2+b^2+c^2)+\frac{1}{6} \ge \frac{2}{3}(a+b+c),$$

$$6s^2-4s-17 \ge 0,$$

hence

$$s \ge k, \qquad k=\frac{1}{3}+\sqrt{\frac{53}{18}} \approx 2.0493. \quad 6k^2-4k-17=0.$$

From

$$k-a \le b+c \le \sqrt{2(b^2+c^2)},$$

we get

$$k-a \le x \le \sqrt{6-2a^2}.$$

Since

$$2bc=(b+c)^2-(b^2+c^2)=x^2-3+a^2,$$

$$2abc+4-2(a+b+c)=a(x^2-3+a^2)+4-2a-2x,$$

we may write the inequality as $f(x) \le 0$, where

$$f(x)=ax^2-2x+a^3-5a+4.$$

Since f is a quadratic convex function, it is enough to show that $f\left(\sqrt{6-2a^2}\right) \le 0$ and $f(k-a) \ge 0$.

The first inequality is equivalent to

$$-a^3+a+4 \le 2\sqrt{6-2a^2}.$$

This is true if

$$(-a^3+a+4)^2 \le 4(6-2a^2),$$

that is

$$a^6-2a^4-8a^3+9a^2+8a-8 \le 0,$$

$$(a-1)^2(a^4+2a^3+a^2-8a-8) \le 0.$$

Since $a < 1$, the inequality is clearly true.

The second inequality is equivalent to

$$2a^3-2ka^2+(k^2-3)a+4-2k \le 0,$$

which can be written as $g(a) \le 0$, where

$$g(a) = 12a^3-12ka^2+(4k-1)a+24-12k,$$

with

$$g'(a) = 36a^2-24ka+4k-1 = (6a-1)(6a-4k+1).$$

Since $6a-1 \ge 0$ and $6a-4k+1 \le 7-4k < 0$, we have $g'(a) \le 0$, $g(a)$ is strictly decreasing, hence $g(a) \le g(1/6)$. So, it suffices to show that $g(1/6) \le 0$. Indeed,

$$g(1/6) = \frac{5(43-105k)}{9} \approx -0.01928 < 0.$$

The equality occurs for $a = b = c = 1$.

□

P 3.107. *If a, b, c are nonnegative real numbers such that*

$$ab+bc+ca+6abc = 9,$$

then

$$2(a+b+c) \ge ab+bc+ca+3.$$

(Vasile Cîrtoaje, 2019)

Solution. For $a = 0$, we need to show that $2(b+c) \ge bc+3$ for $bc = 9$. Indeed,

$$2(b+c)-bc-3 \ge 4\sqrt{bc}-bc-3 = 0.$$

Assume now that $a \ge b \ge c > 0$. According to Remark 2 from P 3.58, for fixed $ab+bc+ca$ and abc, the sum $a+b+c$ is minimal when $a \ge b = c$. Therefore, it suffices to consider this case, when we need to show that

$$2a+4b \ge 2ab+b^2+3$$

for

$$a = \frac{9-b^2}{2b(3b+1)}, \qquad 0 < b < 3.$$

Substituting a, the required inequality becomes

$$(1-b)^2(1+b)(3-b) \ge 0.$$

The inequality is an equality for $a = b = c = 1$, and also for $a = 0$ and $b = c = 3$ (or any cyclic permutation).

Remark. Similarly, we can prove the following generalization:

- *If a, b, c are nonnegative real numbers such that*

$$ab + bc + ca + kabc = 3 + k, \quad k > 0,$$

then

$$\frac{ab + bc + ca - 3}{a + b + c - 3} \le \frac{k}{2\sqrt{k+3} - 3},$$

with equality for $a = 0$ and $b = c = k$ (or any cyclic permutation).

For $k = 1$, we get that $a + b + c + abc = 4$ involves the nice inequality

$$a + b + c \ge ab + bc + ca.$$

□

P 3.108. *If a, b, c are nonnegative real numbers such that*

$$ab + bc + ca + abc = 4,$$

then

$$4(a + b + c) + 3a^2b^2c^2 \ge 15.$$

(Vasile Cîrtoaje, 2019)

Solution. For $a = 0$, we need to show that $4(b + c) \ge 15$ for $bc = 4$. Indeed,

$$4(b + c) - 15 \ge 8\sqrt{bc} - 15 = 1.$$

Assume now that $a \ge b \ge c > 0$. According to Remark 2 from P 3.58, for fixed $ab + bc + ca$ and abc, the sum $a + b + c$ is minimal when $a \ge b = c$. Therefore, it suffices to consider this case, when we need to show that

$$4a + 8b + 3a^2b^4 \ge 15$$

for

$$a = \frac{2 - b}{b}, \quad 0 < b < 2.$$

Substituting a, the required inequality becomes

$$3b^5 - 12b^4 + 12b^3 + 8b^2 - 19b + 8 \ge 0,$$

$$(b - 1)^2(3b^3 - 6b^2 - 3b + 8) \ge 0.$$

This is true since

$$3b^3 - 6b^2 - 3b + 8 = (2 - b)^3 + b(2b - 3)^2 > 0.$$

The inequality is an equality for $a = b = c = 1$.

□

P 3.109. *If* $a, b, c \in \left[0, \frac{5}{3}\right]$ *such that* $a + b + c = 3$, *then*

$$(a+b)(b+c)(c+a) \ge 8\sqrt[3]{abc}.$$

(Vasile Cîrtoaje, 2018)

Solution. Assume that $a \ge b \ge c$ and write the inequality as

$$(a+b+c)(ab+bc+ca) \ge abc + 8\sqrt[3]{abc}.$$

According to Remark 2 from P 3.57, for $a+b+c=3$ and fixed $ab+bc+ca$, the product abc is maximal when $a \ge b = c$ or $a = \frac{5}{3}$. Therefore, it suffices to consider these cases.

Case 1: $a \ge b = c$. For

$$b = c = \frac{3-a}{2}, \quad a \in \left[1, \frac{5}{3}\right],$$

the inequality is equivalent to

$$\left(\frac{a+3}{4}\right)^6 (3-a) \ge 2a.$$

Substituting

$$\frac{a+3}{4} = x, \quad 1 \le x \le \frac{7}{6},$$

we need to show that

$$x^6(3-2x) \ge 4x - 3,$$

which is equivalent to

$$2x^7 - 3x^6 + 4x - 3 \le 0,$$

$$(x-1)^2(2x^5 + x^4 - x^2 - 2x - 3) \le 0.$$

Since

$$2x^5 + x^4 - x^2 - 2x - 3 = x(2x^4 + x^3 + 2x - 8) - 3(x-1)^2 \le xf(x),$$

where

$$f(x) = 2x^4 + x^3 + 2x - 8,$$

it is enough to show that $f(x) \le 0$. Indeed, we have

$$f(x) \le f\left(\frac{7}{6}\right) < 0.$$

Case 2: $a = \frac{5}{3}$. Since $b + c = \frac{4}{3}$, we can write the inequality as

$$5 + bc \ge 5\sqrt[3]{\frac{5bc}{3}}.$$

From

$$\frac{4}{3} = b + c \geq 2\sqrt{bc},$$

we get $bc \leq \frac{4}{9}$. Substituting

$$\sqrt[3]{\frac{5bc}{3}} = x, \qquad x \leq \frac{\sqrt[3]{20}}{3},$$

the inequality becomes $f(x) \geq 0$, where

$$f(x) = 3x^3 - 30x + 25.$$

Since

$$f'(x) = 9x^2 - 30 \leq \sqrt[3]{400} - 30 < 0,$$

f is decreasing, hence

$$f(x) \geq f\left(\frac{\sqrt[3]{20}}{3}\right) = 5\left(\frac{49}{9} - 2\sqrt[3]{20}\right) > 0.$$

The equality holds for $a = b = c = 1$.

Remark. In the absence of the condition $a, b, c \leq \frac{5}{3}$, the following inequality holds:

$$(a+b)(b+c)(c+a) \geq 8\sqrt{abc}.$$

According to P 3.57, it suffices to consider the case $b = c = \frac{3-a}{2}$, when the inequality becomes

$$(a+3)^2 \geq 16\sqrt{a},$$

$$a + 3 \geq 4\sqrt[4]{a}.$$

Indeed, by the AM-GM inequality, we have

$$a + 3 = a + 1 + 1 + 1 \geq 4\sqrt[4]{a \cdot 1 \cdot 1 \cdot 1}.$$

□

P 3.110. *If a, b, c are nonnegative real numbers such that $a \geq b \geq c$ and*

$$a^2 + b^2 + c^2 = 3,$$

then

$$a + b + c + \left(1 - \frac{1}{\sqrt{3}}\right)(a-c)^2 \geq 3.$$

(Vasile Cîrtoaje, 2020)

Proof. Denote

$$k = 1 - \frac{1}{\sqrt{3}},$$

and write the inequality as

$$a + b + c + k(a - c)^2 \geq 3.$$

First Solution. Denote $x = a + c$. Since

$$(a - c)^2 = 2(a^2 + c^2) - (a + c)^2 = 2(3 - b^2) - x^2,$$

we may write the inequality as $f(x) \geq 0$, where

$$f(x) = x + b + k(6 - 2b^2 - x^2) - 3.$$

For fixed b, we have $x \in [m, M]$, where

$$m = \sqrt{3 - b^2}, \qquad M = b + \sqrt{3 - 2b^2}.$$

Indeed, from $(a + c)^2 \geq a^2 + c^2$, we get

$$x \geq \sqrt{3 - b^2} = m,$$

and from $(b^2 - a^2)(b^2 - c^2) \leq 0$, we get

$$a^2c^2 \leq b^2(a^2 + c^2) - b^4 = 3b^2 - 2b^4, \qquad ac \leq b\sqrt{3 - 2b^2},$$

hence

$$x = \sqrt{3 - b^2 + 2ac} \leq \sqrt{3 - b^2 + 2b\sqrt{3 - 2b^2}} = b + \sqrt{3 - 2b^2} = M.$$

We have $x = m$ for $c = 0$, and $x = M$ for $a = b$ or $b = c$. Since $f(x)$ is a quadratic concave function, it suffices to show that $f(m) \geq 0$ and $f(M) \geq 0$, that means to prove the required inequality for $c = 0$, for $a = b$ and for $b = c$.

Case 1: $c = 0$. We need to show that

$$a + b + ka^2 \geq 3$$

for $a \geq b \geq 0$ and $a^2 + b^2 = 3$. Write the inequality as follows:

$$\sqrt{3 - b^2} + b + k(3 - b^2) \geq 3,$$

$$(1 - kb)b \geq \sqrt{3} - \sqrt{3 - b^2},$$

$$(1 - kb)b \geq \frac{b^2}{\sqrt{3} + \sqrt{3 - b^2}},$$

$$(1-kb)\left(\sqrt{3}+\sqrt{3-b^2}\right) \ge b.$$

Since the left hand side is decreasing and the right hand side is increasing with respect to b, it is enough to prove the inequality for $b=\sqrt{\dfrac{3}{2}}$, when it becomes

$$1-\frac{\sqrt{6}-\sqrt{2}}{2} \ge \sqrt{2}-1,$$

$$4 \ge \sqrt{6}+\sqrt{2}.$$

Case 2: $a=b\ge c>0$. We need to show that

$$(2a+c)\sqrt{x}+k(a-c)^2-3x \ge 0$$

for

$$x=\frac{2a^2+c^2}{3}.$$

Due to homogeneity, we may set $c=1$ (which involves $a\ge 1$), when the inequality becomes

$$(2a+1)\sqrt{x}+k(a-1)^2-3x \ge 0,$$

where

$$x=\frac{2a^2+1}{3}.$$

Write the inequality as follows:

$$k(a-1)^2 \ge \sqrt{x}(3\sqrt{x}-2a-1),$$

$$k(a-1)^2 \ge \frac{2(a-1)^2\sqrt{x}}{3\sqrt{x}+2a+1}.$$

It is true if

$$k \ge \frac{2\sqrt{x}}{3\sqrt{x}+2a+1},$$

which is equivalent to

$$k(2a+1) \ge (2-3k)\sqrt{x},$$

$$2a+1 \ge \sqrt{2a^2+1}.$$

Case 3: $a\ge b=c>0$. We need to show that

$$(a+2c)\sqrt{x}+k(a-c)^2-3x \ge 0$$

for

$$x=\frac{a^2+2c^2}{3}.$$

Due to homogeneity, we may set $c=1$ (which involves $a\ge 1$), when the inequality becomes

$$(a+2)\sqrt{x}+k(a-1)^2-3x \ge 0,$$

where
$$x = \frac{a^2+2}{3}.$$
Write the inequality as follows:
$$k(a-1)^2 \geq \sqrt{x}(3\sqrt{x} - a - 2),$$
$$k(a-1)^2 \geq \frac{2(a-1)^2\sqrt{x}}{3\sqrt{x}+a+2}.$$
It is true if
$$k \geq \frac{2\sqrt{x}}{3\sqrt{x}+a+2},$$
which is equivalent to
$$k(a+2) \geq (2-3k)\sqrt{x},$$
$$a+2 \geq \sqrt{a^2+2}.$$

The proof is completed. The equality occurs for $a = b = c = 1$, and also for $a = \sqrt{3}$ and $b = c = 0$ (or any cyclic permutation).

Second Solution. Consider the nontrivial case $a > c$ and write the inequality in the homogeneous form $f(b) \geq 0$, where
$$f(b) = (a+b+c)\sqrt{g(b)} + k(a-c)^2 - 3g(b), \qquad g = \frac{a^2+b^2+c^2}{3}.$$
Since
$$g'(b) = \frac{2b}{3},$$
we have
$$f'(b) = \sqrt{g} + \frac{(a+b+c)b}{3\sqrt{g}} - 2b = bh(b),$$
where
$$h(b) = \frac{\sqrt{g}}{b} + \frac{a+b+c}{3\sqrt{g}} - 2,$$
$$h'(b) = \frac{-\sqrt{g}}{b^2} + \frac{1}{3\sqrt{g}} + \frac{1}{3\sqrt{g}} - \frac{b(a+b+c)}{9g^{3/2}}$$
$$\leq \frac{-\sqrt{g}}{b^2} + \frac{2}{3\sqrt{g}} = \frac{-3g+2b^2}{2b^2\sqrt{g}} \leq 0.$$
Since $h(b)$ is a decreasing function, there are three possible cases: $g(b) \geq 0$ for $b \in [c,a]$, hence $f(b)$ is increasing on $[c,a]$; $g(b) \geq 0$ for $b \in [c,c_1]$ and $g(b) \leq 0$ for $b \in [c_1,a]$, hence $f(b)$ is increasing on $[c,c_1]$ and decreasing on $[c_1,a]$; $g(b) \leq 0$ for $b \in [c,a]$, hence $f(b)$ is decreasing on $[c,a]$. In all these cases $f(b)$ is minimal when $b \in \{c,a\}$. As a consequence, we only need to prove the required inequality for $b = a$ and $b = c$. Both cases were presented in the first solution.

Remark. The inequality is symmetric because it can be written in the form

$$a+b+c+\left(1-\frac{1}{\sqrt{3}}\right)\max\{(a-b)^2,(b-c)^2,(c-a)^2\} \ge 3,$$

without the condition $a \ge b \ge c$.

□

P 3.111. *If a,b,c are nonnegative real numbers such that $a \ge b \ge c$ and*

$$a^2+b^2+c^2=3,$$

then

$$1-abc \;\le\; \sqrt{\frac{2}{3}}\,(a-c).$$

(Vasile C., 2018)

First Solution. Denoting $x=ac$, we need to show that $f(x)\ge 0$, where

$$f(x)=bx+\sqrt{\frac{2(3-b^2-2x)}{3}}-1.$$

For fixed b, $b\in\left[0,\sqrt{\frac{3}{2}}\right]$, we have $x\in[0,M]$, where

$$M=b\sqrt{3-2b^2}.$$

Indeed, from $(b^2-a^2)(b^2-c^2)\le 0$, we get

$$a^2c^2\le b^2(a^2+c^2)-b^4=3b^2-2b^4, \qquad x\le b\sqrt{3-2b^2}=M.$$

From

$$f''(x)=-\sqrt{\frac{2}{3}}\cdot\frac{1}{(3-b^2-2x)^{3/2}}\le 0,$$

it follows that f is a concave function, therefore it suffices to show that $f(0)\ge 0$ and $f(M)\ge 0$. We have

$$f(0)=\sqrt{\frac{2(3-b^2)}{3}}-1\ge 0$$

because

$$3=a^2+b^2+c^2\ge 2b^2.$$

Since

$$\sqrt{3-b^2-2M}=\left|\sqrt{3-2b^2}-b\right|,$$

we may write the inequality $f(M) \ge 0$ as follows:

$$\sqrt{\frac{2}{3}}\left|\sqrt{3-2b^2}-b\right| \ge 1-b^2\sqrt{3-2b^2},$$

$$\frac{\sqrt{6}|1-b^2|}{\sqrt{3-2b^2}+b} \ge \frac{(1-b^2)^2(1+2b^2)}{1+b^2\sqrt{3-2b^2}}.$$

This is true if

$$\frac{\sqrt{6}}{\sqrt{3-2b^2}+b} \ge \frac{|1-b^2|(1+2b^2)}{1+b^2\sqrt{3-2b^2}}.$$

Case 1: $b \in [0,1]$. We claim that

$$\frac{\sqrt{6}}{\sqrt{3-2b^2}+b} > 1 \ge \frac{(1-b^2)(1+2b^2)}{1+b^2\sqrt{3-2b^2}}.$$

Indeed, the left inequality is equivalent to

$$\sqrt{6}-b > \sqrt{3-2b^2},$$

which, by squaring, becomes

$$3b^2-2\sqrt{6}\,b+3 \ge 0,$$

$$3(b-1)^2+2(3-\sqrt{6})b \ge 0,$$

while the right inequality is equivalent to

$$2b^4+b^2(\sqrt{3-2b^2}-1) \ge 0.$$

Case 2: $b \in \left[1, \sqrt{\frac{3}{2}}\right]$. We need to show that

$$\frac{\sqrt{6}}{\sqrt{3-2b^2}+b} \ge \frac{(b^2-1)(1+2b^2)}{1+b^2\sqrt{3-2b^2}},$$

which can be written as

$$A(b)\sqrt{3-2b^2} \ge B(b),$$

where

$$A(b) = \sqrt{6}\,b^2-(b^2-1)(1+2b^2), \qquad B(b) = b(b^2-1)(1+2b^2)-\sqrt{6}.$$

This inequality is true because

$$A(b) = 1+(\sqrt{6}+1)b^2-2b^4 > 3b^2-2b^4 = b^2(3-2b^2) \ge 0$$

and

$$B(b) \le B\left(\sqrt{\frac{2}{3}}\right) = 0.$$

The equality occurs for $a = b = c = 1$, and also for $a = b = \sqrt{\frac{3}{2}}$ and $c = 0$ (or any cyclic permutation).

Second Solution (by Anhduy98). Denoting

$$p = a + b + c, \qquad q = ab + bc + ca,$$

we have

$$q = \frac{p^2 - 3}{2}, \quad p \le 3.$$

Since

$$(a-c)^2 - (a^2 + b^2 + c^2 - ab - bc - ca) = (a-b)(b-c) \ge 0,$$

it is enough to show that

$$abc + \sqrt{\frac{2(a^2 + b^2 + c^2 - ab - bc - ca)}{3}} \ge 1,$$

that is

$$abc \ge 1 - \sqrt{\frac{2(p^2 - 3q)}{3}}$$

or

$$abc \ge 1 - x,$$

where

$$x = \sqrt{\frac{2(p^2 - 3q)}{3}}.$$

From

$$p^2 - 3q = \frac{3x^2}{2}, \qquad p^2 - 2q = 3,$$

we get

$$p = \sqrt{3(3 - x^2)}, \qquad q = \frac{3(2 - x^2)}{2}, \qquad x \in [0, \sqrt{2}].$$

For $x \ge 1$, the inequality $abc \ge 1 - x$ is true because $abc \ge 0 \ge 1 - x$. Consider now that $1 \le x \le \sqrt{2}$. By Schur's inequality

$$p^3 + 9abc \ge 4pq,$$

we get

$$3abc \ge (1 - x^2)\sqrt{3(3 - x^2)}.$$

Thus, the inequality $abc \geq 1 - x$ holds if

$$(1-x^2)\sqrt{3(3-x^2)} \geq 3(1-x),$$

which is true if

$$(1+x)^2(3-x^2) \geq 3.$$

Indeed,

$$3-(1+x)^2(3-x^2) = x[x(x^2-2)+2(x^2-3)] \leq 0.$$

Third Solution. Denoting

$$F(a,b,c) = abc + \sqrt{\frac{2}{3}}\,(a-c)$$

and

$$x = \sqrt{\frac{a^2+b^2}{2}},$$

we can show that

$$F(a,b,c) \geq F(x,x,c) \geq 1.$$

Remark. The inequality is symmetric because it can be written in the form

$$1 - abc \leq \sqrt{\frac{2}{3}}\,\max\{|a-b|,|b-c|,|c-a|\},$$

without the condition $a \geq b \geq c$.

□

P 3.112. *If a,b,c are nonnegative real numbers such that $a \geq b \geq c$ and*

$$a^2+b^2+c^2 = 3,$$

then

$$1 - abc \;\leq\; \frac{7}{10}\,(a-c)^2.$$

(Vasile Cîrtoaje, 2018)

First Solution. Write the inequality as follows:

$$10abc + 7(a-c)^2 \geq 10,$$

$$10abc + 7(3-b^2-2ac) \geq 10,$$

$$11 - 7b^2 \geq 2(7-5b)ac.$$

From

$$3 = a^2 + b^2 + c^2 \geq 2b^2,$$

we get $b \leq \sqrt{\frac{3}{2}}$, hence

$$11 - 7b^2 > 0, \qquad 7 - 2b > 0.$$

On the other hand, from $(b^2 - a^2)(b^2 - c^2) \leq 0$, we get

$$a^2c^2 \leq b^2(a^2 + c^2) - b^4 = 3b^2 - 2b^4, \qquad ac \leq b\sqrt{3 - 2b^2}.$$

Thus, it is enough to show that

$$11 - 7b^2 \geq 2(7 - 5b)b\sqrt{3 - 2b^2}.$$

By squaring, this inequality becomes

$$200b^6 - 560b^5 + 141b^4 + 840b^3 - 742b^2 + 121 \geq 0,$$

or

$$(b-1)^2 f(b) \geq 0,$$

where

$$f(b) = 200b^4 - 160b^3 - 379b^2 + 242b + 121 = 8b(5b-6)^2\left(b + \frac{8}{5}\right)$$

$$+101b^2 - \frac{1094}{5}b + 121 > 100b^2 + 120 - 219b \geq \left(40\sqrt{30} - 219\right)b \geq 0.$$

The proof is completed. The equality occurs for $a = b = c = 1$.

Second Solution (by Nguyen Van Huyen). Since

$$(a-c)^2 - (a^2 + b^2 + c^2 - ab - bc - ca) = (a-b)(b-c) \geq 0,$$

it is enough to show that

$$abc + \frac{7}{10}(a^2 + b^2 + c^2 - ab - bc - ca) \geq 1,$$

which is equivalent to

$$10abc + 11 \geq 7(ab + bc + ca).$$

According to P 3.57, for $a^2 + b^2 + c^2 = 3$ and fixed $ab + bc + ca$ (which involve fixed $a + b + c$ and $ab + bc + ca$), the product abc is minimal when $c = 0$ or $a = b \geq c$. Thus, we only need to consider these cases.

Case 1: $c = 0$. We need to show that $a^2 + b^2 = 3$ involves $11 \ge 7ab$. Indeed, we have

$$2(11 - 7ab) \ge 22 - 7(a^2 + b^2) = 1 > 0.$$

Case 2: $a = b \ge c$. We need to show that $2a^2 + c^2 = 3$ involves

$$10a^2c + 11 \ge 7(a^2 + 2ac),$$

which is equivalent to

$$(10c - 7)a^2 + 11 \ge 14ac,$$

$$1 + 30c + 7c^2 - 10c^3 \ge 14c\sqrt{6 - 2c^2}.$$

Since $c \le 1$, the left side is positive. By squaring, the inequality becomes

$$100c^6 - 140c^5 - 159c^4 + 400c^3 - 262c^2 + 60c + 1 \ge 0,$$

$$(c-1)^2(100c^4 + 60c^3 - 139c^2 + 62c + 1) \ge 0.$$

This is true because

$$100c^4 + 60c^3 - 139c^2 + 62c + 1 \ge 100c^4 + 60c^3 - 140c^2 + 60c + 1$$

$$= (10c^2 - 1)^2 + 60c(c-1)^2 > 0.$$

□

P 3.113. *If* $a \ge b \ge c \ge \frac{1}{3}$ *and* $a^2 + b^2 + c^2 = 3$ *, then*

$$1 - abc \le \frac{11}{18}(a - c).$$

(Vasile C., 2018)

Solution. Denoting $x = ac$, we need to show that $f(x) \ge 0$, where

$$f(x) = bx - 1 + \frac{11}{18}\sqrt{3 - b^2 - 2x}.$$

From

$$(9a^2 - 1)(9c^2 - 1) \ge 0$$

and

$$(b^2 - a^2)(b^2 - c^2) \le 0,$$

we get $x \in [m, M]$, where

$$m = \frac{1}{9}\sqrt{26 - 9b^2}, \qquad M = b\sqrt{3 - 2b^2}.$$

We have $x = m$ for $c = \dfrac{1}{3}$, and $x = M$ for $a = b$ or $b = c$. Since

$$f''(x) = \frac{-11}{18(3-b^2-2x)^{3/}} < 0,$$

f is concave. Therefore, it suffices to show that $f(m) \ge 0$ and $f(M) \ge 0$, that means to prove the required inequality for $c = \dfrac{1}{3}$, for $a = b$ and for $b = c$.

Case 1: $c = \dfrac{1}{3}$. We need to prove that

$$18ab + 33a \ge 65$$

for $a^2 + b^2 = \dfrac{26}{9}$ and $a \ge b \ge \dfrac{1}{3}$. From

$$\frac{26}{9} = a^2 + b^2 \le 2a^2$$

and

$$\frac{26}{9} = a^2 + b^2 \ge a^2 + \frac{1}{9},$$

we get

$$\frac{6}{5} < \frac{\sqrt{13}}{3} \le a \le \frac{5}{3}.$$

Write the required inequality as follows:

$$6a\sqrt{26-9a^2} \ge 65 - 33a,$$

$$36a^2(26-9a^2) \ge (65-33a)^2,$$

$$324a^4 + 153a^2 - 4290a + 4225 \le 0,$$

$$(3a-5)(108a^3 + 180a^2 + 351a - 845) \le 0,$$

$$(3a-5)[(5a-6)(20a^2 + 60a + 141) + 8a^3 + 6a + 1] \le 0.$$

It is true since $3a - 5 \le 0$ and $5a - 6 > 0$.

Case 2: $a = b$. Consider the nontrivial case $a = b > c$, that is $c < 1$. We need to prove that

$$11(a-c) \ge 18(1-a^2c)$$

for $2a^2 + c^2 = 3$, $\dfrac{1}{3} \le c < 1$ and $3 = 2a^2 + c^2 \ge 2a^2 + \dfrac{1}{9}$, hence

$$a \le \frac{\sqrt{13}}{3}.$$

Write the required inequality as follows:

$$\frac{11(a^2-c^2)}{a+c} \ge 9(2-3c+c^3),$$

$$\frac{33(1-c)(1+c)}{a+c} \ge 18(1-c)^2(2+c),$$

$$\frac{11(1+c)}{a+c} \ge 6(1-c)(2+c),$$

$$\frac{11(1+c)}{6(1-c)(2+c)} \ge a+c.$$

We will show that

$$\frac{11(1+c)}{6(1-c)(2+c)} - c \ge \frac{26}{21} > \frac{\sqrt{13}}{3} \ge a.$$

The left inequality is equivalent to

$$42c^3+94c^2+45c-27 \ge 0,$$

$$(3c-1)(14c^2+36c+27) \ge 0.$$

Case 3: $b=c$. Consider the nontrivial case $a>b=c$, that is $a>1$. We need to prove that

$$11(a-c) \ge 18(1-ac^2)$$

for $a^2+2c^2=3$, and $3=a^2+2c^2 \ge a^2+\frac{1}{9}$, hence

$$1 < a \le \frac{5}{3}.$$

Write the required inequality as follows:

$$\frac{11(a^2-c^2)}{a+c} \ge 9(a^3-3a+2),$$

$$\frac{33(a-1)(a+1)}{a+c} \ge 18(a-1)^2(a+2),$$

$$\frac{11(a+1)}{a+c} \ge 6(a-1)(a+2),$$

$$\frac{11(a+1)}{6(a-1)(a+2)} \ge a+c.$$

We will show that

$$\frac{11(a+1)}{6(a-1)(a+2)} - a \ge \frac{16}{3} - 3a \ge c.$$

The left inequality is equivalent to

$$12a^3-20a^2-45a+75\le 0,$$

$$(5-3a)(15-4a^2)\ge 0.$$

The right inequality is equivalent to

$$16-9a\ge 3\sqrt{\frac{3-a^2}{2}},$$

$$2(16-9a)^2\ge 9(3-a^2),$$

$$171a^2-576a+485\ge 0,$$

$$(5-3a)(97-57a)\ge 0.$$

The inequality is an equality for $a=b=c=1$, and also for $a=\frac{5}{3}$, $b=c=\frac{1}{3}$ (or any cyclic permutation).

□

P 3.114. *If a,b,c are nonnegative real numbers such that $a\ge b\ge c$ and*

$$a^2+b^2+c^2=3,$$

then

$$1-\sqrt{abc}\ \le\ \frac{2}{3}\,(a-c)^2.$$

(Vasile C., 2019)

First Solution. Denoting $x=ac$, we need to show that $f(x)\ge 0$, where

$$f(x)=3\sqrt{bx}+2(3-b^2-2x)-3=3\sqrt{bx}-4x+3-2b^2.$$

For fixed b, $b\in\left[0,\sqrt{\frac{3}{2}}\right]$, we have $x\in[0,M]$, where

$$M=b\sqrt{3-2b^2}.$$

Indeed, from $(b^2-a^2)(b^2-c^2)\le 0$, we get

$$a^2c^2\le b^2(a^2+c^2)-b^4=3b^2-2b^4,\qquad x\le b\sqrt{3-2b^2}=M.$$

From

$$f''(x)=-\frac{3}{4}\sqrt{b}x^{-3/2}<0,$$

it follows that f is a concave function, therefore it suffices to show that $f(0) \geq 0$ and $f(M) \geq 0$. We have

$$f(0) = 3 - 2b^2 \geq 0$$

because

$$3 = a^2 + b^2 + c^2 \geq 2b^2.$$

Write the inequality $f(M) \geq 0$ as follows:

$$3\sqrt{bM} - 4M + 3 - 2b^2 \geq 0.$$

Using the substitution

$$x = \sqrt[4]{3 - 2b^2}, \qquad 0 \leq x \leq \sqrt[4]{3},$$

which yields $M = bx^2$, the inequality becomes

$$3bx - 4bx^2 + x^4 \geq 0,$$

$$x^3 \geq b(4x - 3).$$

Consider the nontrivial case $x \geq 4/3$. By squaring, the inequality becomes

$$2x^6 \geq (3 - x^4)(4x - 3)^2,$$

$$6x^6 - 8x^5 + 3x^4 - 16x^2 + 24x - 9 \geq 0,$$

$$(x - 1)^2(6x^4 + 4x^3 + 5x^2 + 6x - 9) \geq 0.$$

It is true because

$$4x^3 + 5x^2 + 6x - 9 = (4x - 3)(x^2 + 2x + 3) \geq 0.$$

The equality occurs for $a = b = c = 1$, and also for $a = b = \sqrt{\dfrac{3}{2}}$ and $c = 0$ (or any cyclic permutation).

Second Solution. Denoting

$$p = a + b + c, \qquad q = ab + bc + ca,$$

we have

$$q = \frac{p^2 - 3}{2}, \quad p \leq 3.$$

Since

$$(a - c)^2 - (a^2 + b^2 + c^2 - ab - bc - ca) = (a - b)(b - c) \geq 0,$$

it is enough to show that

$$1 - \sqrt{abc} \leq \frac{2}{3}(a^2 + b^2 + c^2 - ab - bc - ca),$$

that is

$$\sqrt{abc} \geq 1 - frac23(p^2 - 3q).$$

From

$$3 = a^2 + b^2 + c^2 = p^2 - 2q,$$

we obtain

$$q = \frac{p^2 - 3}{2}.$$

Thus, the required inequality can be written as

$$\sqrt{abc} \geq \frac{p^2 - 6}{3}.$$

Since the inequality is true for $p \leq \sqrt{6}$, consider further $p \geq \sqrt{6}$ and write the inequality as

$$9abc \geq (p^2 - 6)^2.$$

There are two cases to consider: $\sqrt{6} \leq p \leq 3$ and $p \geq 3$.

Case 1: $\sqrt{6} \leq p \leq 3$. By Schur's inequality of third degree, we have

$$9abc \geq 4pq - p^3 = \frac{16p(p^2 - 3) - (p^2 - 3)^3}{8}.$$

Thus, it suffices to show that

$$\frac{16p(p^2 - 3) - (p^2 - 3)^3}{8} \geq (p^2 - 6)^2,$$

which is equivalent to

$$p^6 - p^4 - 16p^3 - 69p^2 + 48p + 261 \leq 0,$$

$$(p - 3)(p^5 + 3p^4 + 8p^3 + 8p^2 - 45p - 29) \leq 0.$$

This is true if

$$p^5 + 3p^4 + 8p^3 + 8p^2 - 45p - 87 \geq 0.$$

Case 2: $p \geq 3$. By Schur's inequality of fourth degree, we have

$$6abcp \geq (p^2 - q)(4q - p^2) = \frac{1}{2}(p^2 + 3)(p^2 - 6).$$

Thus, it suffices to show that

$$\frac{3(p^2 + 3)(p^2 - 6)}{4p} \geq (p^2 - 6)^2,$$

which is true if

$$\frac{3(p^2 + 3)}{4p} \geq (p^2 - 6),$$

$$4p^3 - 3p^2 - 24p - 9 \geq 0,$$

$$(p - 3)(4p^2 + 9p + 3) \geq 0.$$

□

P 3.115. *If a,b,c are nonnegative real numbers such that $a \ge b \ge c$ and*

$$a^2+b^2+c^2=3,$$

then

$$1-abc \;\le\; \frac{2}{3}\,a(a-c)^2.$$

(Vasile Cîrtoaje, 2020)

Solution. Let us denote

$$g=\sqrt{\frac{a^2+b^2+c^2}{3}},$$

and write the inequality in the homogeneous form $f \le 0$, where

$$f=3g^3-3abc-2a(a-c)^2.$$

For fixed a and b, since

$$g'(c)=\frac{c}{3g},$$

we have

$$f'(c)=3cg-3ab+4a(a-c)=c\sqrt{3(a^2+b^2+c^2)}-3ab+4a(a-c).$$

We will show that

$$c\sqrt{3(a^2+b^2+c^2)}-3ab \ge c\sqrt{3(2a^2+c^2)}-3a^2,$$

which is equivalent to

$$3a(a-b) \ge c\sqrt{3(2a^2+c^2)}-c\sqrt{3(a^2+b^2+c^2)},$$

$$3a(a-b) \ge \frac{3c(a-b)(a+b)}{\sqrt{3(2a^2+c^2)}+\sqrt{3(a^2+b^2+c^2)}}.$$

This is true if

$$1 \ge \frac{a+b}{2\sqrt{3(a^2+b^2+c^2)}},$$

which is obvious. So, we have

$$f'(c) \ge c\sqrt{3(2a^2+c^2)}-3a^2+4a(a-c)=c\sqrt{3(2a^2+c^2)}+a^2-4ac$$

$$\ge c(2a+c)+a^2-4ac=(a-c)^2 \ge 0.$$

Since $f(c)$ is increasing, we have $f(c) \le f(b)$, and it remains to prove that $f(b) \le 0$, that is

$$3\left(\frac{a^2+2b^2}{3}\right)^{3/2}-3ab^2-2a(a-b)^2 \le 0.$$

For $b=0$, the inequality is trivial. For $b>0$, due to homogeneity, we may consider $b=1$, when $a \ge 1$. The inequality becomes

$$3\left(\frac{a^2+2}{3}\right)^{3/2}-3a-2a(a-1)^2 \le 0,$$

$$2a^3-4a^2+5a \ge 3\left(\frac{a^2+2}{3}\right)^{3/2},$$

and, by squaring,

$$11a^6-48a^5+102a^4-120a^3+63a^2-8 \ge 0,$$

$$(a-1)^3(11a^3-15a^2+24a+8) \ge 0.$$

The last inequality is clearly true for $a \ge 1$.

The equality occurs for $a=b=c=1$.

Remark. Denoting

$$x=\max\{a,b,c\}, \qquad y=\min\{a,b,c\},$$

we may remove the condition $a \ge b \ge c$ to write the inequality in the symmetric form

$$1-abc \ge \frac{2}{3}\,x(x-y)^2.$$

□

P 3.116. *If a,b,c are nonnegative real numbers such that $a \ge b \ge c$ and*

$$a^2+b^2+c^2=3,$$

then

$$1-abc \ \le\ \frac{1}{9}(5a+c)(a-c)^2.$$

(Vasile Cîrtoaje, 2020)

Solution. Let us denote

$$g=\sqrt{\frac{a^2+b^2+c^2}{3}},$$

and write the inequality in the homogeneous form $f \le 0$, where

$$f=3g^3-3abc-\frac{1}{3}(5a+c)(a-c)^2.$$

For fixed a and b, since

$$g'(c)=\frac{c}{3g},$$

we have

$$f'(c) = 3cg - 3ab + (a-c)(3a+c) = c\sqrt{3(a^2+b^2+c^2)} - 3ab + (a-c)(3a+c).$$

We will show that

$$c\sqrt{3(a^2+b^2+c^2)} - 3ab \geq c\sqrt{3(2a^2+c^2)} - 3a^2,$$

which is equivalent to

$$3a(a-b) \geq c\sqrt{3(2a^2+c^2)} - c\sqrt{3(a^2+b^2+c^2)},$$

$$3a(a-b) \geq \frac{3c(a-b)(a+b)}{\sqrt{3(2a^2+c^2)} + \sqrt{3(a^2+b^2+c^2)}}.$$

This is true if

$$1 \geq \frac{a+b}{2\sqrt{3(a^2+b^2+c^2)}},$$

which is obvious. So, we have

$$f'(c) \geq c\sqrt{3(2a^2+c^2)} - 3a^2 + (a-c)(3a+c) = c\sqrt{3(2a^2+c^2)} - 2ac - c^2$$

$$= c\left[\sqrt{3(2a^2+c^2)} - 2a - c\right] = \frac{2c(a-c)^2}{\sqrt{3(2a^2+c^2)} + 2a + c} \geq 0.$$

Since $f(c)$ is increasing, we have $f(c) \leq f(b)$, and it remains to prove that $f(b) \leq 0$, that is

$$9\left(\frac{a^2+2b^2}{3}\right)^{3/2} - 9ab^2 - (5a+b)(a-b)^2 \leq 0.$$

For $b = 0$, the inequality is trivial. For $b > 0$, due to homogeneity, we may consider $b = 1$, when $a \geq 1$. The inequality becomes

$$9\left(\frac{a^2+2}{3}\right)^{3/2} - 9a - (5a+1)(a-1)^2 \leq 0,$$

$$5a^3 - 9a^2 + 12a + 1 \geq 9\left(\frac{a^2+2}{3}\right)^{3/2},$$

and, by squaring,

$$(5a^3 - 9a^2 + 12a + 1)^2 \geq 3(a^2+2)^3,$$

$$22a^6 - 90a^5 + 183a^4 - 206a^3 + 90a^2 + 24a - 23 \geq 0,$$

$$(a-1)^3(22a^3 - 24a^2 + 45a + 23) \geq 0.$$

The last inequality is clearly true for $a \geq 1$.

The equality occurs for $a = b = c = 1$.

□

P 3.117. *If a,b,c are nonnegative real numbers such that $a \ge b \ge c$ and*

$$a^2+b^2+c^2=3,$$

then

$$1-abc \ \ge \ \frac{2}{3}(b-c)^2.$$

(Vasile Cîrtoaje, 2019)

First Solution. Write the inequality as follows:

$$3-3abc \ge 2(3-a^2-2bc),$$

$$(4-3a)bc \ge 3-2a^2.$$

We have

$$3a^2 \ge a^2+b^2+c^2=3, \qquad a \ge 1.$$

Case 1: $1 \le a \le \sqrt{\frac{3}{2}}$. From $(a^2-b^2)(a^2-c^2) \ge 0$, we get

$$b^2c^2 \ge a^2(b^2+c^2)-a^4 = a^2(3-a^2)-a^4, \qquad bc \ge a\sqrt{3-2a^2}.$$

Thus, it is enough to show that

$$a(4-3a)\sqrt{3-2a^2} \ge 3-2a^2.$$

This is true if

$$a(4-3a) \ge \sqrt{3-2a^2},$$

which, by squaring, becomes

$$3a^4-8a^3+6a^2-1 \ge 0,$$

$$(a-1)^3(3a+1) \ge 0.$$

Case 2: $\sqrt{\frac{3}{2}} \le a \le \frac{4}{3}$. We have

$$3-2a^2 \le 0 \le (4-3a)bc.$$

Case 3: $\frac{4}{3} \le a \le \sqrt{3}$. Since

$$2bc \le b^2+c^2 = 3-a^2,$$

it is enough to show that

$$(3a-4)(3-a^2) \le 2(2a^2-3).$$

which is equivalent to

$$a^3 - 3a + 2 \geq 0,$$

$$(a-1)^2(a+2) \geq 0.$$

The equality occurs for $a = b = c = 1$, and also for $a = b = \sqrt{\frac{3}{2}}$ and $c = 0$.

Second Solution (by *Ali3985*). From

$$3 = a^2 + b^2 + c^2 \geq 2ab + c^2 \geq 2b^2 + c^2,$$

we get

$$ab \leq \frac{1}{2}(3 - c^2), \qquad b \leq \frac{3 - c^2}{2}\ .$$

Thus,

$$abc + \frac{2}{3}(b-c)^2 \leq \frac{1}{2}\left(3 - c^2\right)c + \frac{2}{3}\left(\sqrt{\frac{3-c^2}{2}} - c\right)^2,$$

and it remains to show that

$$\frac{1}{2}\left(3 - c^2\right)c + \frac{2}{3}\left(\sqrt{\frac{3-c^2}{2}} - c\right)^2 \leq 1,$$

which is equivalent to

$$8c\sqrt{\frac{3-c^2}{2}} \geq -3c^3 + 2c^2 + 9c.$$

By squaring, the inequality becomes

$$c^2(1-c)^3(3c+5) \geq 0.$$

This is true because

$$3c^2 \leq a^2 + b^2 + c^2 = 3, \quad c \leq 1.$$

□

P 3.118. *Let a, b, c be nonnegative real numbers such that*

$$a^2 + b^2 + c^2 = 3.$$

If $a \geq b \geq c$, then

$$1 - abc \ \geq\ \frac{2}{3}(1 + \sqrt{2})(a-b)(b-c).$$

(Vasile Cîrtoaje, 2019)

Solution. Write the inequality in the homogeneous form $F \ge 0$, where

$$F = g^{3/2} - abc - k(a-b)(b-c)g^{1/2}, \quad k = \frac{2}{3}(1+\sqrt{2}), \quad g = \frac{a^2+b^2+c^2}{3}.$$

Denote

$$x = a-c, \quad y = b-c, \quad x \ge y \ge 0,$$

and consider that x and y are fixed. We need to show that $F(c) \ge 0$. Since $a' = b' = c' = 1$ and

$$g' = \frac{2}{3}(a+b+c),$$

we have

$$F'(c) = (a+b+c)g^{1/2} - (bc+ca+ab) - \frac{k}{3}(a-b)(b-c)(a+b+c)g^{-1/2} \ge E(c),$$

where

$$E(c) = (a+b+c)g^{1/2} - (ab+bc+ca) - k(a-b)(b-c).$$

By the AM-GM inequality, we have

$$E'(c) = 3g^{1/2} + \frac{1}{3}(a+b+c)^2 g^{-1/2} - 2(a+b+c) \ge 2(a+b+c) - 2(a+b+c) = 0.$$

As a consequence, $E(c)$ is increasing, hence $E(c) \ge E(0)$, where

$$E(0) = (a+b)\sqrt{\frac{a^2+b^2}{3}} - ab - k(a-b)b.$$

We will show that $E(0) \ge 0$. Since $a^2+b^2 \ge \frac{1}{2}(a+b)^2$, it is enough to show that

$$\frac{1}{\sqrt{6}}(a+b)^2 \ge ab + k(a-b)b.$$

Since $\sqrt{6} < 32/13$ and $k < 13/8$, it suffices to prove that

$$\frac{13}{32}(a+b)^2 \ge ab + \frac{13}{8}(a-b)b,$$

which is equivalent to

$$13(a^2+5b^2) \ge 58ab.$$

Indeed,

$$13(a^2+5b^2) \ge 26\sqrt{5}\, ab \ge 58ab.$$

From $E(c) \ge E(0) \ge 0$, it follows that $F'(c) \ge 0$, $F(c)$ is increasing, therefore $F(c) \ge F(0)$. So, we only need to show that $F(0) \ge 0$, that is

$$\frac{a^2+b^2}{3} - k(a-b)b \ge 0,$$

$$a^2+(3k+1)b^2 \geq 3kab.$$

Indeed, we have

$$a^2+(3k+1)b^2 \geq 2\sqrt{3k+1}\,ab \geq 3kab.$$

The proof is completed. The equality occurs for $a=b=c=1$, and also for $a=\sqrt{3k+1}\,b$ and $c=0$, that is

$$a=\frac{1}{2}\sqrt{6+3\sqrt{2}}, \quad b=\frac{1}{2}\sqrt{6-3\sqrt{2}}, \quad c=0.$$

□

P 3.119. *If a,b,c are nonnegative real numbers such that $a \geq b \geq c$ and*

$$a^2+b^2+c^2=3,$$

then

(a) $$1-abc \geq 2b(a-b)(b-c);$$

(b) $$1-abc \geq (a-c)(a-b)(b-c);$$

(c) $$1-abc \geq a(a-b)(b-c);$$

(d) $$1-abc \geq (a+c)(a-b)(b-c).$$

(Vasile Cîrtoaje, 2020)

Solution. (a) Since

$$4(a-b)(b-c) \leq (a-b+b-c)^2=(a-c)^2,$$

it is enough to show that

$$1-abc \geq \frac{1}{2}b(a-c)^2,$$

which is equivalent to

$$2 \geq b(a^2+c^2),$$

$$2 \geq b(3-b^2),$$

$$(b-1)^2(b+2) \geq 0.$$

The equality occurs for $a=b=c=1$.

(b) Write the inequality in the homogeneous form

$$\left(\frac{a^2+b^2+c^2}{3}\right)^{3/2} - abc-(a-b)(b-c)(a-c) \geq 0.$$

Use the substitution

$$a = x + c, \quad b = y + c, \quad x \ge y \ge 0,$$

and, for fixed x and y, write the inequality as $f(c) \ge 0$, where

$$f(c) = \left(\frac{a^2 + b^2 + c^2}{3}\right)^{3/2} - abc - (a-b)(b-c)(a-c).$$

Since

$$a' = b' = c' = 1,$$

we have

$$f'(c) = (a+b+c)\frac{a^2+b^2+c^2}{3} - ab - bc - ca$$
$$\ge \frac{1}{3}(a+b+c)^2 - ab - bc - ca \ge 0.$$

Since $f(c)$ is increasing, we have $f(c) \ge f(0)$. Therefore, it is enough to show that $f(0) \ge 0$, that is

$$\left(\frac{a^2+b^2}{3}\right)^{3/2} \ge ab(a-b).$$

By squaring, we need to show that

$$(a^2+b^2)^3 \ge 27(a^2+b^2-2ab)a^2b^2,$$

$$(a^2+b^2-3ab)^2(a^2+b^2+6ab) \ge 0.$$

The equality occurs for $a = b = c = 1$, and also for $a^2 + b^2 - 3ab = 0$ and $c = 0$, that is

$$a = \frac{\sqrt{5}+1}{2}, \qquad b = \frac{\sqrt{5}-1}{2}, \qquad c = 0.$$

(c) According to (a), the inequality is true if $a \le 2b$. Consider further the case $a \ge 2b$, denote

$$g = \sqrt{\frac{a^2+b^2+c^2}{3}}, \qquad g \le a,$$

and write the inequality in the homogeneous form $f \ge 0$, where

$$f = g^3 - abc - a(a-b)(b-c).$$

For fixed a and b $(a \ge 2b)$, since

$$g'(c) = \frac{c}{3g},$$

we have

$$f'(c) = cg - ab + a(a-b) = cg + a(a-2b) \ge 0.$$

Thus, $f(c)$ is increasing, hence

$$f(c) \geq f(0) = \left(\frac{a^2+b^2}{3}\right)^{3/2} - ab(a-b).$$

We only need to show that

$$(a^2+b^2)^3 \geq 27a^2b^2(a-b)^2,$$

which is equivalent to

$$(a^2+b^2)^3 \geq 27a^2b^2(a^2+b^2) - 54a^3b^3 \geq 0,$$

$$(a^2+b^2-3ab)^2(a^2+b^2+6ab) \geq 0.$$

The equality occurs for $a=b=c=1$, and also for

$$a = \frac{\sqrt{5}+1}{2}, \qquad b = \frac{\sqrt{5}-1}{2}, \qquad c = 0.$$

(d) Write the required inequality as follows:

$$1 \geq abc + (a+c)(-b^2+ab+bc-ca),$$

$$\left(\frac{a^2+b^2+c^2}{3}\right)^{3/2} \geq (a+c-b)(ab+bc-ca),$$

$$(a^2+b^2+c^2)^3 \geq 27(a-b+c)^2(ab+bc-ca)^2,$$

$$(a^2+b^2+c^2)^3 \geq 27\left[a^2+b^2+c^2-2(ab+bc-ca)\right](ab+bc-ca)^2,$$

$$\left[a^2+b^2+c^2-3(ab+bc-ca)\right]^2\left[a^2+b^2+c^2+6(ab+bc-ac)\right] \geq 0.$$

The equality occurs for $a=b=c=1$, and also for $a^2+b^2+c^2=3$ and $(a+c)b = 1+ac$.

Remark Denoting

$$x = \max\{a,b,c\}, \qquad y = \min\{a,b,c\}, \qquad s = a+b+c,$$

we may remove the condition $a \geq b \geq c$ to write the inequalities in the symmetric forms:

(a) $$1 - abc \ \geq\ 2(s-x-y)(2x+y-s))(s-x-2y);$$

(b) $$1 - abc \ \geq\ (x-y)(2x+y-s))(s-x-2y);$$

(c) $$1 - abc \ \geq\ x(2x+y-s))(s-x-2y);$$

(d) $$1 - abc \ \geq\ (x+y)(2x+y-s))(s-x-2y).$$

□

P 3.120. *If a,b,c are nonnegative real numbers such that $a \ge b \ge c$ and*

$$a^2+b^2+c^2=3,$$

then

(a) $$1-abc \ \ge\ \frac{2}{3}\,b(a-b)^2;$$

(b) $$1-abc \ \ge\ \frac{2}{27}\,(2a+7b)(a-b)^2.$$

(Vasile Cîrtoaje, 2020)

Solution. Let us denote

$$g=\sqrt{\frac{a^2+b^2+c^2}{3}}$$

and

$$f=g^3-abc.$$

For fixed a and b, since

$$g'(c)=\frac{c}{3g},$$

we have

$$f'(c)=cg-ab\le bg-ab=b(g-a)\le 0.$$

Since $f(c)$ is decreasing, we have $f(c)\ge f(b)$.

(a) We need to prove the homogeneous inequality

$$3f(b)\ge 2b(a-b)^2,$$

that is

$$3\left(\frac{a^2+2b^2}{3}\right)^{3/2}-3ab^2-2b(a-b)^2\ge 0.$$

For $b=0$, the inequality is trivial. For $b>0$, due to homogeneity, we may consider $b=1$, when $a\ge 1$. The inequality becomes

$$3\left(\frac{a^2+2}{3}\right)^{3/2}-3a-2(a-1)^2\ge 0,$$

$$3\left(\frac{a^2+2}{3}\right)^{3/2}\ge 2a^2-a+2,$$

and, by squaring,

$$(a^2+2)^3\ge 3(2a^2-a+2)^2,$$

$$a^6-6a^4+12a^3-15a^2+12a-4\ge 0,$$

$$(a-1)^3(a^3+3a^2+4)\ge 0.$$

The equality occurs for $a=b=c=1$.

(b) We need to prove the homogeneous inequality

$$27f(b) \ge 2b(2a+7b)(a-b)^2,$$

that is

$$27\left(\frac{a^2+2b^2}{3}\right)^{3/2} - 27ab^2 - 2(2a+7b)(a-b)^2 \ge 0.$$

For $b=0$, the inequality is trivial. For $b>0$, due to homogeneity, we may consider $b=1$, when $a \ge 1$. The inequality becomes

$$27\left(\frac{a^2+2}{3}\right)^{3/2} - 27a - 2(2a+7)(a-1)^2 \ge 0,$$

$$27\left(\frac{a^2+2}{3}\right)^{3/2} \ge 4a^3+6a^2+3a+14,$$

and, by squaring,

$$27(a^2+2)^3 \ge (4a^3+6a^2+3a+14)^2,$$

$$11a^6 - 48a^5 + 102a^4 - 148a^3 + 147a^2 - 84a + 20 \ge 0,$$

$$(a-1)^4(11a^2-4a+20) \ge 0.$$

The equality occurs for $a=b=c=1$.

□

P 3.121. *If a,b,c are nonnegative real numbers such that $a \ge b \ge c$ and*

$$a^2+b^2+c^2=3,$$

then

(a) $$1-abc \ge \frac{1}{3}(b+c)(b-c)^2;$$

(b) $$1-abc \ge \frac{2}{27}(7b+2c)(b-c)^2.$$

(Vasile Cîrtoaje, 2020)

Solution. Let us denote

$$g = \sqrt{\frac{a^2+b^2+c^2}{3}}$$

and

$$f = g^3 - abc.$$

For fixed b and c, since
$$g'(a)=\frac{a}{3g},$$
we have
$$f'(a)=ag-bc\le bg-bc=b(g-c)\ge 0.$$
Since $f(a)$ is increasing, we have $f(a)\ge f(b)$.

(a) We need to prove the homogeneous inequality
$$3f(b)\ge (b+c)(b-c)^2,$$
that is
$$3\left(\frac{2b^2+c^2}{3}\right)^{3/2}-3b^2c-(b+c)(b-c)^2\ge 0.$$
For $c=0$, the inequality is trivial. For $c>0$, due to homogeneity, we may consider $c=1$, when $b\ge 1$. The inequality becomes
$$3\left(\frac{2b^2+1}{3}\right)^{3/2}-3b^2-(b+1)(b-1)^2\ge 0,$$
$$3\left(\frac{2b^2+1}{3}\right)^{3/2}\ge b^3+2b^2-b+1,$$
and, by squaring,
$$(2b^2+1)^3\ge 3(b^3+2b^2-b+1)^2,$$
$$5b^6-12b^5+6b^4+6b^3-9b^2+6b-2\ge 0,$$
$$(b-1)^3(b+1)(5b^2-2b+2)\ge 0.$$
The equality occurs for $a=b=c=1$.

(b) We need to prove the homogeneous inequality
$$27f(b)\ge 2(7b+2c)(b-c)^2,$$
that is
$$27\left(\frac{2b^2+c^2}{3}\right)^{3/2}-27b^2c-2(7b+2c)(b-c)^2\ge 0.$$
For $c=0$, the inequality is trivial. For $c>0$, due to homogeneity, we may consider $c=1$, when $b\ge 1$. The inequality becomes
$$27\left(\frac{2b^2+1}{3}\right)^{3/2}-27b^2-2(7b+2)(b-1)^2\ge 0,$$
$$27\left(\frac{2b^2+1}{3}\right)^{3/2}\ge 14b^3+3b^2+6b+4,$$

and, by squaring,

$$27(2b^2+1)^3 \ge (14b^3+3b^2+6b+4)^2,$$

$$20b^6-84b^5+147b^4-148b^3+102b^2-48b+11 \ge 0,$$

$$(b-1)^4(20b^2-4b+11) \ge 0.$$

The equality occurs for $a=b=c=1$.

□

P 3.122. *If a,b,c are nonnegative real numbers such that $a^2+b^2+c^2=3$, then*

(a) $$1-\sqrt{abc} \ \ge\ (a-b)(b-c);$$

(b) $$1-\sqrt[3]{a^2b^2c^2} \ \ge\ \frac{4}{3}(a-b)(b-c).$$

(Vasile Cîrtoaje, 2020)

Solution. (a) Write the inequality as follows:

$$4-4\sqrt{abc} \ge 4(ab+bc-b^2-ac),$$

$$a^2+b^2+c^2+1-4\sqrt{abc} \ge 4(ab+bc-b^2-ac),$$

$$(a-2b+c)^2+1+b^2+2ac \ge 4\sqrt{abc},$$

$$(a-2b+c)^2+(1-b)^2+2(\sqrt{b}-\sqrt{ac})^2 \ge 0.$$

The equality occurs for $a=b=c=1$.

(b) Write the inequality as follows (*Kiyoras*_2001):

$$a^2+b^2+c^2-3\sqrt[3]{a^2b^2c^2}-4(a-b)(b-c) \ge 0,$$

$$(a-2b+c)^2+b^2+2ac-3\sqrt[3]{a^2b^2c^2} \ge 0.$$

Since

$$b^2+ac+ac \ge 3\sqrt[3]{a^2b^2c^2}.$$

(by AM-GM), the proof is completed. The equality occurs for $a=b=c=1$.

Remark. The inequality (b) is stronger than (a) because

$$1-\sqrt{abc} \ge \frac{3}{4}\left(1-\sqrt[3]{a^2b^2c^2}\right),$$

which is equivalent to

$$(x-1)^2(3x^2+2x+1) \ge 0,$$

where $x=\sqrt[6]{abc}$.

□

P 3.123. *If a,b,c are nonnegative real numbers such that $a \ge b \ge c$ and*

$$a^2+b^2+c^2=3,$$

then

$$1-abc \;\ge\; \frac{2}{3}b(\sqrt{a}-\sqrt{c})^2.$$

(Vasile Cîrtoaje, 2020)

Solution. Let us denote

$$g=\sqrt{\frac{a^2+b^2+c^2}{3}}, \qquad \sqrt{bc}\le g\le a,$$

and write the inequality in the homogeneous form $f\ge 0$, where

$$f=a^2+b^2+c^2-\frac{3abc}{g}-2b(\sqrt{a}-\sqrt{c})^2.$$

For fixed b and c, since

$$g'(a)=\frac{a}{3g}\ ,$$

we have

$$f'(a)=2a-\frac{3bc}{g}+\frac{a^2bc}{g^3}-\frac{2b(\sqrt{a}-\sqrt{c})}{\sqrt{a}}$$

$$\ge 2a-\frac{2bc}{g}-2b\left(1-\sqrt{\frac{c}{a}}\right)\ge 2a-\frac{2bc}{\sqrt{bc}}-2b\left(1-\sqrt{\frac{c}{a}}\right)$$

$$=2a-2\sqrt{bc}-2b\left(1-\sqrt{\frac{c}{a}}\right)\ge 2a-2\sqrt{ac}-2a\left(1-\sqrt{\frac{c}{a}}\right)=0.$$

Since $f(a)$ is increasing, we have $f(a)\ge f(b)$, and it remains to prove that $f(b)\ge 0$, that is

$$2b^2+c^2-\frac{3b^2c}{g}\ge 2b(\sqrt{b}-\sqrt{c})^2,$$

where

$$g=\sqrt{\frac{2b^2+c^2}{3}}.$$

For $c=0$, the inequality is an equality. For $c>0$, due to homogeneity, we may consider $c=1$, when $b\ge 1$. The inequality becomes

$$2b^2+1-\frac{3b^2}{g}\ge 2b(\sqrt{b}-1)^2,$$

where

$$g=\sqrt{\frac{2b^2+1}{3}}.$$

Since

$$g \geq \frac{2b+1}{3},$$

it is enough to show that

$$2b^2 + 1 - \frac{9b^2}{2b+1} \geq 2b(\sqrt{b} - 1)^2,$$

which is equivalent to

$$\frac{(b-1)^2(4b+1)}{2b+1} \geq 2b(\sqrt{b} - 1)^2.$$

This is true if

$$\frac{(\sqrt{b}+1)^2(4b+1)}{2b+1} \geq 2b,$$

which is equivalent to

$$8b(\sqrt{b} - 1) + 3(b-1) + 2\sqrt{b} \geq 0.$$

The equality occurs for $a = b = c = 1$, and also for $a = b = \sqrt{\frac{3}{2}}$ and $c = 0$.

Remark. We claim that, in the same conditions, the following stronger inequality holds:

$$1 - abc \geq \frac{2}{3}(b + 2c)(\sqrt{a} - \sqrt{c})^2.$$

□

P 3.124. *If a, b, c are nonnegative real numbers such that $a \geq b \geq c$ and*

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \ \geq \ \frac{8}{3}b(\sqrt{a} - \sqrt{b})^2.$$

(Vasile Cîrtoaje, 2020)

Solution. Let us denote

$$g = \sqrt{\frac{a^2 + b^2 + c^2}{3}}, \qquad g \leq a,$$

and write the inequality in the homogeneous form $f \geq 0$, where

$$f = a^2 + b^2 + c^2 - \frac{3abc}{g} - 8b(\sqrt{a} - \sqrt{b})^2.$$

For fixed a and b, since
$$g'(c) = \frac{c}{3g},$$
we have
$$f'(c) = 2c - \frac{3ab}{g} + \frac{abc^2}{g^3} = 2c - \frac{3ab}{g} + \frac{3abc^2}{g(a^2+b^2+c^2)}$$
$$= 2c - \frac{3ab(a^2+b^2)}{g(a^2+b^2+c^2)} \le 2c - \frac{2ab}{g}$$
$$\le 2b - \frac{2ab}{g} = 2b\left(1 - \frac{a}{g}\right) \le 0.$$
Since $f(c)$ is decreasing, we have $f(c) \ge f(b)$, and it remains to prove that $f(b) \ge 0$, that is
$$a^2 + 2b^2 - \frac{3ab^2}{g} \ge 8b(\sqrt{a} - \sqrt{b})^2,$$
where
$$g = \sqrt{\frac{a^2+2b^2}{3}}.$$
For $b = 0$, the inequality is true. For $b > 0$, due to homogeneity, we may consider $b = 1$, when $a \ge 1$. The inequality becomes
$$a^2 + 2 - \frac{3a}{g} \ge 8(\sqrt{a} - 1)^2,$$
where
$$g = \sqrt{\frac{a^2+2}{3}}.$$
Since
$$a^2 + 2 - \frac{3a}{g} = \frac{(a^2+2)^2 - \frac{9a^2}{g^2}}{a^2 + 2 + \frac{3a}{g}} \ge \frac{(a^2+2)^2 - \frac{9a^2}{g^2}}{2(a^2+2)}$$
$$= \frac{(a^2+2)^3 - 27a^2}{2(a^2+2)^2} = \frac{(a^2-1)^2(a^2+8)}{2(a^2+2)^2},$$
it suffices to show that
$$\frac{(a^2-1)^2(a^2+8)}{2(a^2+2)^2} \ge 8(\sqrt{a} - 1)^2,$$
which is true if
$$(\sqrt{a} + 1)^2(a+1)^2(a^2+8) \ge 16(a^2+2)^2.$$
Denoting $x = \sqrt{a}$, $x \ge 1$, we can write this inequality as follows:
$$(x+1)^2(x^2+1)^2(x^4+8) \ge 16(x^4+2)^2,$$

$$x^{10}+2x^9-13x^8+4x^7+11x^6+18x^5-39x^4+32x^3+24x^2+16x-56\ge 0,$$
$$(x-1)h(x)\ge 0,$$

where

$$h(x)=x^9+3x^8-10x^7-6x^6+5x^5+23x^4-16x^3+16x^2+40x+56$$
$$=x^4h_1(x)+4x^2(x-2)^2+40x+56,$$
$$h_1(x)=x^5+3x^4-10x^3-6x^2+5x+19.$$

To complete the proof, we need to show that $h_1(x)\ge 0$. Indeed, we have

$$54h_1(x)=(3x-5)^2(6x^3+38x^2+41)+xh_2(x)+1,$$

where

$$h_2(x)=450x^2+1500-1643x\ge\left(300\sqrt{30}-1643\right)x>0.$$

The equality occurs for $a=b=c=1$.

□

P 3.125. *If a,b,c are nonnegative real numbers such that $a\ge b\ge c$ and*

$$a^2+b^2+c^2=3,$$

then

$$1-abc\ \ge\ \frac{1}{3}(a+3c)(\sqrt{a}-\sqrt{c})^2.$$

(Vasile Cîrtoaje, 2020)

Solution. Let us denote

$$g=\sqrt{\frac{a^2+b^2+c^2}{3}},\qquad g\le a,$$

and write the inequality in the homogeneous form $f\ge 0$, where

$$f=a^2+b^2+c^2-\frac{3abc}{g}-(a+3c)(\sqrt{a}-\sqrt{c})^2.$$

For fixed b and c, since

$$g'(a)=\frac{a}{3g},$$

we have

$$f'(a)=2a-\frac{3bc}{g}+\frac{a^2bc}{g^3}-(\sqrt{a}-\sqrt{c})^2-(a+3c)\left(1-\sqrt{\frac{c}{a}}\right)$$

$$= -\frac{3bc}{g} + \frac{a^2bc}{g^3} - 4c + 3\left(\sqrt{ac} + c\sqrt{\frac{c}{a}}\right)$$

$$\geq -\frac{3bc}{g} + \frac{a^2bc}{g^3} + 2c.$$

Due to homogeneity, we may consider $g = 1$, when

$$f'(a) \geq -3bc + a^2bc + 2c = 2c - bc(3 - a^2) \geq 2c - ac(3 - a^2)$$

$$= c(2 - 3a + a^3) = c(1 - a)^2(2 + a) \geq 0.$$

Since $f(a)$ is increasing, we have $f(a) \geq f(b)$, and it remains to prove that $f(b) \geq 0$, that is

$$2b^2 + c^2 - \frac{3b^2c}{g} \geq (b + 3c)(\sqrt{b} - \sqrt{c})^2,$$

where

$$g = \sqrt{\frac{2b^2 + c^2}{3}}.$$

For $c = 0$, the inequality is an identity. For $c > 0$, due to homogeneity, we may consider $c = 1$, when $b \geq 1$. The inequality becomes

$$2b^2 + 1 - \frac{3b^2}{g} \geq (b + 3)(\sqrt{b} - 1)^2,$$

where

$$g = \sqrt{\frac{2b^2 + 1}{3}}.$$

Since

$$g \geq \frac{2b + 1}{3},$$

it suffices to show that

$$2b^2 + 1 - \frac{9b^2}{2b + 1} \geq (b + 3)(\sqrt{b} - 1)^2,$$

which is equivalent to

$$\frac{(b - 1)^2(4b + 1)}{2b + 1} \geq (b + 3)(\sqrt{b} - 1)^2.$$

This is true if

$$(\sqrt{b} + 1)^2(4b + 1) \geq (b + 3)(2b + 1).$$

Since

$$(\sqrt{b} + 1)^2 > b + 2,$$

we have

$$(\sqrt{b} + 1)^2(4b + 1) - (b + 3)(2b + 1) > (b + 2)(4b + 1) - (b + 3)(2b + 1)$$

$$= 2b^2 + 2b - 1 > 0.$$

The equality occurs for $a = b = c = 1$, and also for $a = \sqrt{3}$ and $b = c = 0$.

Remark. We claim that, in the same conditions, the following stronger inequality holds:

$$1 - abc \geq \frac{1}{3}(a+5c)(\sqrt{a} - \sqrt{c})^2.$$

□

P 3.126. *If a, b, c are nonnegative real numbers such that $a \geq b \geq c$ and*

$$a^2 + b^2 + c^2 = 3,$$

then

$$1 - abc \;\geq\; \frac{2}{3}(a+3c)(\sqrt{b} - \sqrt{c})^2.$$

(Vasile Cîrtoaje, 2020)

Solution. Let us denote

$$g = \sqrt{\frac{a^2+b^2+c^2}{3}}, \qquad g \leq a,$$

and write the inequality in the homogeneous form $f \geq 0$, where

$$f = a^2 + b^2 + c^2 - \frac{3abc}{g} - 2(a+3c)(\sqrt{b} - \sqrt{c})^2.$$

For fixed b and c, since

$$g'(a) = \frac{a}{3g},$$

we have

$$f'(a) = 2a - \frac{3bc}{g} + \frac{a^2bc}{g^3} - 2(\sqrt{b} - \sqrt{c})^2$$

$$\geq 2a - \frac{2bc}{g} - 2(\sqrt{a} - \sqrt{c})^2 = 2(a-b) + 2c\left(2\sqrt{\frac{b}{c}} - \frac{b}{g} - 1\right).$$

To prove that $f'(a) \geq 0$, it is sufficient to show that

$$2\sqrt{\frac{b}{c}} \geq \frac{b}{g} + 1.$$

Since

$$\frac{b}{g} = \frac{3b}{\sqrt{3(a^2+b^2+c^2)}} \leq \frac{3b}{a+b+c} \leq \frac{3b}{2b+c} = \frac{3}{2+\frac{c}{b}},$$

it suffices to show that

$$2\sqrt{\frac{b}{c}} \geq \frac{3}{2+\frac{c}{b}}+1.$$

Denoting

$$x=\sqrt{\frac{c}{b}}, \qquad 0 \leq x \leq 1,$$

the inequality becomes

$$\frac{2}{x} \geq \frac{3}{2+x^2}+1,$$

that is

$$4-5x+2x^2-x^3 \geq 0,$$

$$(1-x)(4-x+x^2) \geq 0.$$

Since $f(a)$ is increasing, we have $f(a) \geq f(b)$, and it remains to prove that $f(b) \geq 0$, that is

$$2b^2+c^2-\frac{3b^2c}{g} \geq 2(b+3c)(\sqrt{b}-\sqrt{c})^2 \geq 0,$$

where

$$g=\sqrt{\frac{2b^2+c^2}{3}}.$$

For $c=0$, the inequality is an identity. For $c>0$, due to homogeneity, we may consider $c=1$, when $b \geq 1$. The inequality becomes

$$2b^2+1-\frac{3b^2}{g} \geq 2(b+3)(\sqrt{b}-1)^2,$$

where

$$g=\sqrt{\frac{2b^2+1}{3}}.$$

Since

$$2b^2+1-\frac{3b^2}{g}=\frac{(2b^2+1)^2-\frac{9b^4}{g^2}}{2b^2+1+\frac{3b^2}{g}}$$

$$=\frac{(2b^2+1)^3-27b^4}{(2b^2+1)(2b^2+1+\frac{3b^2}{g})}=\frac{(b^2-1)^2(8b^2+1)}{(2b^2+1)(2b^2+1+\frac{3b^2}{g})}$$

and

$$g \geq 2b+1,$$

we have

$$2b^2+1-\frac{3b^2}{g} \geq \frac{(b^2-1)^2(8b^2+1)}{(2b^2+1)(2b^2+1+\frac{3b^2}{2b+1})}$$

$$=\frac{(b^2-1)^2(8b^2+1)(2b+1)}{(2b^2+1)((4b^3+11b^2+2b+1)}.$$

Thus, it suffices to show that

$$\frac{(b^2-1)^2(8b^2+1)(2b+1)}{(2b^2+1)((4b^3+11b^2+2b+1)} \ge 2(b+3)(\sqrt{b}-1)^2,$$

which is true if

$$(\sqrt{b}+1)^2(b+1)^2(8b^2+1)(2b+1) \ge 2(b+3)(2b^2+1)(4b^3+11b^2+2b+1).$$

Consider two cases: $1 \le b \le 3$ and $b \ge 3$.

Case 1: $1 \le b \le 3$. Since

$$(\sqrt{b}+1)^2-(b+3) = 2(\sqrt{b}-1) \ge 0,$$

it suffices to show that

$$(b+1)^2(8b^2+1)(2b+1) \ge 2(2b^2+1)(4b^3+11b^2+2b+1),$$

which is equivalent to

$$-b^4+18b^3-13b^2-1 \ge 0,$$

$$(b-1)(-4b^3+14b^2+b+1) \ge 0.$$

This is true because

$$-4b^3+14b^2+b+1 = 4b^2(3-b)+2b^2+b+1 > 0.$$

Case 2: $b \ge 3$. Since

$$(\sqrt{b}+1)^2-(b+4) = 2\sqrt{b}-3 \ge 2\sqrt{3}-3 > 0,$$

it suffices to show that

$$(b+4)(b+1)^2(8b^2+1)(2b+1) \ge 2(b+3)(2b^2+1)(4b^3+11b^2+2b+1).$$

Moreover, since

$$(8b^2+1)(2b+1) > 8b^2(2b+1) = 8b(2b^2+b) \ge 8b(2b^2+1),$$

it suffices to show that

$$4b(b+4)(b+1)^2 \ge (b+3)(4b^3+11b^2+2b+1),$$

which reduces to the obvious inequality

$$b^3+b^2+9b-3 \ge 0.$$

The equality occurs for $a=b=c=1$, and also for $a=\sqrt{3}$ and $b=c=0$.

□

P 3.127. *Let*

$$F(a,b,c)=3(a^2+b^2+c^2)-(a+b+c)^2,$$

where a,b,c are positive real numbers such that $a\le b\le c$ and

$$a^2(b^2+c^2)\ge 2.$$

Then,

$$F(a,b,c)\ge F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2020)

Solution. From

$$2\le a^2(b^2+c^2)\le 2a^2c^2,$$

it follows that

$$ac\ge 1,\qquad bc\ge 1.$$

We need to show that

$$3(a^2+b^2+c^2)-(a+b+c)^2\ge 3\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2,$$

which is equivalent to

$$(a-b)^2+(b-c)^2+(c-a)^2\ge\frac{(a-b)^2}{a^2b^2}+\frac{(b-c)^2}{b^2c^2}+\frac{(c-a)^2}{c^2a^2},$$

$$(a-b)^2\left(1-\frac{1}{a^2b^2}\right)+(b-c)^2\left(1-\frac{1}{b^2c^2}\right)+(c-a)^2\left(1-\frac{1}{c^2a^2}\right)\ge 0.$$

Since $bc\ge 1$, it is enough to show that

$$(a-b)^2\left(1-\frac{1}{a^2b^2}\right)+(c-a)^2\left(1-\frac{1}{c^2a^2}\right)\ge 0,$$

that is

$$\left(1-\frac{a}{c}\right)^2\left(c^2-\frac{1}{a^2}\right)\ge\left(1-\frac{a}{b}\right)^2\left(\frac{1}{a^2}-b^2\right).$$

Since

$$1-\frac{a}{c}\ge 1-\frac{a}{b}\ge 0$$

it suffices to show that

$$c^2-\frac{1}{a^2}\ge\frac{1}{a^2}-b^2,$$

which is just the hypothesis $a^2(b^2+c^2)\ge 2$.

The proof is completed. The equality occurs for $a=b=c\ge 1$, and also for $\frac{1}{a}=b=c\ge 1$.

Remark. Since $a(b+c)\ge 2$ implies $a^2(b^2+c^2)\ge 2$, the inequality is true for $a(b+c)\ge 2$. In addition, it is true in the particular case $a,b,c\ge 1$.

□

P 3.128. *Let*

$$F(a,b,c) = a+b+c-3\sqrt[3]{abc},$$

where a,b,c are positive real numbers. If

$$\min\{a,b,c\} \geq \frac{1}{abc},$$

then

$$F(a,b,c) \geq F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2020)

Solution. Assume that $a \leq b \leq c$. Then, we have

$$a^2bc \geq 1, \quad bc \geq 1.$$

Write the inequality as $E(a,b,c) \geq 0$, where

$$E(a,b,c) = a+b+c-3\sqrt[3]{abc}-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}+\frac{3}{\sqrt[3]{abc}},$$

and show that

$$E(a,b,c) \geq E(a,\sqrt{bc},\sqrt{bc}) \geq 0.$$

The left inequality is equivalent to

$$(\sqrt{b}-\sqrt{c})^2(bc-1) \geq 0.$$

Substituting

$$a = x^3, \quad bc = y^6, \qquad x \leq y, \quad xy \geq 1,$$

the right inequality becomes as follows:

$$E(x^3,y^3,y^3) \geq 0,$$

$$x^3+2y^3-3xy^2-\left(\frac{1}{x^3}+\frac{2}{y^3}-\frac{3}{xy^2}\right) \geq 0,$$

$$(x-y)^2(x+2y)-\left(\frac{1}{x}-\frac{1}{y}\right)^2\left(\frac{1}{x}+\frac{2}{y}\right) \geq 0.$$

After dividing by $(x-y)^2$, we need to show that

$$x+2y-\frac{2x+y}{x^3y^3} \geq 0.$$

Indeed,

$$x+2y-\frac{2x+y}{x^3y^3} \geq x+2y-(2x+y) = y-x \geq 0.$$

The equality holds for $a=b=c \geq 1$.

Remark. The inequality is true in the particular case $a,b,c \geq 1$.

□

P 3.129. *Let*

$$F(a,b,c)=a+b+c-3\sqrt[3]{abc},$$

where a,b,c are positive real numbers such that $a\le b\le c$ and

$$a(b+c)\ge 2.$$

Then,

$$F(a,b,c)\ge F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c}\right).$$

(Vasile Cîrtoaje, 2020)

Solution. Since both side of the inequality are nonnegative, it suffices to prove the homogeneous inequality

$$2\left(a+b+c-3\sqrt[3]{abc}\right)\ge a(b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{3}{\sqrt[3]{abc}}\right),$$

that is

$$2(a+b+c)-6\sqrt[3]{abc}\ge b+c+\frac{(b+c)^2}{bc}a-\frac{3(b+c)}{\sqrt[3]{bc}}a^{2/3}.$$

For fixed b and c, write the inequality as $f(a)\ge 0$, $a\in(0,b]$. We will show that

$$f(a)\ge f(b)\ge 0.$$

To prove that $f(a)\ge f(b)$, we show that $f'(a)\le 0$, which is equivalent to

$$\frac{2(b+c)}{\sqrt[3]{abc}}\le\frac{b^2+c^2}{bc}+2\sqrt[3]{\frac{bc}{a^2}}.$$

Since

$$\frac{b^2+c^2}{bc}+2\sqrt[3]{\frac{bc}{a^2}}\ge\frac{2\sqrt{2(b^2+c^2)}}{\sqrt[3]{abc}},$$

it suffices to show that

$$(b+c)\le\sqrt{2(b^2+c^2)},$$

which is true. The inequality $f(b)\ge 0$ has the form

$$2\left(2b+c-3\sqrt[3]{b^2c}\right)\ge b(b+c)\left(\frac{2}{b}+\frac{1}{c}-\frac{3}{\sqrt[3]{b^2c}}\right).$$

Due to homogeneity, we may set $b=1$, when the inequality has the form

$$2\left(2+c-3\sqrt[3]{c}\right)\ge(1+c)\left(2+\frac{1}{c}-\frac{3}{\sqrt[3]{c}}\right).$$

Denoting

$$t=\sqrt[3]{c},\quad t\ge 1,$$

the inequality becomes as follows:

$$2c^3(c^3+2-3c) \ge (c^3+1)(2c^3-3c^2+1),$$

$$2c^3(c-1)^2(c+2) \ge (c^3+1)(c-1)^2(2c+1),$$

$$(c-1)^2(3c^3-2c-1) \ge 0,$$

$$(c-1)^3(3c^2+3c+1) \ge 1.$$

The equality holds for $a = b = c \ge 1$.

Remark. Because $a^2bc \ge 1$ yields $a(b+c) \ge 2$, the inequality in P 3.128 follows from the inequality in P 3.129.

□

P 3.130. *Let*

$$F(a,b,c,d) = a+b+c+d-4\sqrt[4]{abcd},$$

where a,b,c,d are positive real numbers. If

$$\min\{a^2,b^2,c^2,d^2\} \ge \frac{1}{abcd},$$

then

$$F(a,b,c,d) \ge F\left(\frac{1}{a},\frac{1}{b},\frac{1}{c},\frac{1}{d}\right).$$

(Vasile Cîrtoaje, 2020)

Solution. By the AM-GM inequality, both sides of the inequality are nonnegative. Assume that $a \le b \le c \le d$. Then, we have

$$a^3bcd \ge 1, \quad a \le x = \sqrt[3]{bcd} \ge 1, \quad a^3x^3 \ge 1.$$

Write the inequality as $E(a,b,c,d) \ge 0$, where

$$E(a,b,c,d) = a+b+c+d-4\sqrt[4]{abcd} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{d} + \frac{4}{\sqrt[4]{abcd}},$$

and show that

$$E(a,b,c,d) \ge E(a,x,x,x) \ge 0.$$

The left inequality shows that if a and bcd are fixed, then $E(a,b,c,d)$ is minimal for $b = c = d$. Using the contradiction method, assume for the sake of contradiction that $E(a,b,c,d)$ is minimal for $b < d$. This is false if

$$E(a,b,c,d) > E(a,\sqrt{bd},c,\sqrt{bd}).$$

Since

$$E(a,b,c,d)-E(a,\sqrt{bd},c,\sqrt{bd})=b+d-2\sqrt{bd}-\frac{1}{b}-\frac{1}{d}+\frac{2}{\sqrt{bd}}$$

$$=(\sqrt{b}-\sqrt{d})^2\left(1-\frac{1}{bd}\right),$$

we need to show that $bd>1$. Indeed, we have

$$(bd)^3-1\geq (bd)^3-a^3bcd\geq a^3(d^3-bcd)>0.$$

Write now the right inequality $E(a,x,x,x)\geq 0$ as

$$a+3x-4\sqrt[4]{ax^3}\geq \frac{1}{a}+\frac{3}{x}-\frac{4}{ax^3}.$$

It suffices to prove the homogeneous inequality

$$a+3x-4\sqrt[4]{ax^3}\geq ax\left(\frac{1}{a}+\frac{3}{x}-\frac{4}{ax^3}\right),$$

which is equivalent to

$$2(x-a)\geq 4(\sqrt[4]{ax^3}-\sqrt[4]{a^3x}),$$

$$(\sqrt{x}-\sqrt{a})(\sqrt[4]{x}-\sqrt[4]{a})^2\geq 0.$$

The equality holds for $a=b=c=d\geq 1$.

Remark. The inequality is true in the particular case $a,b,c,d\geq 1$, which implies $\min\{a^2,b^2,c^2,d^2\}\geq \dfrac{1}{abcd}$.

□

P 3.131. *Let a,b,c,d be nonnegative real numbers such that*

$$a^2+b^2+c^2+d^2=1.$$

Prove that

$$(1-a)(1-b)(1-c)(1-d)\ \geq\ abcd.$$

(Vasile Cîrtoaje, 2001)

Solution. The desired inequality follows by multiplying the inequalities

$$(1-a)(1-b)\geq cd,$$

$$(1-c)(1-d)\geq ab.$$

With regard to the first inequality, we have

$$2cd \le c^2 + d^2 = 1 - a^2 - b^2,$$

and hence

$$\begin{aligned} 2(1-a)(1-b) - 2cd &\ge 2(1-a)(1-b) - 1 + a^2 + b^2 \\ &= (1-a-b)^2 \ge 0. \end{aligned}$$

The second inequality can be proved similarly. The equality holds for

$$a = b = c = d = 1/2,$$

and also for

$$a, b, c, d) = (1, 0, 0, 0)$$

or any cyclic permutation.

□

P 3.132. *Let a, b, c, d and x be positive real numbers such that*

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} = \frac{4}{x^2}.$$

If $x \ge 2$, then

$$(a-1)(b-1)(c-1)(d-1) \ \ge \ (x-1)^4.$$

(Vasile Cîrtoaje, 2001)

Solution. The desired inequality follows by multiplying the inequalities

$$2(a-1)(b-1) \ge (x-1)\left(\frac{ab}{cd}x + x - 2\right),$$

$$2(c-1)(d-1) \ge (x-1)\left(\frac{cd}{ab}x + x - 2\right),$$

$$\left(\frac{ab}{cd}x + x - 2\right)\left(\frac{cd}{ab}x + x - 2\right) \ge 4(x-1)^2.$$

With regard to the first inequality, we write it as

$$2ab - 2(a+b) + x(3-x) \ge x(x-1)\frac{ab}{cd}.$$

Since

$$\frac{2}{cd} \le \frac{1}{c^2} + \frac{1}{d^2} = \frac{4}{x^2} - \frac{1}{a^2} - \frac{1}{b^2},$$

it suffices to show that

$$4ab - 4(a+b) + 2x(3-x) \ge x(x-1)ab\left(\frac{4}{x^2} - \frac{1}{a^2} - \frac{1}{b^2}\right).$$

This is equivalent to

$$4a^2b^2 - 4ab(a+b)x + 2x^2(3-x)ab + x^2(x-1)(a^2+b^2) \ge 0,$$

which can be written in the obvious form

$$[2ab - (a+b)x]^2 + x^2(x-2)(a-b)^2 \ge 0.$$

The second inequality can be proved similarly. With regard to the third inequality, we have

$$\left(\frac{ab}{cd}x + x - 2\right)\left(\frac{cd}{ab}x + x - 2\right) =$$

$$= 2x^2 - 4x + 4 + \left(\frac{ab}{cd} + \frac{cd}{ab}\right)x(x-2)$$

$$\ge 2x^2 - 4x + 4 + 2x(x-2) = 4(x-1)^2.$$

The equality holds for $a = b = c = d = x$.

Remark. Setting $x = 2$ and substituting a, b, c, d by $1/a, 1/b, 1/c, 1/d$, respectively, we get the inequality from P 3.131.

□

P 3.133. *If a, b, c, d are positive real numbers, then*

$$\frac{(1+a^3)(1+b^3)(1+c^3)(1+d^3)}{(1+a^2)(1+b^2)(1+c^2)(1+d^2)} \ge \frac{1+abcd}{2}.$$

(Vasile Cîrtoaje, 1992)

Solution. For $a = b = c = d$, the inequality can be written as

$$\left(\frac{1+a^3}{1+a^2}\right)^4 \ge \frac{1+a^4}{2}.$$

We will show that

$$\left(\frac{1+a^3}{1+a^2}\right)^4 \ge \left(\frac{1+a^3}{1+a}\right)^2 \ge \frac{1+a^4}{2}.$$

The left side inequality is equivalent to

$$(1+a^3)(1+a) \ge (1+a^2)^2,$$

$$a(1-a)^2 \ge 0,$$

while the right side inequality is equivalent to

$$2(1-a+a^2)^2 \geq 1+a^4,$$

$$(1-a)^4 \geq 0.$$

Multiplying the inequalities

$$\left(\frac{1+a^3}{1+a^2}\right)^4 \geq \frac{1+a^4}{2}, \quad \left(\frac{1+b^3}{1+b^2}\right)^4 \geq \frac{1+b^4}{2},$$

$$\left(\frac{1+c^3}{1+c^2}\right)^4 \geq \frac{1+c^4}{2}, \quad \left(\frac{1+d^3}{1+d^2}\right)^4 \geq \frac{1+d^4}{2},$$

yields

$$\frac{(1+a^3)(1+b^3)(1+c^3)(1+d^3)}{(1+a^2)(1+b^2)(1+c^2)(1+d^2)} \geq \frac{1}{2}\sqrt[4]{(1+a^4)(1+b^4)(1+c^4)(1+d^4)}.$$

Applying the Cauchy-Schwarz inequality produces

$$(1+a^4)(1+b^4)(1+c^4)(1+d^4) \geq (1+a^2b^2)^2(1+c^2d^2)^2 \geq (1+abcd)^4,$$

from which the desired inequality follows. The equality holds for

$$a=b=c=d=1.$$

□

P 3.134. *Let a,b,c,d be positive real numbers such that*

$$a+b+c+d=4.$$

Prove that

$$\left(a+\frac{1}{a}-1\right)\left(b+\frac{1}{b}-1\right)\left(c+\frac{1}{c}-1\right)\left(d+\frac{1}{d}-1\right)+3 \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}.$$

Solution. Write the inequality as

$$\prod\left[1+\left(a+\frac{1}{a}-2\right)\right] \geq \sum\frac{1}{a}-3.$$

Since

$$a+\frac{1}{a}-2 \geq 0, \quad b+\frac{1}{b}-2 \geq 0,$$

$$c+\frac{1}{c}-2 \geq 0, \quad d+\frac{1}{d}-2 \geq 0,$$

applying Bernoulli's inequality, it suffices to show that

$$1+\sum\left(a+\frac{1}{a}-2\right)\ge\sum\frac{1}{a}-3.$$

This is an identity, and then the proof is completed. The equality holds for

$$a=b=c=d=1.$$

□

P 3.135. *If a,b,c,d are nonnegative real numbers, then*

$$4(a^3+b^3+c^3+d^3)+15(abc+bcd+cda+dab)\ge(a+b+c+d)^3.$$

Solution. Let

$$E(a,b,c,d)=4(a^3+b^3+c^3+d^3)+15(abc+bcd+cda+dab)-(a+b+c+d)^3.$$

Without loss of generality, assume that $a\le b\le c\le d$. We will show that

$$E(a,b,c,d)\ge E(0,a+b,c,d)\ge 0.$$

We have

$$\begin{aligned}E(a,b,c,d)-E(0,a+b,c,d)&=4[a^3+b^3-(a+b)^3]+15ab(c+d)\\&=3ab[5(c+d)-4(a+b)]\ge 0.\end{aligned}$$

Now, putting $x=a+b$, we need to show that $E(0,x,c,d)\ge 0$, where

$$E(0,x,c,d)=4(x^3+c^3+d^3)+15xcd-(x+c+d)^3.$$

This is equivalent to Schur's inequality

$$x^3+c^3+d^3+3xcd\ge xc(x+c)+cd(c+d)+dx(d+x).$$

The equality holds for $a=0$ and $b=c=d$ (or any cyclic permutation), and also for $a=b=0$ and $c=d$ (or any permutation thereof).

□

P 3.136. *Let a,b,c,d be positive real numbers such that*

$$a+b+c+d=4.$$

Prove that

$$1+2(abc+bcd+cda+dab)\ge 9\min\{a,b,c,d\}.$$

(Vasile Cîrtoaje, 2008)

Solution. Assume that

$$a = \min\{a, b, c, d\}$$

and write the inequality in the homogeneous forms

$$(a+b+c+d)^3 + 128bcd + 128a(bc+cd+db) \ge 36a(a+b+c+d)^2.$$

Use now the substitution

$$b = a + x, \quad c = a + y, \quad d = a + z, \quad t = x + y + z,$$

where $x, y, z, t \ge 0$. Since

$$a + b + c + d = 4a + t,$$

$$bcd = (a+x)(a+y)(a+z) \ge a^3 + a^2 t$$

and

$$\begin{aligned} bc + cd + db &= (a+x)(a+y) + (a+y)(a+z) + (a+z)(a+x) \\ &= 3a^2 + 2at + xy + yz + zx \ge 3a^2 + 2at, \end{aligned}$$

it suffices to prove that

$$(4a+t)^3 + 128(a^3 + a^2 t) + 128a(3a^2 + 2at) \ge 36a(4a+t)^2.$$

This inequality is equivalent to

$$t(t - 12a)^2 \ge 0.$$

The equality holds for $a = b = c = d = 1$, and also for $(a, b, c, d) = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{13}{4}\right)$ or any cyclic permutation.

□

P 3.137. *Let a, b, c, d be nonnegative real numbers such that*

$$a + b + c + d = 4.$$

Prove that

$$5(a^2 + b^2 + c^2 + d^2) \ge a^3 + b^3 + c^3 + d^3 + 16.$$

(Vasile Cîrtoaje, 2005)

Solution. Assume that

$$a \ge b \ge c \ge d.$$

First Solution. Use the mixing variables technique. Setting

$$x = \frac{b+c+d}{3},$$

we have

$$a + 3x = 4, \qquad x \le 1.$$

We will show that

$$E(a,b,c,d) \ge E(a,x,x,x) \ge 0,$$

where

$$E(a,b,c,d) = 5(a^2+b^2+c^2+d^2) - a^3 - b^3 - c^3 - d^3 - 16.$$

The left side inequality is equivalent to

$$5(b^2+c^2+d^2-3x^2) - (b^3+c^3+d^3-3x^3) \ge 0.$$

Since $b^2+c^2+d^2-3x^2 \ge 0$ and $x \le 1$, it suffices to prove the homogeneous inequality

$$5x(b^2+c^2+d^2-3x^2) - (b^3+c^3+d^3-3x^3) \ge 0,$$

which is equivalent to

$$2(b^3+c^3+d^3) + 3b(c^2+d^2) + 3c(d^2+b^2) + 3d(b^2+c^2) \ge 24bcd.$$

This is true, since $b^3+c^3+d^3 \ge 3bcd$ and

$$b(c^2+d^2) + c(d^2+b^2) + d(b^2+c^2 \ge 2bcd + 2cdb + 2dbc = 6bcd.$$

The right side inequality is also true, since

$$\begin{aligned} E(a,x,x,x) &= 5(a^2+3x^2) - a^3 - 3x^3 - 16 \\ &= 5(4-3x)^2 + 15x^2 - (4-3x)^3 - 3x^3 - 16 \\ &= 24x(x-1)^2 \ge 0. \end{aligned}$$

This completes the proof. The equality holds for $a=b=c=d=1$, and also for $(a,b,c,d) = (0,0,0,4)$ or any cyclic permutation.

Second Solution. Write the inequality as

$$\sum (5a^2 - a^3 - 7a + 3) \ge 0,$$

$$\sum (1-a)^2(3-a) \ge 0.$$

For $a \le 3$, the inequality is clearly true. Otherwise, for $3 < a \le 4$, which involves

$$1 > b \ge c \ge d \ge 0,$$

we get the required inequality by summing the inequalities

$$5a^2 \geq a^3 + 16$$

and

$$5(b^2 + c^2 + d^2) \geq b^3 + c^3 + d^3.$$

We have

$$5a^2 - a^3 - 16 = (4-a)(a^2 - a - 4) = (4-a)[a(a-3) + 2(a-2)] \geq 0,$$

and

$$5(b^2 + c^2 + d^2) \geq b^2 + c^2 + d^2 \geq b^3 + c^3 + d^3.$$

Third Solution. Write the inequality as

$$\sum (a^3 - 5a^2 + 4a) \leq 0,$$

or

$$\sum (a-1)f(a) \leq 0,$$

where

$$f(a) = a^2 - 4a.$$

Since $a + b \leq 4$, we have

$$f(a) - f(b) = (a-b)(a+b-4) \leq 0,$$

and, similarly,

$$f(b) - f(c) \leq 0, \quad f(c) - f(d) \leq 0.$$

Since

$$a - 1 \geq b - 1 \geq c - 1 \geq d - 1$$

and

$$f(a) \leq f(b) \leq f(c) \leq f(d),$$

by Chebyshev's inequality, we get

$$4\sum (a-1)f(a) \leq \left[\sum (a-1)\right]\left[\sum f(a)\right] = 0.$$

Remark. Similarly, we can prove the following generalization:

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n,$$

then

$$(n+1)(a_1^2 + a_2^2 + \cdots + a_n^2) \geq a_1^3 + a_2^3 + \cdots + a_n^3 + n^2.$$

□

P 3.138. *Let a, b, c, d be nonnegative real numbers such that*

$$a+b+c+d=4.$$

Prove that

$$3(a^2+b^2+c^2+d^2)+4abcd \ge 16.$$

(Vasile Cîrtoaje, 2004)

Solution. We use the mixing variables method. Assume that

$$a=\min\{a,b,c,d\}, \qquad a \le 1.$$

Setting

$$x=\frac{b+c+d}{3},$$

we have

$$a+3x=4, \qquad 1 \le x \le \frac{4}{3}.$$

We will show that

$$E(a,b,c,d) \ge E(a,x,x,x) \ge 0,$$

where

$$E(a,b,c,d)=3(a^2+b^2+c^2+d^2)+4abcd-16.$$

The left side inequality is equivalent to

$$3(b^2+c^2+d^2-3x^2) \ge 4a(x^3-bcd),$$

$$3(3x^2-bc-cd-db) \ge 2a(x^3-bcd).$$

From Schur's inequality

$$(b+c+d)^3+9bcd \ge 4(b+c+d)(bc+cd+db),$$

we get

$$9x^3+3bcd \ge 4x(bc+cd+db),$$

$$x^3-bcd \le \frac{4x}{3}(3x^2-bc-cd-db).$$

Therefore, it suffices to prove that

$$3(3x^2-bc-cd-db) \ge \frac{8ax}{3}(3x^2-bc-cd-db);$$

that is,

$$(3x^2-bc-cd-db)(9-8ax) \ge 0.$$

This is true since

$$6(3x^2-bc-cd-db)=(b-c)^2+(c-d)^2+(d-b)^2 \ge 0,$$

and

$$3(9-8ax) = 27-8a(4-a) = 8(1-a)^2+16(1-a)+3 > 0.$$

The right side inequality is also true, since

$$\begin{aligned} E(a,x,x,x) &= 3a^2+9x^2+4ax^3-16 \\ &= 3(4-3x)^2+9x^2+4(4-3x)x^3-16 \\ &= 4(8-18x+9x^2+4x^3-3x^4) \\ &= 4(1-x)^2(2+x)(4-3x) \ge 0. \end{aligned}$$

This completes the proof. The equality holds for $a=b=c=d=1$, and also for $(a,b,c,d) = \left(0,\frac{4}{3},\frac{4}{3},\frac{4}{3}\right)$ or any cyclic permutation.

Remark. The following generalization holds (*Vasile Cîrtoaje*, 2005):

- *Let $a_1,a_2,\ldots,a_n$ $(n \ge 3)$ be nonnegative real numbers such that*

$$a_1+a_2+\cdots+a_n = n.$$

If k is a positive integer satisfying

$$2 \le k \le n+2,$$

then

$$\frac{a_1^k+a_2^k+\cdots+a_n^k}{n} - 1 \ge m(1-a_1a_2\cdots a_n), \qquad m = \left(\frac{n}{n-1}\right)^{k-1} - 1.$$

□

P 3.139. *Let a,b,c,d be nonnegative real numbers such that*

$$a+b+c+d = 4.$$

Prove that

$$27(abc+cd+cda+dab) \le 44abcd+64.$$

Solution. Use the mixing variables method. Without loss of generality, assume that

$$a \ge b \ge c \ge d.$$

Setting $x = (a+b+c)/3$, we have

$$3x+d = 4, \quad d \le x \le 4/3, \quad x^3 \ge abc.$$

We will show that

$$E(a,b,c,d) \ge E(x,x,x,d) \ge 0.$$

The left inequality is equivalent to

$$(x^3 - abc) + d(3x^2 - ab - bc - ca) \geq \frac{44}{27}d(x^3 - abc).$$

By Schur's inequality

$$(a+b+c)^3 + 9abc \geq 4(a+b+c)(ab+bc+ca),$$

we get

$$9x^3 + 3abc \geq 4x(ab+bc+ca),$$

and hence

$$3x^2 - ab - bc - ca \geq \frac{3(x^3 - abc)}{4x} \geq 0.$$

Therefore, it suffices to prove that

$$1 + \frac{3d}{4x} \geq \frac{44}{27}d.$$

Write this inequality in the homogeneous form

$$27(3x+d)(4x+3d) \geq 704xd,$$

or, equivalently,

$$81(4x^2 + d^2) \geq 353xd.$$

This inequality is true, since

$$81(4x^2 + d^2) - 353xd \geq 81(4x^2 + d^2 - 5xd) = 81(x-d)(4x-d) \geq 0.$$

The right inequality $E(x,x,x,t) \geq 0$ is also true, since

$$\begin{aligned} E(x,x,x,d) &= (44x^3 - 81x^2)d - 27x^3 + 64 \\ &= 4(16 - 81x^2 + 98x^3 - 33x^4) \\ &= 4(1-x)^2(16 + 32x - 33x^2) \\ &= 4(1-x)^2(4-3x)(4+11x) \geq 0. \end{aligned}$$

This completes the proof. The equality holds for $a = b = c = d = 1$, and also for $(a,b,c,d) = \left(0, \frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$ or any cyclic permutation.

□

P 3.140. *Let a,b,c,d be positive real numbers such that*

$$a + b + c + d = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}.$$

Prove that

$$(1 - abcd)\left(a^2 + b^2 + c^2 + d^2 - \frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2} - \frac{1}{d^2}\right) \geq 0.$$

(Vasile Cîrtoaje, 2007)

Solution. From

$$(a+b+c+d)^2 = \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)^2,$$

we get

$$\begin{aligned}\sum a^2 - \sum \frac{1}{a^2} &= 2\sum_{sym}\frac{1}{ab} - 2\sum_{sym} ab \\ &= 2\sum_{sym}\left(\frac{1}{ab} - cd\right) \\ &= 2(1-abcd)\sum_{sym}\frac{1}{ab}.\end{aligned}$$

Thus, the inequality can be restated as

$$2(1-abcd)^2\sum_{sym}\frac{1}{ab} \ge 0,$$

which is obviously true. The equality holds for $ab = cd = 1$, or $ac = bd = 1$, or $ad = bc = 1$.

□

P 3.141. *Let a, b, c, d be positive real numbers such that*

$$a+b+c+d = 1.$$

Prove that

$$(1-a)(1-b)(1-c)(1-d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \ge \frac{81}{16}.$$

(Keira, 2007)

Solution. Write the inequality as

$$E(a,b,c,d) \ge \frac{81}{16},$$

where

$$E(a,b,c,d) = (1-a)(1-b)(1-c)(1-d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right).$$

Without loss of generality, assume that

$$a \le b \le c \le d.$$

First, we show that for

$$a \le b \le c \le d, \qquad a+b+c+d = 1,$$

$F(a,b,c,d)$ is minimal when $a=c$. This is true if

$$E(a,b,c,d) \ge E\left(\frac{a+c}{2}, b, \frac{a+c}{2}, d\right).$$

Since

$$(1-a)(1-c) = 1-a-c+ac = b+d+ac$$

and

$$(1-b)(1-d) = 1-b-d+bd = a+c+bd,$$

we have

$$E(a,b,c,d) = (b+d+ac)(a+c+bd)\left(\frac{a+c}{ac}+\frac{b+d}{bd}\right),$$

and the inequality is equivalent to

$$(b+d+ac)\left(\frac{a+c}{ac}+\frac{b+d}{bd}\right) \ge \left[b+d+\left(\frac{a+c}{2}\right)^2\right]\left(\frac{4}{a+c}+\frac{b+d}{bd}\right),$$

$$(a-c)^2\left(\frac{4bd}{ac}-a-c\right) \ge 0.$$

Since

$$\frac{4bd}{ac}-a-c \ge 4-a-c = 3+b+d > 0,$$

the last inequality is clearly true. Since $E(a,b,c,d)$ is minimal when $a=c$, from $a \le b \le c \le d$ it follows that $E(a,b,c,d)$ is minimal when $a=b=c$. Therefore, it suffices to prove that $3a+d=4$ involves

$$E(a,a,a,d) \ge \frac{81}{16}.$$

This is equivalent to

$$21d^4+61d^3-57d^2-153d+128 \ge 0,$$

$$(d-1)^2(21d^2+103d+128) \ge 0.$$

The equality holds for $a=b=c=d=1/4$.

□

P 3.142. *Let a,b,c,d be nonnegative real numbers such that*

$$a+b+c+d = a^3+b^3+c^3+d^3 = 2.$$

Prove that

$$a^2+b^2+c^2+d^2 \ge \frac{7}{4}.$$

(Vasile Cîrtoaje, 2010)

Solution. Let us denote

$$x = a^2 + b^2 + c^2 + d^2.$$

From

$$2 = a^3 + b^3 + c^3 + d^3 \ge a^3 + \frac{1}{9}(b+c+d)^3 = a^3 + \frac{1}{9}(2-a)^3,$$

it follows that

$$(4a-5)(a+1)^2 \le 0,$$

hence $a \le \frac{5}{4}$. Similarly, we have $b, c, d \le \frac{5}{4}$. On the other hand,

$$5x = 5\sum a^2 = 4\sum a^3 + \sum(5a^2 - 4a^3) = 8 + \sum a^2(5-4a)$$

and, by the Cauchy-Schwarz inequality,

$$\sum a^2(5-4a) \ge \frac{[\sum a(5-4a)]^2}{\sum(5-4a)} = \frac{(5\sum a - 4\sum a^2)^2}{20 - 4\sum a} = \frac{(5-2x)^2}{3}.$$

Therefore, we have

$$5x \ge 8 + \frac{(5-2x)^2}{3}.$$

This is equivalent to

$$(4x-7)(x-7) \le 0,$$

which involves $x \ge \frac{7}{4}$. The equality holds for $(a,b,c,d) = \left(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{7}{4}\right)$ or any cyclic permutation.

□

P 3.143. *Let $a, b, c, d \in (0, 4]$ such that*

$$abcd = 1.$$

Prove that

$$(1+2a)(1+2b)(1+2c)(1+2d) \ge (5-2a)(5-2b)(5-2c)(5-2d).$$

(Vasile Cîrtoaje, 2011)

Solution. Assume that

$$a \ge b \ge c \ge d.$$

For the nontrivial case where the right side of the inequality is positive, there are two cases to consider.

Case 1: $a < 5/2$. In virtue of the AM-GM inequality, we have

$$(1+2a)(1+2b)(1+2c)(1+2d) \ge (3\sqrt[3]{a^2})(3\sqrt[3]{b^2})(3\sqrt[3]{c^2})(3\sqrt[3]{d^2}) = 81,$$

$$(5-2a)(5-2b)(5-2c)(5-2d) \le \left[\frac{(5-2a)+(5-2b)+(5-2c)+(5-2d)}{4}\right]^4$$
$$= \left[\frac{10-(a+b+c+d)}{2}\right]^4$$
$$\le \left(\frac{10-4\sqrt[4]{abcd}}{2}\right)^4 = 81,$$

from which the conclusion follows.

Case 2: $a \ge b > 5/2 > c \ge d$. Write the inequality as

$$\frac{(1+2a)(1+2b)}{(2a-5)(2b-5)} \ge \frac{(5-2c)(5-2d)}{(1+2c)(1+2d)},$$

$$\frac{1+4ab+2(a+b)}{25+4ab-10(a+b)} \ge \frac{25+4cd-10(c+d)}{1+4cd+2(c+d)}.$$

According to the AM-GM inequality, it suffices to prove that

$$\frac{1+4ab+4\sqrt{ab}}{25+4ab-20\sqrt{ab}} \ge \frac{25+4cd-20\sqrt{cd}}{1+4cd+4\sqrt{cd}}.$$

This is equivalent to

$$\frac{2\sqrt{ab}+1}{2\sqrt{ab}-5} \ge \frac{5-2\sqrt{cd}}{1+2\sqrt{cd}},$$

$$\frac{2\sqrt{ab}+1}{2\sqrt{ab}-5} \ge \frac{5\sqrt{ab}-2}{\sqrt{ab}+2},$$

$$(4\sqrt{ab}-1)(4-\sqrt{ab}) \ge 0.$$

The last inequality is true, since $a, b \in (5/2, 4]$ involves $4-\sqrt{ab} \ge 0$.
The equality holds for $a = b = c = d = 1$, and for $(a, b, c, d) = \left(4, 4, \frac{1}{4}, \frac{1}{4}\right)$ (or any permutation thereof).

□

P 3.144. *If* $a, b, c, d \in \left[0, 1+\frac{1}{\sqrt{6}}\right]$ *and* $a^2+b^2+c^2+d^2 = 4$, *then*

$$a+b+c+d \ \ge \ abcd+3.$$

(Vasile Cîrtoaje, 2018)

Solution. Assume that $a \ge b \ge c \ge d$ and denote

$$k = 1 + \frac{1}{\sqrt{6}}.$$

According to Remark 2 from P 3.57, *if* $k \ge a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$ *and*

$$a_1 + a_2 + \cdots + a_n = constant, \qquad a_1^2 + a_2^2 + \cdots + a_n^2 = constant,$$

then the product $a_1 a_1 \cdots a_n$ *is maximal for* $a_1 \ge a_2 = \cdots = a_n$ *or* $a_1 = k$. Thus, it is enough to consider the cases $a \ge b = c = d$ and $a = k$. In addition, for $a = k$, it suffices to consider the cases $b = k$ and $b \ge c = d$.

Case 1: $a \ge b = c = d$. We need to show that

$$a + 3b \ge ab^3 + 3$$

for

$$a^2 + 3b^2 = 4, \qquad k \ge a \ge b \ge 0.$$

From

$$4 - 3b^2 = a^2 \le \left(1 + \frac{1}{\sqrt{6}}\right)^2,$$

we get

$$b \ge \sqrt{\frac{17 - 2\sqrt{6}}{18}} > \frac{1}{\sqrt{3}},$$

hence

$$\frac{1}{\sqrt{3}} < b \le 1.$$

Write the inequality as

$$a(1 - b^3) - 3(1 - b) \ge 0,$$

$$(1 - b)[a(1 + b + b^2) - 3] \ge 0.$$

Since $1 - b \ge 0$ and $1 + b + b^2 \ge 3b$, it suffices to show that $ab \ge 1$. Indeed, we have

$$a^2 b^2 - 1 = (4 - 3b^2)b^2 - 1 = (1 - b^2)(3b^2 - 1) \ge 0.$$

Case 2: $a = b = k$. We need to show that

$$2k + c + d \ge k^2 cd + 3$$

for

$$c^2 + d^2 = 4 - 2k^2, \qquad k \ge c \ge d \ge 0.$$

Denoting

$$x = c + d,$$

from $c^2+d^2=4-2k^2$, we get

$$2cd = x^2-4+2k^2.$$

Thus, the required inequality becomes as follows:

$$4k+2x \ge k^2(x^2+2k^2-4)+6,$$

$$k^2x^2-2x+2k^4-4k^2-4k+6 \le 0.$$

$$3(7+2\sqrt{6})x^2-36x+25-4\sqrt{6} \le 0,$$

$$75x^2-36(7-2\sqrt{6})x+271-106\sqrt{6} \le 0,$$

$$(3x-3+\sqrt{6})(75x-177+47\sqrt{6}) \le 0.$$

This is true if

$$\frac{3-\sqrt{6}}{3} \le x \le \frac{177-47\sqrt{6}}{75}.$$

Indeed, from

$$(c+d)^2 \ge c^2+d^2 = 4-2k^2,$$

we get

$$x = c+d \ge \sqrt{4-2k^2} = \frac{3-\sqrt{6}}{3},$$

and from

$$(c+d)^2 \le 2(c^2+d^2) = 2(4-2k^2),$$

we get

$$x \le \sqrt{2(4-2k^2)} = \frac{3\sqrt{2}-2\sqrt{3}}{3} < \frac{1}{3} < \frac{177-47\sqrt{6}}{75}.$$

Case 3: $a=k,\ b\ge c=d$. We need to show that

$$k+b+2d \ge kbd^2+3$$

for

$$b^2+2d^2=4-k^2, \qquad k\ge b\ge d\ge 0.$$

Write the inequality as

$$(1-kd^2)b \ge 3-k-2d.$$

From

$$3d^2 \le b^2+2d^2 = 4-k^2,$$

we get

$$1-kd^2 \ge 1-\frac{k(4-k^2)}{3} = \frac{3\sqrt{6}-5}{18\sqrt{6}} > 0.$$

Therefore, consider further the nontrivial case $3-k-2d \geq 0$, when $d \leq \frac{3-k}{2} < \frac{4}{5}$. On the other hand, from

$$k^2+2d^2 \geq b^2+2d^2 = 4-k^2,$$

we get

$$d \geq \sqrt{2-k^2} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} > \frac{1}{8}.$$

Therefore, it suffices to show that

$$(1-kd^2)^2(4-k^2-2d^2)-(3-k-2d)^2 \geq 0$$

for

$$\frac{1}{8} \leq d \leq \frac{4}{5}.$$

After many calculations, we can show that this inequality is true.

The inequality is an equality for $a=b=c=d=1$, and also for

$$a=b=1+\frac{1}{\sqrt{6}}, \quad c=1-\sqrt{\frac{2}{3}}, \quad d=0$$

(or any permutation).

Open Problem. If

$$1+\frac{1}{\sqrt{(n-1)(n-2)}} \geq a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$$

and

$$a_1^2+a_2^2+\cdots+a_n^2 = n,$$

then

$$a_1+a_2+\cdots+a_n \geq a_1a_2\cdots a_n+n-1.$$

□

P 3.145. *Let a,b,c,d be positive real numbers such that*

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \leq (1+\sqrt{10}\,)^2.$$

Prove that any three of a,b,c,d are the lengths of the sides of a triangle (non-degenerate or degenerate).

Solution. Without loss of generality, assume that

$$a \ge b \ge c \ge d.$$

Clearly, any three of a, b, c, d are the lengths of the sides of a triangle if and only if $a \le c + d$. By virtue of the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(1+\sqrt{10})^2 &\ge (b+a+c+d)\left(\frac{1}{b}+\frac{1}{a}+\frac{1}{c}+\frac{1}{d}\right)\\ &\ge \left[1+\sqrt{(a+c+d)\left(\frac{1}{a}+\frac{1}{c}+\frac{1}{d}\right)}\,\right]^2\\ &\ge \left[1+\sqrt{(a+c+d)\left(\frac{1}{a}+\frac{4}{c+d}\right)}\,\right]^2,\end{aligned}$$

hence

$$(a+c+d)\left(\frac{1}{a}+\frac{4}{c+d}\right) \le (1+\sqrt{10})-1 = 10.$$

Writing this inequality as

$$(a-c-d)(4a-c-d) \le 0,$$

we get $a \le c + d$. Thus, the proof is completed.

Remark 1. Notice that $11 + 2\sqrt{10}$ is the largest value of the product

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

such that any three of the positive real numbers a, b, c, d are the lengths of the sides of a triangle. In order to prove this, assume that

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) = k, \qquad k > 11 + 2\sqrt{10}.$$

This relation is satisfied for

$$a = p + \sqrt{p^2-1}, \qquad b = \sqrt{\frac{a(a+2)}{2a+1}}, \qquad c = d = 1,$$

where

$$p = \frac{(\sqrt{k}-1)^2-5}{4}.$$

For $k > 11 + 2\sqrt{10}$, we get $p > 5/4$, then $a > 2$. Clearly, the numbers a, c and d are not the lengths of the sides of a triangle.

Remark 2. In the same manner, we can prove the following generalization:

- *If $a_1, a_2, \ldots, a_n$ $(n \ge 3)$ are positive real numbers such that*

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \le (n + \sqrt{10} - 3)^2,$$

then any three of $a_1, a_2, \ldots, a_n$ are the lengths of the sides of a triangle.

□

P 3.146. *Let a, b, c, d be positive real numbers such that*

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \le \frac{119}{6}.$$

Prove that there exist three of a, b, c, d which are the lengths of the sides of a triangle (non-degenerate or degenerate).

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that

$$a \ge b \ge c \ge d.$$

We need to show that either $a \le b + c$ or $b \le c + d$. For the sake of contradiction, consider that

$$a > b + c, \qquad b > c + d.$$

It suffices to show that

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) > \frac{119}{6}.$$

Notice that for $a = 3$, $b = 2$ and $c = d = 1$, we have $a = b + c$, $b = c + d$ and

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) = \frac{119}{6}.$$

Therefore, we apply the Cauchy-Schwarz inequality in the following manner

$$[a + (b+c+d)]\left(\frac{9}{a} + \frac{16}{b+c+d}\right) \ge (3+4)^2 = 49;$$

that is,

$$a+b+c+d \ge \frac{49}{\dfrac{9}{a} + \dfrac{16}{b+c+d}}.$$

Thus, it suffices to show that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} > \frac{17}{42}\left(\frac{9}{a} + \frac{16}{b+c+d}\right).$$

From

$$(b+c+d)\left(\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)-10 \geq (b+c+d)\left(\frac{1}{b}+\frac{4}{c+d}\right)-10$$
$$=\frac{(b-c-d)(4b-c-d)}{b(c+d)}>0,$$

we get

$$\frac{1}{b}+\frac{1}{c}+\frac{1}{d}>\frac{10}{b+c+d}.$$

So, it suffices to show that

$$\frac{1}{a}+\frac{10}{b+c+d} \geq \frac{17}{42}\left(\frac{9}{a}+\frac{16}{b+c+d}\right).$$

This is equivalent to

$$4a \geq 3(b+c+d),$$

which is true, since

$$4a-3(b+c+d)=4(a-b-c)+(b-c-d)+2(c-d)>0.$$

□

P 3.147. *Let a,b,c,d be positive real numbers such that*

$$3(a+b+c+d)^2 \geq 11(a^2+b^2+c^2+d^2).$$

Prove that any three of a,b,c,d are the lengths of the sides of a triangle (non-degenerate or degenerate).

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that

$$a \geq b \geq c \geq d.$$

Then, any three of a,b,c,d are the lengths of the sides of a triangle if and only if

$$a \leq c+d.$$

To prove this, let us denote

$$x=\frac{c+d}{2}, \quad x \leq b.$$

Since $c^2+d^2 \geq 2x^2$, we have

$$3(a+b+2x)^2 \geq 11(a^2+b^2+2x^2),$$

which can be written as

$$8b^2-6(a+2x)b+11a^2+22x^2-3(a+2x)^2\le 0,$$

$$8\left[b-\frac{3(a+2x)}{8}\right]^2+11a^2+22x^2-\frac{33}{8}(a+2x)^2\le 0.$$

This involves

$$a^2+2x^2-\frac{3}{8}(a+2x)^2\le 0,$$

$$(a-2x)(5a-2x)\le 0,$$

from which we get $a\le 2x$; that is,

$$a-c-d\le 0.$$

Thus, the proof is completed.

Remark 1. Notice that 11/3 is the smallest value of the ratio

$$\frac{(a+b+c+d)^2}{a^2+b^2+c^2+d^2}$$

such that any three of the positive real numbers a,b,c,d are the lengths of the sides of a triangle. In order to prove this, assume that

$$(a+b+c+d)^2=k(a^2+b^2+c^2+d^2),\qquad 1<k<\frac{11}{3}.$$

This relation is satisfied for

$$a=\frac{7+\sqrt{k(66-17k)}}{k-1},\qquad b=3,\qquad c=d=2.$$

Since $a>4$ for $1<k<11/3$, the numbers a, c and d are not the lengths of the sides of a triangle.

Remark 2. In the same manner, we can prove the following generalization:

- *If $a_1,a_2,\dots,a_n$ $(n\ge 3)$ are positive real numbers such that*

$$3(a_1+a_2+\cdots+a_n)^2\ge(3n-1)(a_1^2+a_2^2+\cdots+a_n^2).$$

then any three of $a_1,a_2,\dots,a_n$ are the lengths of the sides of a triangle.

Notice that $n-1/3$ is the smallest value of the ratio

$$\frac{(a_1+a_2+\cdots+a_n)^2}{a_1^2+a_2^2+\cdots+a_n^2}$$

such that any three of the positive real numbers $a_1,a_2,\dots,a_n$ are the lengths of the sides of a triangle. To prove this, assume that

$$(a_1+a_2+\cdots+a_n)^2=k(a_1^2+a_2^2+\cdots+a_n^2),\qquad 1<k<n-\frac{1}{3}.$$

This relation is satisfied for

$$a_1 = \frac{3n-5+\sqrt{k[3(n-2)(3n-1)-k(9n-19)]}}{k-1},$$

$$a_2 = \cdots = a_{n-2} = 3, \quad a_{n-1} = a_n = 2.$$

Since $a_1 > 4$ for $1 < k < n-1/3$, the numbers a_1, a_{n-1} and a_n are not the lengths of the sides of a triangle.

□

P 3.148. *Let a,b,c,d be positive real numbers such that*

$$15(a+b+c+d)^2 \ge 49(a^2+b^2+c^2+d^2).$$

Prove that there exist three of a,b,c,d which are the lengths of the sides of a triangle (non-degenerate or degenerate).

(Vasile Cîrtoaje, 2010)

Solution. Without loss of generality, assume that

$$a \ge b \ge c \ge d.$$

We need to show that either $a \le b+c$ or $b \le c+d$. For the sake of contradiction, consider that

$$a > b+c, \qquad b > c+d.$$

To complete the proof, it suffices to show that

$$15(a+b+c+d)^2 < 49(a^2+b^2+c^2+d^2).$$

Notice that for $a=3$, $b=2$ and $c=d=1$, we have

$$a = b+c, \qquad b = c+d,$$

and

$$15(a+b+c+d)^2 = 49(a^2+b^2+c^2+d^2).$$

Therefore, we apply the Cauchy-Schwarz inequality in the following manner

$$(3+4)\left[\frac{a^2}{3}+\frac{(b+c+d)^2}{4}\right] \ge (a+b+c+d)^2.$$

Based on this result, it suffices to show that

$$\frac{a^2}{3}+\frac{(b+c+d)^2}{4} < \frac{7}{15}(a^2+b^2+c^2+d^2),$$

which is equivalent to

$$2a^2+7(b^2+c^2+d^2)>\frac{15}{4}(b+c+d)^2.$$

Since

$$\begin{aligned}8(b^2+c^2+d^2)-3(b+c+d)^2 &= 5b^2-6b(c+d)+5(c^2+d^2)-6cd\\ &\ge 5b^2-6b(c+d)+(c+d)^2\\ &=(b-c-d)(5b-c-d)>0,\end{aligned}$$

it is enough to prove that

$$2a^2+\frac{21}{8}(b+c+d)^2\ge\frac{15}{4}(b+c+d)^2,$$

which is equivalent to

$$4a\ge 3(b+c+d).$$

Indeed, we have

$$4a-3(b+c+d)=4(a-b-c)+(b-c-d)+2(c-d)>0.$$

□

P 3.149. *Let a,b,c,d be nonnegative real numbers such that*

$$a^2+b^2+c^2+d^2=4.$$

If $a\ge b\ge c\ge d$, then

$$a+b+c+d+(2-\sqrt{2})(a-d)^2\ \ge\ 4.$$

(Vasile Cîrtoaje, 2019)

Solution. For $a=d$, the inequality is a trivial equality. Consider next that $a>d$, give up the condition $b\ge c$ (consider only that $b,c\in[d,a]$) and write the inequality in the homogeneous form $f\ge 0$, where

$$f=(a+b+c+d)\sqrt{g}+(2-\sqrt{2})(a-d)^2-4g,\qquad g=\frac{a^2+b^2+c^2+d^2}{4}.$$

For fixed a, c and d, f and g are functions of b, $b\in[d,a]$. Since

$$g'(b)=\frac{b}{2},$$

we have

$$f'(b) = \sqrt{g} + \frac{(a+b+c+d)b}{4\sqrt{g}} - 2b = bh(b),$$

where

$$h(b) = \frac{\sqrt{g}}{b} + \frac{a+b+c+d}{4\sqrt{g}} - 2,$$

$$h'(b) = \frac{-\sqrt{g}}{b^2} + \frac{1}{4\sqrt{g}} + \frac{1}{4\sqrt{g}} - \frac{b(a+b+c+d)}{16g^{3/2}}$$

$$\leq \frac{-\sqrt{g}}{b^2} + \frac{1}{2\sqrt{g}} = \frac{-2g+b^2}{2b^2\sqrt{g}} \leq 0.$$

Since $h(b)$ is a decreasing function, there are three possible cases: (1) $h(b) \geq 0$ for $b \in [d,a]$, hence $f(b)$ is increasing on $[d,a]$; (2) $h(b) \geq 0$ for $b \in [d,d_1]$ and $h(b) \leq 0$ for $b \in [d_1,a]$, hence $f(b)$ is increasing on $[d,d_1]$ and decreasing on $[d_1,a]$; (3) $h(b) \leq 0$ for $b \in [d,a]$, hence $f(b)$ is decreasing on $[d,a]$. In all these cases $f(b)$ is minimal when $b \in \{d,a\}$. As a consequence, we only need to prove the required inequality for $b \in \{d,a\}$. Similarly, we only need to prove the required inequality for $c \in \{d,a\}$. So, we need to show that

$$[ka+(4-k)d]\sqrt{g} + (2-\sqrt{2})(a-d)^2 - 4g \geq 0,$$

where

$$g = \frac{ka^2+(4-k)d^2}{4}, \qquad k \in \{1,2,3\}.$$

For $d=0$, the inequality reduces to

$$(k\sqrt{k} - 2k + 4 - 2\sqrt{2})a^2 \geq 0,$$

which is true for $k \in \{1,2,3\}$. Next, due to homogeneity, we may set $d=1$ (which involves $a>1$). The inequality becomes

$$(ka+4-k]\sqrt{x} + (2-\sqrt{2})(a-1)^2 - 4x \geq 0,$$

with

$$x = \frac{ka^2+4-k}{4}, \qquad x \geq 1.$$

Write the inequality as

$$(2-\sqrt{2})(a-1)^2 \geq \sqrt{x}\left(4\sqrt{x} - ka - 4 + k\right),$$

$$(2-\sqrt{2})(a-1)^2 \geq \frac{k(4-k)(a-1)^2\sqrt{x}}{4\sqrt{x}+ka+4-k}.$$

This is true if

$$(2-\sqrt{2})(ka+4-k) \geq (4k-k^2-8+4\sqrt{2})\sqrt{x}.$$

Case 1: $k=1$. Since

$$x=\frac{a^2+3}{4},$$

the inequality becomes

$$(4-2\sqrt{2})(a+3)\geq(4\sqrt{2}-5)\sqrt{a^2+3}.$$

It follows by multiplying the inequalities

$$4-2\sqrt{2}>4\sqrt{2}-5$$

and

$$a+3>\sqrt{a^2+3}.$$

Case 2: $k=2$. Since

$$x=\frac{a^2+1}{2},$$

the inequality becomes

$$a+1\geq\sqrt{a^2+1},$$

which is clearly true.

Case 3: $k=3$. Since

$$x=\frac{3a^2+1}{4},$$

the inequality becomes

$$(4-2\sqrt{2})(3a+1)\geq(4\sqrt{2}-5)\sqrt{3a^2+1}.$$

It follows by multiplying the inequalities

$$4-2\sqrt{2}>4\sqrt{2}-5$$

and

$$3a+1>\sqrt{3a^2+1}.$$

The proof is completed. The equality occurs for $a=b=c=d=1$, and also for $a=b=\sqrt{2}$ and $c=d=0$.

Remark. Similarly, we can prove the following statements:

- *Let a_1,a_2,a_3,a_4,a_5 be nonnegative real numbers such that*

$$a_1^2+a_2^2+a_3^2+a_4^2+a_5^2=5.$$

If $a_1\geq a_2\geq a_3\geq a_4\geq a_5$, then

$$1-\frac{a_1+a_2+a_3+a_4+a_5}{5}\leq\frac{2}{5}\left(1-\sqrt{\frac{2}{5}}\right)(a_1-a_5)^2,$$

with equality for $a_1 = a_2 = a_3 = a_4 = a_5 = 1$*, and also for* $a_1 = a_2 = \sqrt{\frac{5}{2}}$ *and* $a_3 = a_4 = a_5 = 0$.

- *Let* $a_1, a_2, a_3, a_4, a_5, a_6$ *be nonnegative real numbers such that*

$$a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 = 6.$$

If $a_1 \ge a_2 \ge a_3 \ge a_4 \ge a_5 \ge a_6$*, then*

$$1 - \frac{a_1 + a_2 + a_3 + a_4 + a_5 + a_6}{6} \le \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)(a_1 - a_6)^2,$$

with equality for $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 1$*, and also for* $a_1 = a_2 = a_3 = \sqrt{2}$ *and* $a_4 = a_5 = a_6 = 0$.

- *Let* $a_1, a_2, \dots, a_n$ *be nonnegative real numbers such that*

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n.$$

If $a_1 \ge a_2 \ge \cdots \ge a_n$*, then*

$$1 - \frac{a_1 + a_2 + \cdots + a_n}{n} \le \frac{4}{27}(a_1 - a_n)^2,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 3.150. *Let* a, b, c, d *be nonnegative real numbers such that*

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If $a \ge b \ge c \ge d$*, then*

$$1 - abcd \le \frac{\sqrt{3}}{2}(a - d).$$

(Vasile Cîrtoaje, 2019)

Solution. For $a = d$, the inequality is a trivial equality. It is also true for $d = 0$, because we have

$$4 = a^2 + b^2 + c^2 \le 3a^2,$$

hence

$$\frac{\sqrt{3}}{2}\,a \ge 1.$$

Consider next that $a > d > 0$, give up the condition $b \ge c$ (consider only that $b, c \in [d, a]$) and write the inequality in the homogeneous form $f \ge 0$, where

$$f = abcd + \frac{\sqrt{3}}{2}(a - d)g^{3/2} - g^2, \qquad g = \frac{a^2 + b^2 + c^2 + d^2}{4}.$$

For fixed a, c and d, f and g are functions of b, $b \in [d,a]$. Since

$$g'(b) = \frac{b}{2},$$

we have

$$f'(b) = acd + \frac{3\sqrt{3}}{8}(a-d)b\sqrt{g(b)} - bg(b) = b\sqrt{g(b)}\,h(b),$$

where

$$h(b) = \frac{acd}{b\sqrt{g(b)}} + \frac{3\sqrt{3}}{8}(a-d) - \sqrt{g(b)}.$$

Since $h(b)$ is a decreasing function, there are three possible cases: $g(b) \ge 0$ for $b \in [d,a]$, hence $f(b)$ is increasing on $[a,d]$; $g(b) \ge 0$ for $b \in [d,d_1]$ and $g(b) \le 0$ for $b \in [d_1,a]$, hence $f(b)$ is increasing on $[d,d_1]$ and decreasing on $[d_1,a]$; $g(b) \le 0$ for $b \in [d,a]$, hence $f(b)$ is decreasing on $[a,d]$. In all these cases $f(b)$ is minimal when $b \in \{d,a\}$. As a consequence, we only need to prove the required inequality for $b \in \{d,a\}$. Similarly, we only need to prove the required inequality for $c \in \{d,a\}$. So, we need to show that

$$a^k d^{4-k} + \frac{\sqrt{3}}{2}(a-d)g^{3/2} - g^2 \ge 0,$$

where

$$g = \frac{ka^2 + (4-k)d^2}{4}, \qquad k \in \{1,2,3\}.$$

Due to homogeneity, we may set $d = 1$ (which involves $a > 1$), when the inequality becomes

$$a^k + \frac{\sqrt{3}}{2}(a-1)x^{3/2} \ge x^2.$$

with

$$x = \frac{ka^2 + 4 - k}{4} > 1.$$

Case 1: $k = 1$. Since

$$x = \frac{a^2+3}{4},$$

the inequality becomes

$$16a + \sqrt{3}(a-1)(a^2+3)^{3/2} \ge (a^2+3)^2,$$

$$\sqrt{3}(a-1)(a^2+3)^{3/2} \ge (a-1)^2(a^2+2a+9).$$

and, by squaring,

$$2a(a-1)^2(a^5 - a^4 + 6a^3 + 2a^2 + 25a + 63) \ge 0.$$

Clearly, it is true for $a > 1$.

Case 2: $k = 2$. Since

$$x = \frac{a^2+1}{2},$$

the inequality becomes

$$8a^2 + \sqrt{6}(a-1)(a^2+1)^{3/2} \ge 2(a^2+1)^2,$$

$$\sqrt{6}(a-1)(a^2+1)^{3/2} \ge 2(a^2-1)^2,$$

and, by squaring,

$$2(a-1)^2(a^6 - 4a^5 + 11a^4 + 8a^3 + 11a^2 - 4a + 1) \ge 0.$$

This is true because

$$a^6 - 4a^5 + 11a^4 + 8a^3 + 11a^2 - 4a + 1 = a^2(a-1)^4 + 5a^4 + 12a^3 + 6a^2 + (2a-1)^2 > 0.$$

Case 3: $k = 3$. Since

$$x = \frac{3a^2+1}{4},$$

the inequality becomes

$$16a^3 + \sqrt{3}(a-1)(3a^2+1)^{3/2} \ge (3a^2+1)^2,$$

$$\sqrt{3}(a-1)(3a^2+1)^{3/2} \ge (a-1)^2(9a^2+2a+1).$$

and, by squaring,

$$2(a-1)^2(63a^5 + 25a^4 + 2a^3 + 6a^2 - a + 1) \ge 0.$$

Clearly, it is true for $a > 1$.

The proof is completed. The equality occurs for $a = b = c = d = 1$, and also for $a = b = c = \frac{2}{\sqrt{3}}$ and $d = 0$.

Remark. The following more general statement holds for $n \le 6$:

- *If $a_1, a_2, \ldots, a_n$ ($n \le 6$) are nonnegative real numbers such that*

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n, \qquad a_1 \ge a_2 \ge \cdots \ge a_n.$$

then

$$1 - a_1 a_2 \cdots a_n \le \sqrt{1 - \frac{1}{n}}\,(a_1 - a_n),$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = a_2 = \cdots = a_{n-1} = \sqrt{\frac{n}{n-1}}$ and $a_n = 0$.

To prove this inequality, we need to show that

$$1-a^kb^{n-k} \le \sqrt{1-\frac{1}{n}}\,(a-b)$$

for $a \ge 1 \ge b \ge 0$ and

$$ka^2+(n-k)b^2=n, \qquad k\in\{1,2,\ldots,n-1\}.$$

□

P 3.151. *Let a,b,c,d be nonnegative real numbers such that*

$$a^2+b^2+c^2+d^2=4.$$

If $a \ge b \ge c \ge d$, then

$$1-\sqrt{abcd} \le \frac{3}{4}(a-d)^2.$$

(Vasile Cîrtoaje, 2019)

Proof. For $a=d$, the inequality is a trivial equality. Consider next that $a>d$, give up the condition $b \ge c$ (consider only that $b,c\in[d,a]$) and write the inequality in the homogeneous form $f \ge 0$, where

$$f=4\sqrt{abcd}+3(a-d)^2-(a^2+b^2+c^2+d^2)^2.$$

First Solution. For fixed a, c and d, f is a function of b, $b\in[d,a]$. We have

$$f'(b)=2\sqrt{\frac{acd}{b}}-2b.$$

Since $f'(b)$ is a decreasing function, there are three possible cases: (1) $f'(b)\ge 0$ for $b\in[d,a]$, hence $f(b)$ is increasing on $[d,a]$; (2) $f'(b)\ge 0$ for $b\in[d,d_1]$ and $f'(b)\le 0$ for $b\in[d_1,a]$, hence $f(b)$ is increasing on $[d,d_1]$ and decreasing on $[d_1,a]$; (3) $f'(b)\le 0$ for $b\in[d,a]$, hence $f(b)$ is decreasing on $[d,a]$. In all these cases, if the inequality $f(b)$ holds for the extreme values of b, then it holds for all possible values of b. Thus, we only need to prove the required inequality for $b\in\{a,d\}$. Similarly, we only need to prove the required inequality for $c\in\{a,d\}$. So, we need to show that

$$4\sqrt{a^kd^{4-k}}+3(a-d)^2 \ge ka^2+(4-k)d^2,$$

where

$$k\in\{1,2,3\}.$$

For $d = 0$, the inequality reduces to

$$(3-k)a^2 \geq 0,$$

which is true for $k \in \{1,2,3\}$. Next, due to homogeneity, we may set $d = 1$. Substituting $x = \sqrt{a}$, $x \geq 1$, the required inequality becomes

$$3(x^2-1)^2 \geq kx^4 - 4x^k + 4 - k.$$

Case 1: $k = 1$. We need to show that

$$3(x^2-1)^2 \geq x^4 - 4x + 3,$$

that is

$$3(x^2-1)^2 \geq (x-1)^2(x^2+2x+3),$$

$$x(x-1)^2(3x+1) \geq 0.$$

Case 2: $k = 2$. We need to show that

$$3(x^2-1)^2 \geq 2x^4 - 4x^2 + 2,$$

that is

$$3(x^2-1)^2 \geq 2x^4 - 4x^2 + 2,$$

$$3(x^2-1)^2 \geq 2(x^2-1)^2.$$

Case 3: $k = 3$. We need to show that

$$3(x^2-1)^2 \geq 3x^4 - 4x^3 + 1,$$

that is

$$3(x^2-1)^2 \geq (x-1)^2(3x^2+2x+1),$$

$$(x-1)^2(2x+1) \geq 0.$$

The proof is completed. The equality occurs for $a = b = c = d = 1$, and also for $a = b = c = \dfrac{2}{\sqrt{3}}$ and $d = 0$.

Second Solution (by *Marius Stanean*). Replacing a, b, c, d with a^2, b^2, c^2, d^2, respectively, we need to sow that $a \geq b \geq c \geq d \geq 0$ yields the homogeneous inequality $F(a,b,c,d) \geq 0$, where

$$F(a,b,c,d) = 4abcd - 3(a^2-d^2)^2 - a^4 - b^4 - c^4 - d^4.$$

We will sow that

$$F(a,b,c,d) \geq F(a,a,c,d) \geq 0.$$

The left inequality is equivalent to

$$a^4 - b^4) \geq 4acd(a-b,$$

which is true if

$$(a+b)(a^2+b^2) \geq 4acd.$$

Indeed,

$$(a+b)(a^2+b^2)-4acd \geq 2ab(a+b)-4ab^2 = 2ab(a-b) \geq 0.$$

For fixed a and d, we have

$$G(c) = F(a,a,c,d) = -c^4+4a^2cd+a^4-6a^2d^2+2d^4, \quad c \in [d,a].$$

Since G is concave, it is enough to show that $G(a) \geq 0$ and $G(d) \geq 0$. We have

$$G(a) = 2d(a-d)^2(2a+d) \geq 0,$$

$$G(d) = (a^2-d^2)^2 \geq 0.$$

Third Solution. Write the inequality in the homogeneous form

$$a^2+b^2+c^2+d^2-4\sqrt{abcd} \leq \frac{3}{4}(a-d)^2.$$

As known (see Remark from the proof of P 2.104), for fixed a and d, if $b,c \in [d,a]$, then Jensen's difference

$$a^2+b^2+c^2+d^2-4\sqrt{abcd}$$

is maximal when $b,c \in \{a,d\}$, as shown in the first solution.

Remark. The following more general statement holds for $n \leq 6$:

- *If $a_1,a_2,\ldots,a_n$ ($n \leq 6$) are nonnegative real numbers such that*

$$a_1^2+a_2^2+\cdots+a_n^2 = n, \qquad a_1 \geq a_2 \geq \cdots \geq a_n.$$

then

$$1-\sqrt{a_1a_2\cdots a_n} \leq \left(1-\frac{1}{n}\right)(a_1-a_n)^2,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = \cdots = a_{n-1} = \sqrt{\dfrac{n}{n-1}}$ and $a_n = 0$.

To prove this inequality, we need to show that

$$1-\sqrt{a^kb^{n-k}} \leq \left(1-\frac{1}{n}\right)(a-b)^2$$

for $a \geq 1 \geq b \geq 0$ and

$$ka^2+(n-k)b^2 = n, \qquad k \in \{1,2,\ldots,n-1\}.$$

□

P 3.152. *If a, b, c, d are nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$a^2 + b^2 + c^2 + d^2 = 4,$$

then

$$1 - (abcd)^{3/4} \le \frac{3}{4}(a-d)^2.$$

(Vasie Cirtoaje, 2020)

Solution. For $a = d$, the inequality is a trivial equality. Consider next that $a > d$, give up the condition $b \ge c$ (consider only that $b, c \in [d, a]$) and write the inequality in the homogeneous form $f \ge 0$, where

$$f = \frac{4(abcd)^{3/4}}{g^{1/2}} + 3(a-d)^2 - (a^2 + b^2 + c^2 + d^2),$$

where

$$g = \frac{a^2 + b^2 + c^2 + d^2}{4}.$$

For fixed a, c and d, f is a function of b, $b \in [d, a]$. We have

$$g'(b) = \frac{b}{2}$$

and

$$f'(b) = \frac{3(acd)^{3/4}}{b^{1/4}g^{1/2}} - \frac{2(abcd)^{3/4}g'}{g^{3/2}} - 2b$$

$$= \frac{3(acd)^{3/4}}{b^{1/4}g^{1/2}} - \frac{b^{7/4}(acd)^{3/4}}{g^{3/2}} - 2b = \frac{(acd)^{3/4}g(b)}{b^{1/4}h^{1/2}},$$

where

$$h(b) = 3 - \frac{1}{1 + \frac{a^2+c^2+d^2}{b^2}} - 2b^{5/4}g^{1/2}.$$

Since $h(b)$ is a decreasing function, there are three possible cases: (1) $h(b) \ge 0$ for $b \in [d, a]$, hence $f(b)$ is increasing on $[d, a]$; (2) $h(b) \ge 0$ for $b \in [d, d_1]$ and $h(b) \le 0$ for $b \in [d_1, a]$, hence $f(b)$ is increasing on $[d, d_1]$ and decreasing on $[d_1, a]$; (3) $h(b) \le 0$ for $b \in [d, a]$, hence $f(b)$ is decreasing on $[d, a]$. In all these cases, if the inequality $f(b)$ holds for the extreme values of b, then it holds for all possible values of b. Thus, we only need to prove the required inequality for $b \in \{a, d\}$. Similarly, we only need to prove the required inequality for $c \in \{a, d\}$. So, we need to show that $f \ge 0$ for $b, c \in \{a, d\}$, that is

$$\frac{8(a^k d^{4-k})^{3/4}}{\sqrt{ka^2 + (4-k)d^2}} + 3(a-d)^2 - ka^2 - (4-k)d^2 \ge 0,$$

or

$$8(a^k d^{4-k})^{3/4} \ge [(k-3)a^2 + 6ad + (1-k)d^2]\sqrt{ka^2 + (4-k)d^2},$$

where
$$k \in \{1, 2, 3\}.$$
For $d = 0$, the inequality reduces to
$$(3-k)a^2 \geq 0,$$
which is true for $k \in \{1, 2, 3\}$. Next, due to homogeneity, we may set $d = 1$. So, we need to show that $a \geq 1$ yields
$$8a^{3k/4} \geq [(k-3)a^2 + 6a + 1 - k]\sqrt{ka^2 + 4 - k}.$$
This is true if
$$64a^{3k/2} \geq [(k-3)a^2 + 6a + 1 - k]^2(ka^2 + 4 - k)$$
for $a \geq 1$ and $(k-3)a^2 + 6a + 1 - k \geq 0$.

Case 1: $k = 1$. We need to show that
$$16 \geq \sqrt{a}(3-a)^2(a^2+3)$$
for $1 \leq a \leq 3$. Since $2\sqrt{a} \leq a + 1$, it suffices to show that
$$32 \geq (a+1)(a-3)^2(a^2+3),$$
which is equivalent to
$$a^5 - 5a^4 + 6a^3 - 6a^2 + 9a - 5 \leq 0,$$
$$(a-1)^2[a^2(a-3) - a - 5] \leq 0.$$

Case 2: $k = 2$. We need to show that
$$32a^3 \geq (-a^2 + 6a - 1)^2(a^2 + 1)$$
for $1 \leq a \leq 3 + \sqrt{10} < 7$. Write the required inequality as follows:
$$a^6 - 12a^5 + 39a^2 - 56a^3 + 39a^2 - 12a + 1 \leq 0,$$
$$(a-1)^4(a^2 - 8a + 1) \leq 0.$$
It is true because
$$a^2 - 8a + 1 = -a(7-a) - (a-1) < 0.$$

Case 3: $k = 3$. We need to show that
$$16a^{9/2} \geq (3a-1)^2(a^2+1)$$
for $a \geq 1$. Since
$$a^{9/2} \geq \frac{2a^5}{a+1},$$

it suffices to show that

$$32a^5 \geq (a+1)(3a-1)^2(3a^2+1),$$

which is equivalent to

$$5a^5 - 9a^4 + 6a^3 - 6a^2 + 5a - 1 \geq 0,$$

$$(a-1)^2(5a^3 + a^2 + 3a - 1) \geq 0.$$

The proof is completed. The equality occurs for $a = b = c = d = 1$, and also for $a = b = c = \dfrac{2}{\sqrt{3}}$ and $d = 0$.

Remark. Since $abcd \leq 1$, the inequality in P 3.152 is stronger than the inequality in the preceding P 3.151.

□

P 3.153. *Let a, b, c, d be nonnegative real numbers such that*

$$a^2 + b^2 + c^2 + d^2 = 4.$$

If $a \geq b \geq c \geq d$, then

(a) $$1 - \sqrt{abcd} \geq \frac{1}{2}(b-c)^2;$$

(b) $$1 - \sqrt{abcd} \geq \frac{1}{4}(a-d)^2.$$

(Vasile Cîrtoaje, 2019)

Proof. (a) Since

$$b - c \leq \sqrt{ab} - \sqrt{cd},$$

it suffices to show that

$$1 - \sqrt{abcd} \geq \frac{1}{2}(\sqrt{ab} - \sqrt{cd})^2,$$

which is equivalent to

$$a^2 + b^2 + c^2 + d^2 - 4\sqrt{abcd} \geq 2ab + 2cd - 4\sqrt{abcd},$$

$$(a-b)^2 + (c-d)^2 \geq 0.$$

The equality occurs for

$$a = b = x, \quad c = d = y,$$

where x and y are nonnegative numbers such that $x \geq y$ and $x^2 + y^2 = 2$.

(b) Write the inequality as follows:

$$a^2+b^2+c^2+d^2-4\sqrt{abcd} \ge (a-d)^2,$$

$$b^2+c^2+2ad \ge 4\sqrt{abcd},$$

$$(b-c)^2+2\left(\sqrt{bc}-\sqrt{ad}\right)^2 \ge 0.$$

The equality occurs for

$$b=c=\sqrt{ad}, \quad a+d=2.$$

□

P 3.154. *Let a,b,c,d be nonnegative real numbers such that*

$$a^2+b^2+c^2+d^2=4.$$

If $a \ge b \ge c \ge d$, then

$$1-abcd \ \ge \ \frac{3}{4}(c-d)^2.$$

(Vasile Cîrtoaje, 2019)

Solution. Write the inequality in the homogeneous form

$$(a^2+b^2+c^2+d^2)^2-16abcd \ge 3(c-d)^2(a^2+b^2+c^2+d^2),$$

$$(a^2+b^2+c^2+d^2)(a^2+b^2+6cd-2c^2-2d^2)-16abcd \ge 0.$$

Denoting

$$x=\sqrt{ab}, \qquad x \ge b \ge c,$$

since $a^2+b^2 \ge 2x^2$, it is enough to show that $F(x) \ge 0$, where

$$F(x)=(2x^2+c^2+d^2)(x^2+3cd-c^2-d^2)-8x^2cd.$$

We will show that

$$F(x) \ge F(c) \ge 0.$$

We have

$$F(x)-F(c)=2(x^4-c^4)+(6cd-c^2-d^2)(x^2-c^2)-8cd(x^2-c^2)$$

$$=(x^2-c^2)(2x^2+c^2-d^2-2cd) \ge (x^2-c^2)(3c^2-d^2-2cd)$$

$$=(x^2-c^2)(c-d)(3c+d) \ge 0$$

and

$$F(c)=(2c^2+d^2)(3cd-d^2)-8c^3d=d(c-d)^3 \ge 0.$$

The proof is completed. The equality occurs for $a=b=c=d=1$, and also for $a=b=c=\dfrac{2}{\sqrt{3}}$, $d=0$.

□

P 3.155. *Let a,b,c,d be nonnegative real numbers such that*

$$a^2+b^2+c^2+d^2=4.$$

If $a \ge b \ge c \ge d$, then

(a) $$1-abcd \ \ge\ (a-b)(c-d);$$

(b) $$1-abcd \ \ge\ \frac{1+\sqrt{3}}{2}\,(a-b)(c-d).$$

(Vasile Cîrtoaje, 2019)

Solution. (a) After homogenization, we need to prove that ($kiyoras$_2001)

$$a^2+b^2+c^2+d^2-4\sqrt{abcd} \ge 4(a-b)(c-d),$$

which is equivalent to

$$(a-b)^2+(c-d)^2+2(ab+cd)-4\sqrt{abcd} \ge 4(a-b)(c-d),$$

$$(a-b-c+d)^2+2(ab+cd)-4\sqrt{abcd} \ge 2(a-b)(c-d).$$

Thus, it suffices to prove that

$$ab+cd-2\sqrt{abcd} \ge ac+bd-ad-bc,$$

that is

$$(ad+bc-2\sqrt{abcd}+(ab+cd-ac-bd) \ge 0,$$

$$\left(\sqrt{ad}-\sqrt{bc}\right)^2+(a-d)(b-c) \ge 0,$$

which is true. The equality occurs for $a=b=c=d=1$.

(b) Denote

$$k=\frac{1+\sqrt{3}}{2},$$

and write the inequality in the homogeneous form $f(a,b,c,d) \ge 0$, where

$$f(a,b,c,d)=a^2+b^2+c^2+d^2-4k(a-b)(c-d)-\frac{16abcd}{a^2+b^2+c^2+d^2}.$$

We will show that

$$f(a,b,c,d) \ge f(a,z,z,d) \ge 0,$$

where

$$z=\frac{b+c}{2}.$$

We have

$$f(a,b,c,d)-f(a,z,z,d)=A+4kB+16adC,$$

where

$$A = b^2 + c^2 - 2z^2 = \frac{1}{2}(b-c)2 \geq 0,$$

$$B = (a-z)(z-d) - (a-b)(c-d) = (a+d)z - ac - bd - (z^2 - bc)$$
$$= \frac{1}{2}(b-c)(a-d) - \frac{1}{4}(b-c)^2 = \frac{1}{4}(b-c)(2a-b+c-2d)$$
$$\geq \frac{1}{4}(b-c)(b+c-2d) \geq 0,$$

$$C = \frac{z^2}{a^2+2z^2+d^2} - \frac{bc}{a^2+b^2+c^2+d^2}$$
$$= \frac{(a^2+d^2)(z^2-bc)+z^2(b-c)^2}{(a^2+2z^2+d^2)(a^2+b^2+c^2+d^2)} \geq 0.$$

Since $A \geq 0$, $B \geq 0$ and $C \geq 0$, the inequality $f(a,b,c,d) \geq f(a,z,z,d)$ is true. Further, we need to show that $a \geq z \geq c$ yields $f(a,z,z,d) \geq 0$, that is

$$a^2 + 2z^2 + d^2 - \frac{16az^2d}{a^2+2z^2+d^2} \geq 4k(a-z)(z-d).$$

Denote

$$x = ad, \qquad y = \frac{a^2+d^2}{2}.$$

Since

$$(a-z)(z-d) = z(a+d) - ad - z^2 = z\sqrt{2(x+y)} - x - z^2,$$

the inequality is equivalent to

$$y + z^2 - \frac{4xz^2}{y+z^2} \geq 2k\left[z\sqrt{2(x+y)} - x - z^2\right].$$

For $z = 0$, which involves $x = 0$, the inequality is clearly true. For $z > 0$, due to homogeneity, we may consider $z = 1$. From

$$0 \leq (a-z)(z-d) = \sqrt{2(x+y)} - x - 1,$$

it follows

$$x \leq \sqrt{2y-1}, \quad y \geq \frac{1}{2}.$$

So, we need to show that for fixed $y \in \left[\frac{1}{2}, \infty\right)$, we have $f(x) \geq 0$, where

$$f(x) = y + 1 - \frac{4x}{y+1} - (1+\sqrt{3})\left[\sqrt{2(x+y)} - x - 1\right], \qquad 0 \leq x \leq \sqrt{2y-1}.$$

There are two cases to consider: $\frac{1}{2} \leq y \leq 2$ and $y \geq 2$

Case 1: $\frac{1}{2} \le y \le 2.$ Since $2k < 3$, it suffices to show that

$$y+1-\frac{4x}{y+1} \ge 3\left[\sqrt{2(x+y)}-x-1\right].$$

In addition, by Bernoulli's Inequality, we have

$$\sqrt{2(x+y)} = 2\sqrt{\frac{x+y}{2}} = 2\sqrt{1+\frac{x+y-2}{2}}$$

$$\le 2\left(1+\frac{x+y-2}{4}\right) = \frac{x+y+2}{2}.$$

Thus, it is enough to show that

$$y+1-\frac{4x}{y+1} \ge 3\left(\frac{x+y+2}{2}-x-1\right),$$

that is

$$(2-y)(y+1) \ge (5-3y)x.$$

For the nontrivial case $\frac{1}{2} \le y \le \frac{5}{3}$, it suffices to show that

$$(2-y)(y+1) \ge (5-3y)\sqrt{2y-1}.$$

By squaring, the inequality becomes

$$y^4-20y^3+66y^2-76y+29 \ge 0,$$

$$(y-1)^2(y^2-18y+29) \ge 0.$$

It is true because

$$y^2-18y+29 = (y-3)^2+4(5-3y) > 0.$$

Case 2: $y \ge 2.$ Since $x+y \ge y \ge 2$, we have

$$f'(x) = \frac{-4}{y+1}+(1+\sqrt{3})\left(1-\frac{1}{\sqrt{2(x+y)}}\right)$$

$$\ge \frac{-4}{3}+\frac{1}{2}(1+\sqrt{3}) = \frac{3\sqrt{3}-5}{6} > 0.$$

Therefore, f is increasing, hence

$$f(x) \ge f(0) = y+1-(1+\sqrt{3})\left(\sqrt{2y}-1\right) = \frac{1}{2}\left(\sqrt{2y}-1-\sqrt{3}\right)^2 \ge 0.$$

The proof is completed. The equality occurs for $a=b=c=d=1$, and also for

$$a = \sqrt{2\left(1+\frac{1}{\sqrt{3}}\right)}, \quad b=c=\sqrt{1-\frac{1}{\sqrt{3}}}, \quad d=0\,.$$

□

P 3.156. *Let a, b, c, d be nonnegative real numbers such that*

$$a^2+b^2+c^2+d^2=4.$$

If $a \ge b \ge c \ge d$, then

$$1-abcd \;\ge\; 3(a-b)(b-c)(c-d)(a-d).$$

(Vasile Cîrtoaje, 2019)

Solution. Write the inequality in the homogeneous form

$$(a^2+b^2+c^2+d^2)^2-16abcd-48(a-b)(b-c)(c-d)(a-d) \ge 0.$$

Using the substitution

$$a=x+d, \quad b=y+d, \quad c=z+d, \quad x \ge y \ge z \ge 0,$$

for fixed x, y, z, we may write the inequality as $f(d) \ge 0$. Since

$$a'=b'=c=d'=1,$$

we have

$$\frac{1}{4}f'(d)=(a+b+c+d)(a^2+b^2+c^2+d^2)-4(bcd+acd+abd+abc).$$

Since

$$\left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) \ge \frac{2}{3}\left(\sum_{cyc} a\right)\left(\sum_{sym} ab\right)=2\sum_{cyc} abc+\frac{2}{3}\sum_{cyc} a(b^2+c^2+d^2)$$

$$\ge 2\sum_{cyc} abc+\frac{2}{3}\sum_{cyc} a(bc+cd+da)=4\sum_{cyc} abc,$$

we have $f'(d) \ge 0$. Therefore, $f(d)$ is increasing and hence $f(d) \ge f(0)$. So, it is enough to show that $f(0) \ge 0$, that is

$$(a^2+b^2+c^2)^2 \ge 48(a-b)(b-c)ac.$$

Since

$$a^2+b^2+c^2 \ge a^2+b^2$$

and

$$4(b-c)c \le b^2,$$

it suffices to show that

$$(a^2+b^2)^2 \ge 12ab^2(a-b),$$

which is equivalent to

$$a^4-10a^2b^2+12ab^3+b^4\ge 0,$$

$$(a-2b)^2(a-b)(a+5b)+7a^2-24ab+21b^2\ge 0.$$

$$(a-2b)^2(a-b)(a+5b)+7\left(a-\frac{12}{7}b\right)^2+\frac{3}{7}b^2\ge 0.$$

The proof is completed. The equality occurs for $a=b=c=d=1$.

□

P 3.157. *Let a,b,c,d be nonnegative real numbers such that*

$$a^2+b^2+c^2+d^2=4.$$

If $a\ge b\ge c\ge d$, then

$$(a)\qquad 1-\sqrt{abcd}\ \ge\ \frac{1}{3}(b-d)^2;$$

$$(b)\qquad 1-(abcd)^{3/4}\ \ge\ \frac{1}{2}(b-d)^2.$$

(Vasile Cîrtoaje, 2019)

Solution. (a) Write the inequality in the homogeneous form

$$3(a^2+b^2+c^2+d^2)^2-12\sqrt{abcd}\ge 4(b-d)^2,$$

or $F(a,b,c,d)\ge 0$, where

$$F(a,b,c,d)=3a^2-b^2+3c^2-d^2+8bd-12\sqrt{abcd}.$$

We will show that

$$F(a,b,c,d)\ge F(b,b,c,d)\ge 0.$$

We have

$$F(a,b,c,d)-F(b,b,c,d)=3(a^2-b^2)-12\sqrt{bcd}\,(\sqrt{a}-\sqrt{b})$$

$$=3(\sqrt{a}-\sqrt{b})\Big[(\sqrt{a}+\sqrt{b})(a+b)-4\sqrt{bcd}\Big]\ge 0.$$

In addition,

$$F(b,b,c,d)=2b^2+4(2d-3\sqrt{cd})b+3c^2-d^2$$

$$=2(b+2d-3\sqrt{cd}\,)^2+3(c^2-6cd+8d\sqrt{cd}-3d^2)$$

$$=2(b+2d-3\sqrt{cd}\,)^2+3(\sqrt{c}-\sqrt{d}\,)^3(\sqrt{c}+3\sqrt{d})\ge 0.$$

The equality occurs for $a=b=c=d=1$.

(b) Let us denote

$$g = \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}, \quad h = \sqrt[4]{abcd},$$

which satisfy

$$a \ge g \ge h.$$

For fixed b, c, d, write the inequality in the homogeneous form $f(a) \ge 0$, where

$$f(a) = a^2 + b^2 + c^2 + d^2 - \frac{4h^3}{g} - 2(b-d)^2.$$

Since

$$g'(a) = \frac{a}{4g}, \qquad h'(a) = \frac{h}{4a},$$

we have

$$f'(a) = 2a - \frac{12h^2h'}{g} + \frac{4h^3g'}{g^2} = 2a - \frac{3h^3}{ag} + \frac{ah^3}{g^3}$$

$$= \frac{a^2(2g^3+h^3) - 3g^2h^3}{ag^3} \ge \frac{g^2(2g^3+h^3) - 3g^2h^3}{ag^3} = \frac{2(g^3-h^3)}{ag} \ge 0.$$

hence f is increasing, therefore $f(a) \ge f(b)$. So, we only need to prove that $f(b) \ge 0$, that is

$$2b^2 + c^2 + d^2 - \frac{8(b^2cd)^{3/4}}{\sqrt{2b^2+c^2+d^2}} - 2(b-d)^2 \ge 0,$$

$$(4bd + c^2 - d^2)\sqrt{2b^2+c^2+d^2} \ge 8(b^2cd)^{3/4}.$$

Due to homogeneity, we may set (for the nontrivial case $d > 0$)

$$b = \frac{1}{cd},$$

when the inequality becomes

$$\left(c^2 - d^2 + 4\sqrt{\frac{d}{c}}\right)\sqrt{c^2 + d^2 + \frac{2}{cd}} \ge 8.$$

Denoting

$$d = cx^2, \quad 0 \le x \le 1,$$

we need to show that

$$(c^2 - c^2x^4 + 4x)\sqrt{c^2 + c^2x^4 + \frac{2}{c^2x^2}} \ge 8,$$

or, by squaring,

$$(c^2-c^2x^4+4x)^2\left(c^2+c^2x^4+\frac{2}{c^2x^2}\right)\geq 8,$$

that is

$$x^2(1-x^4)^2(1+x^4)c^8+8x^3(1-x^4)(1+x^4)c^6+2[(1-x^4)^2+8x^4(1+x^4)]c^4+$$
$$+16x(1-x^4-4x)c^2+32x^2\geq 0.$$

Write the inequality in the form

$$(1+x^4)\left[x(1-x^4)c^4+4x^2c^2-4x\right]^2+Ac^4+Bc^2+C,$$

where

$$A=2(1-x^4)(4x^6-x^4+4x^2+1\geq 0,$$
$$B=16x(2x^6-x^4+2x^2-4x+1),$$
$$C=16x^2(1-x^4)\geq 0.$$

The inequality is true for $0\leq x\leq \frac{1}{4}$ because

$$B\geq 16x[x^2(1-x^2)+1-4x]>0.$$

For $\frac{1}{4}\leq x\leq 1$, since

$$Ac^4+Bc^2+C\geq (2\sqrt{AC}+B)c^2,$$

it suffices to show that $2\sqrt{AC}+B\geq 0$. This is true if $4AC\geq B^2$, that is

$$4x^{14}-9x^{12}+4x^{10}-15x^8+32x^7-4x^6-16x^5-7x^4+32x^3-36x^2+16x-1\geq 0,$$

which is equivalent to

$$(x-1)^4f(x)\geq 0,$$

where

$$f(x)=4x^{10}+16x^9+31x^8+44x^7+54x^6+60x^5+46x^4+28x^3+18x^2+12x-1\geq 0.$$

The equality occurs for $a=b=c=d=1$, and also for $a=b=\sqrt{2}$ and $c=d=0$.

□

P 3.158. *Let a,b,c,d be nonnegative real numbers such that*

$$a^4+b^4+c^4+d^4=4.$$

If $a\geq b\geq c\geq d$, then

(a) $$1-\sqrt{abcd}\ \geq\ \frac{1}{2}(ac-bd)^2;$$

(b) $$1-abcd\ \geq\ \frac{1}{\sqrt{2}}(ac-bd)^2.$$

(Vasile Cîrtoaje, 2019)

Solution. Firstly, we show that

$$a^4c^4 + b^4d^4 \le 2. \tag{*}$$

We have:

$$\begin{aligned}2 - a^4c^4 - b^4d^4 &= 2 + (a^4 - b^4)(b^4 - c^4) - b^4(a^4 + c^4 + d^4 - b^4)\\ &= 2 + (a^4 - b^4)(b^4 - c^4) - b^4(4 - 2b^4)\\ &= (a^4 - b^4)(b^4 - c^4) + 2(b^4 - 1)^2 \ge 0.\end{aligned}$$

(a) Using the notation

$$x = \sqrt{ac}, \qquad y = \sqrt{bd},$$

and having in view (*), it suffices to show that $x^8 + y^8 \le 2$ yields

$$2 - 2xy \ge (x^2 - y^2)^2,$$

that is

$$2(1 - xy + x^2y^2) \ge x^4 + y^4.$$

By squaring, the inequality becomes

$$4(1 - xy + x^2y^2)^2 \ge x^8 + y^8 + 2x^4y^4.$$

This is true if

$$2(1 - xy + x^2y^2)^2 \ge 1 + x^4y^4,$$

that is equivalent to

$$(1 - xy)^4 \ge 0.$$

The equality occurs for $a = b = c = d = 1$.

(b) ***First Solution.*** Using the notation

$$x = ac, \qquad y = bd,$$

and having in view (*), it suffices to show that $x^4 + y^4 \le 2$ yields

$$\sqrt{2}\,(1 - xy) \ge (x - y)^2,$$

that is

$$\sqrt{2} + \sqrt{2}(\sqrt{2} - 1)xy \ge x^2 + y^2.$$

Since

$$x^2 + y^2 = \sqrt{x^4 + y^4 + 2x^2y^2} \le \sqrt{2 + 2x^2y^2},$$

we only need to show that

$$1 + (\sqrt{2} - 1)xy \ge \sqrt{1 + x^2y^2}.$$

By squaring, the inequality becomes

$$xy(1-xy) \ge 0.$$

This is true because

$$2 \ge x^4 + y^4 \ge 2x^2y^2.$$

The equality occurs for $a = b = c = d = 1$, and also for $a = \sqrt[4]{2}$, $b = c = 1$ and $d = 0$.

Second Solution (by *kiyoras*_2001). Write the inequality in the homogeneous form $f(a,b,c,d) \ge 0$, where

$$f(a,b,c,d) = a^4 + b^4 + c^4 + d^4 - 2\sqrt{2}(a^2c^2 + b^2d^2) + 4(\sqrt{2}-1)abcd.$$

We will show that

$$f(a,b,c,d) \ge f(a,c,c,d).$$

We have:

$$f(a,b,c,d) - f(a,c,c,d) = b^4 - c^4 - 2\sqrt{2}d^2(b^2-c^2) + 4(\sqrt{2}-1)acd(b-c)$$

$$= (b-c)\left[(b+c)(b^2+c^2) - 2\sqrt{2}d^2(b+c) + 4(\sqrt{2}-1)acd\right]$$

$$\ge (b-c)\left[2d^2(b+c) - 2\sqrt{2}d^2(b+c) + 2(\sqrt{2}-1)(b+c)d^2\right] = 0.$$

Further, write the inequality $f(a,c,c,d) \ge 0$ as

$$2c^4 + \left[4(\sqrt{2}-1)ad - 2\sqrt{2}(a^2+d^2)\right]c^2 + a^4 + d^4 \ge 0.$$

Since

$$2c^4 + a^4 + d^4 \ge 2c^2\sqrt{2(a^4+d^4)} = 2c^2\sqrt{(2(a^2+d^2)^2 - 4a^2d^2},$$

we only need to show that

$$\sqrt{2(a^2+d^2)^2 - 4a^2d^2} \ge \sqrt{2}(a^2+d^2) - 2(\sqrt{2}-1)ad.$$

This is true because, by squaring, it becomes

$$ad(a-d)^2 \ge 0.$$

Third Solution. Write the inequality in the homogeneous form

$$a^4 + b^4 + c^4 + d^4 - 4abcd - 2\sqrt{2}\,(ac-bd)^2 \ge 0,$$

and use the substitution

$$a = x + d, \quad b = y + d, \quad c = z + d, \quad x \ge y \ge z \ge 0.$$

For fixed x, y, z, we may write the inequality as $f(d) \ge 0$. Since

$$a' = b' = c = d' = 1,$$

we have

$$\frac{1}{4}f'(d) = a^3 + b^3 + c^3 + d^3 - (abc + bcd + cda + dab) - \sqrt{2}\,(ac - bd)(a + c - b - d),$$

$$\frac{1}{4}f''(d) = 3(a^2 + b^2 + c^2 + d^2) - 2\sum_{sym} ab - \sqrt{2}\,(a + c - b - d)^2.$$

Since

$$3(a^2 + b^2 + c^2 + d^2) - 2\sum_{sym} ab - 2(a + c - b - d)^2 =$$

$$= (a - b - c + d)^2 + 4(a - d)(b - c) \ge 0,$$

it follows that $f'' \ge 0$, f' is increasing, hence

$$\frac{1}{4}f'(d) \ge \frac{1}{4}f'(0) = a^3 + b^3 + c^3 - abc - \sqrt{2}\,ac(a + c - b)$$

$$= a^3 + b^3 + c^3 + (\sqrt{2} - 1)abc - \sqrt{2}\,ac(a + c)$$

$$\ge a^3 + 2c^3 + (\sqrt{2} - 1)ac^2 - \sqrt{2}\,ac(a + c)$$

$$= a(a - c)^2 + (2 - \sqrt{2}\,)a^2c - 2ac^2 + 2c^3$$

$$\ge a(a - c)^2 + \frac{1}{2}a^2c - 2ac^2 + 2c^3 = a(a - c)^2 + \frac{1}{2}c(a - 2c)^2 \ge 0.$$

Since $f' \ge 0$, f is increasing, therefore

$$f(d) \ge f(0) = a^4 + b^4 + c^4 - 2\sqrt{2}\,a^2c^2 \ge a^4 + 2c^4 - 2\sqrt{2}\,a^2c^2$$

$$= (a^2 - \sqrt{2}\,c^2)^2 \ge 0.$$

□

P 3.159. *If a, b, c, d are nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$a^4 + b^4 + c^4 + d^4 = 4,$$

then

$$1 - abcd \ \ge\ \frac{3}{4}(ad - bc)^2.$$

(Vasie Cirtoaje, 2020)

Solution. Write the inequality in the homogeneous form $f(a,b,c,d) \ge 0$, where

$$f(a,b,c,d) = a^4 + b^4 + c^4 + d^4 + 2abcd - 3(a^2d^2 + b^2d^2).$$

We will show that

$$f(a,b,c,d) \ge f(b,b,c,d) \ge f(c,c,c,d) \ge 0.$$

Since

$$f(a,b,c,d) - f(b,b,c,d) = (a^4 - b^4) + 2bcd(a-b) - 3d^2(a^2 - b^2),$$

we have $f(a,b,c,d) \ge f(b,b,c,d)$ if

$$a^2 + b^2 + \frac{2bcd}{a+b} - 3d^2 \ge 0,$$

which is true if

$$a^2 + b^2 + \frac{bcd}{a} - 3d^2 \ge 0.$$

We have

$$a^2 + b^2 + \frac{bcd}{a} - 3d^2 \ge a^2 + \frac{bcd}{a} - 2d^2$$
$$\ge 2\sqrt{abcd} - 2d^2 \ge 0.$$

The inequality $f(b,b,c,d) \ge f(c,c,c,d)$ is equivalent to

$$2(b^4 - c^4) + 2cd(b^2 - c^2) - 3(c^2 + d^2)(b^2 - c^2) \ge 0,$$

which is true if

$$2(b^2 + c^2) + 2cd - 3(c^2 + d^2) \ge 0.$$

Indeed,

$$2(b^2 + c^2) + 2cd - 3(c^2 + d^2) \ge 2(c^2 + c^2) + 2cd - 3(c^2 + d^2)$$
$$= (c-d)(c+3d) \ge 0.$$

Finally, we have

$$f(c,c,c,d) = 3c^4 + d^4 + 2c^3d - 3c^2(c^2 + d^2) = d(2c^3 - 3c^2d + d^3)$$
$$= d(c-d)^2(2c+d) \ge 0.$$

The proof is completed. The equality occurs for $a = b = c = d = 1$, and also for $a = b = c = \sqrt[4]{\frac{4}{3}}$ and $d = 0$.

□

P 3.160. *If a,b,c,d are nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$a+b+c+d=4,$$

then

$$(a) \qquad \frac{a^4+b^4+c^4+d^4}{4} - abcd \ge 2(b-c)^2,$$

$$(b) \qquad \frac{a^4+b^4+c^4+d^4}{4} - abcd \ge \frac{3}{2}(a-b)^2;$$

$$(c) \qquad \frac{a^4+b^4+c^4+d^4}{4} - abcd \ge \frac{4}{3}(a-c)^2;$$

$$(d) \qquad \frac{a^4+b^4+c^4+d^4}{4} - abcd \ge \frac{4}{3}(c-d)^2.$$

(Vasile Cîrtoaje, 2020)

Solution. (a) For fixed b,c,d. write the left inequality in the homogeneous form $f(a) \ge 0$, where

$$f(a) = 2(a^4+b^4+c^4+d^4) - 8abcd - (b-c)^2(a+b+c+d)^2.$$

Since

$$\frac{1}{2}f'(a) = 4a^3 - 4bcd - (b-c)^2(a+b+c+d) \ge 4a^3 - 4acd - (a-c)^2(2a+c+d)$$

$$\ge 4a^3 - 4ac^2 - (a-c)^2(2a+2c) = 2(a-c)(a^2+2ac+c^2) \ge 0,$$

$f(a)$ is increasing, hence

$$f(a) \ge f(b) = 2(2b^4+c^4+d^4) - 8b^2cd - (b-c)^2(2b+c+d)^2.$$

So, we need to show that $g(d) \ge 0$, where

$$g(d) = 2(2b^4+c^4+d^4) - 8b^2cd - (b-c)^2(2b+c+d)^2.$$

Since

$$\frac{1}{2}g'(d) = 4(d^3-b^2c) - (b-c)^2(2b+c+d) \le 0,$$

$g(d)$ is decreasing, hence

$$g(d) \ge g(c) = 4(b^4+c^4) - 8b^2c^2 - 4(b-c)^2(b+c)^2 = 0.$$

The equality occurs for $a=b$, $c=d$, $a+c=2$.

(b) Write the inequality as

$$f(a,b,c,d) \ge \frac{3}{8}(a-b)^2(a+b+c+d)^2,$$

where $f(a,b,c,d) = a^4 + b^4 + c^4 + d^4 - 4abcd$. Since

$$f(a,b,c,d) - f(a,b,c,c) = -(c^4 - d^4) + 4abc(c-d)$$

$$= (c-d)(-c^3 - c^2d - cd^2 - d^3 + 4abc) \ge 0,$$

it suffices to show that

$$f(a,b,c,c) \ge \frac{3}{8}(a-b)^2(a+b+2c)^2.$$

Since

$$f(a,b,c,c) - f(a,b,b,b) = -2(b^4 - c^4) + 4ab(b^2 - c^2)$$
$$= 2(b^2 - c^2)(2ab - b^2 - c^2) \ge 0,$$

it suffices to show that

$$f(a,b,b,b) \ge \frac{3}{8}(a-b)^2(a+3b)^2,$$

which is equivalent to

$$8(a-b)^2(a^2 + 2ab + 3b^2) \ge 3(a-b)^2(a+3b)^2,$$

$$(a-b)^3(5a+3b) \ge 0.$$

The equality occurs for $a = b = c = d = 1$.

(c) Write the inequality as

$$f(a,b,c,d) \ge \frac{1}{3}(a-c)^2(a+b+c+d)^2,$$

where $f(a,b,c,d) = a^4 + b^4 + c^4 + d^4 - 4abcd$. Since

$$f(a,b,c,d) - f(a,b,c,c) = -(c^4 - d^4) + 4abc(c-d)$$

$$= (c-d)(-c^3 - c^2d - cd^2 - d^3 + 4abc) \ge 0,$$

it suffices to show that

$$f(a,b,c,c) \ge \frac{1}{3}(a-c)^2(a+b+2c)^2,$$

that is

$$3(a^4 + b^4 + 2c^4) - 12abc^2 \ge (a-c)^2(a+b+2c)^2.$$

Use the substitution

$$a = x + c, \quad b = y + c, \qquad x \ge y \ge 0,$$

and, for fixed x, y, write the inequality in the homogeneous form $f(c) \ge 0$, where

$$f(c) = 3(a^4 + b^4 + 2c^4) - 12abc^2 - (a-c)^2(a+b+2c)^2.$$

Since

$$a' = b' = c' = 1,$$

we have

$$\frac{1}{4}f'(c) = 3(a^3 + b^3 + 2c^3) - 3(2abc + ac^2 + bc^2) - (a-c)^2(a+b+2c),$$

$$\frac{1}{4}f''(c) = 9(a^2 + b^2 + 2c^2) - 6(ab + 2ac + 2bc + c^2) - 4(a-c)^2,$$

$$\frac{1}{24}f''(c) = 3(a+b+2c) - 3(a+b+2c) = 0,$$

therefore

$$\frac{1}{4}f''(c) = \frac{1}{4}f''(0) = 9(a^2+b^2) - 6ab - 4a^2 \geq 5a^2 - 6ab + 9b^2$$

$$= 4a^2 + (a-3b)^2 \geq 0,$$

$f'(c)$ is increasing, hence

$$\frac{1}{4}f'(c) \geq \frac{1}{4}f'(0) = 3(a^3+b^3) - a^2(a+b) \geq 0,$$

$f(c)$ is increasing, hence

$$f(c) \geq f(0) = 3(a^4+b^4) - a^2(a+b)^2 = 2a^4 - 2a^3b - a^2b^2 + 3b^4$$

$$= (a-b)^2(2a^2 + 2ab + b^2) + 2b^2 \geq 0.$$

The equality occurs for $a = b = c = d = 1$.

(d) Write the inequality as $f(a,b,c,d) \geq 0$, where

$$F(a,b,c,d) = 3(a^4+b^4+c^4+d^4) - 12abcd - (c-d)^2(a+b+c+d)^2.$$

Denote

$$x = \frac{a+b}{2}, \qquad a \geq x \geq b,$$

and show that

$$F(a,b,c,d) \geq F(x,x,c,d) \geq F(c,c,c,d) \geq 0.$$

Since $a^4 + b^4 \geq 2x^4$ and $x^2 \geq abc$, we have

$$F(a,b,c,d) - F(x,x,c,d) = 3(a^4+b^4-2x^4) + 12cd(x^2-ab) \geq 0.$$

Further, we have

$$F(x,x,c,d) = 6x^4 + 3(c^4+d^4) - 12x^2cd - (c-d)^2(2x+c+d)^2,$$

$$F(x,x,c,d)-F(c,c,c,d)=6(x^4-c^4)-12cd(x^2-c^2)-2(c-d)^2(x-c)(x+2c+d)$$
$$=6(x^2-c^2)(x^2+c^2-2cd)-2(c-d)^2(x-c)(x+2c+d)$$
$$\geq 6(x^2-c^2)(c-d)^2-2(c-d)^2(x-c)(x+2c+d)$$
$$=2(x-c)(c-d)^2(2x+c-d)\geq 0$$

and

$$F(c,c,c,d)=9c^4+3d^4-12c^3d-(c-d)^2(3c+d)^2$$
$$=3(c-d)^2(3c^3+2cd+d^2)-(c-d)^2(3c+d)^2$$
$$=2(c-d)^2d^2\geq 0.$$

The equality occurs for $a=b=c=d=1$, an also for $a=b=c=4/3$ and $d=0$.

□

P 3.161. *If a,b,c,d are nonnegative real numbers such that $a\geq b\geq c\geq d$ and $a+d\geq b+c$, then*

$$a+b+c+d-4\sqrt[4]{abcd}\ \leq 2\left(\sqrt{a}-\sqrt{d}\right)^2.$$

(Vasile Cîrtoaje, 2020)

Solution. For fixed b, c and d, write the inequality as $f(a)\geq 0$, where

$$f(a)=2\left(\sqrt{a}-\sqrt{d}\right)^2-a-b-c-d+4\sqrt[4]{abcd}$$
$$=a-b-c+d+4\sqrt[4]{abcd}-4\sqrt{ad}.$$

Denote

$$x=\sqrt[4]{\frac{d}{a}},\qquad x\leq 1.$$

Since

$$f'(a)=1+\sqrt[4]{\frac{bcd}{a^3}}-2\sqrt{\frac{d}{a}}\ \geq 1+\sqrt[4]{\frac{d^3}{a^3}}-2\sqrt{\frac{d}{a}}$$
$$=1+x^3-2x^2=(1-x)(1+x-x^2)\geq 0,$$

f is increasing, therefore $f(a)\geq f(b+c-d)$. So, we only need to show that $f(b+c-d)\geq 0$, that is

$$\sqrt[4]{(b+c-d)bcd}\geq\sqrt{(b+c-d)d}.$$

This is true if

$$bc\geq(b+c-d)d,$$

which is equivalent to the obvious inequality

$$(b-d)(c-d)\geq 0.$$

The equality occurs for $a=b=c=d$, and also for $a=b+c$ and $d=0$.

□

P 3.162. *If a,b,c,d are nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$a+kd \ge b+c, \qquad k=(1+\sqrt{2}\,)^4 \approx 33.970,$$

then

$$a+b+c+d-4\sqrt[4]{abcd} \le 2\left(\sqrt{a}-\sqrt{d}\right)^2.$$

(Vasile Cîrtoaje, 2020)

Solution. For fixed b, c and d, write the inequality as $f(a) \ge 0$, where

$$f(a)=2\left(\sqrt{a}-\sqrt{d}\right)^2-a-b-c-d+4\sqrt[4]{abcd}$$

$$=a-b-c+d+4\sqrt[4]{abcd}-4\sqrt{ad}.$$

Denote

$$x=\sqrt[4]{\frac{d}{a}}, \qquad x \le 1.$$

Since

$$f'(a)=1+\sqrt[4]{\frac{bcd}{a^3}}-2\sqrt{\frac{d}{a}} \ge 1+\sqrt[4]{\frac{d^3}{a^3}}-2\sqrt{\frac{d}{a}}$$

$$=1+x^3-2x^2=(1-x)(1+x-x^2) \ge 0,$$

it follows that $f(a)$ is increasing, therefore $f(a) \ge f(b+c-kd)$ (if $b+c-kd \ge b$) and $f(a) \ge f(b)$. There are two cases to consider: $c \ge kd$ and $d \le c \le kd$.

Case 1: $c \ge kd$. Since $f(a) \ge f(b+c-kd)$, we need to show that

$$f(b+c-kd) \ge 0,$$

that is

$$-(k-1)d+4\sqrt[4]{(b+c-kd)bcd}-4\sqrt{(b+c-kd)d} \ge 0.$$

From

$$(b-kd)(c-kd) \ge 0,$$

we get

$$bc \ge k(b+c-kd)d.$$

Thus, it is enough to show that

$$-(k-1)d+4\sqrt[4]{k(b+c-kd)^2d^2}-4\sqrt{(b+c-kd)d} \ge 0,$$

which is equivalent to

$$-(k-1)d+4\left(\sqrt[4]{k}-1\right)\sqrt{(b+c-kd)d} \ge 0.$$

For $d = 0$, the inequality is an equality. For $d > 0$, due to homogeneity, we may set $d = 1$ (hence $c \ge k$), when the inequality becomes

$$4\left(\sqrt[4]{k}-1\right)\sqrt{b+c-k} \ge k-1.$$

Since $b + c \ge 2c \ge 2k$, we only need to show that

$$4\left(\sqrt[4]{k}-1\right)\sqrt{k} \ge k-1.$$

Denoting

$$m = \sqrt[4]{k} = 1+\sqrt{2},$$

we have

$$4\left(\sqrt[4]{k}-1\right)\sqrt{k}-(k-1) = 4(m-1)m^2-(m^4-1)$$
$$= (m-1)^2(1+2m-m^2) = 0.$$

Case 2: $d \le c \le kd$. Since $f(a) \ge f(b)$, we need to show that $f(b) \ge 0$, that is $g(b) \ge 0$, where

$$g(b) = -c+d+4\sqrt[4]{b^2cd}-4\sqrt{bd}.$$

We will show that $g(b) \ge g(c) \ge 0$. Since

$$g'(b) = \frac{2\left(\sqrt[4]{cd}-\sqrt{d}\right)}{\sqrt{b}} \ge 0,$$

$g(b)$ is increasing, therefore $g(b) \ge g(c)$. The inequality $g(c) \ge 0$ has the form

$$-c+d+4\sqrt[4]{c^3d}-4\sqrt{c} \ge 0.$$

Due to homogeneity, we may set $d = 1$, when $1 \le c \le k$. We need to show that $h(c) \ge 0$, where

$$h(c) = -c+1+4\sqrt[4]{c^3}-4\sqrt{c} \ge 0.$$

Since $h(c)$ is concave, it is enough to show that $h(1) \ge 0$ and $h(k) \ge 0$. Indeed, $h(1) = 0$ and

$$h(k) = -k+1+4\sqrt[4]{k^3}-4\sqrt{k} = 0.$$

The proof is completed. The equality occurs for $a = b = c = d$, for $a = b + c$ and $d = 0$, and for $a = b = c = kd$.

Remark. Let

$$k_1 \le k = (1+\sqrt{2}\,)^4.$$

Since $a + k_1d \ge b + c$ yields $a + kd \ge b + c$, it follows that the inequality is true for all nonnegative real numbers a, b, c, d such that $a \ge b \ge c \ge d$ and $a + k_1d \ge b + c$. For $k_1 < k$, the equality occurs when $a = b = c = d$, and when $a = b + c$ and $d = 0$. The particular cases $k_1 = -1$, $k_1 = 0$ and $k_1 = 1$ are relevant.

□

P 3.163. *If a,b,c,d are nonnegative real numbers such that $a \ge b \ge c \ge d$ and*

$$a+d \ge 2c,$$

then

$$a+b+c+d-4\sqrt[4]{abcd} \le \frac{5}{2}\left(\sqrt{a}-\sqrt{d}\right)^2.$$

(Vasile Cîrtoaje, 2020)

Solution. For fixed b, c and d, write the inequality as $f(a) \ge 0$, where

$$f(a) = 5\left(\sqrt{a}-\sqrt{d}\right)^2 - 2(a+b+c+d) + 8\sqrt[4]{abcd}$$

$$= 3a - 2b - 2c + 3d + 8\sqrt[4]{abcd} - 10\sqrt{ad}.$$

Denote

$$x = \sqrt[4]{\frac{d}{a}}, \qquad x \le 1.$$

Since

$$f'(a) = 3 + 2\sqrt[4]{\frac{bcd}{a^3}} - 5\sqrt{\frac{d}{a}} \ge 3 + 2\sqrt[4]{\frac{d^3}{a^3}} - 5\sqrt{\frac{d}{a}}$$
$$= 3 + 2x^3 - 5x^2 = (1-x)(3+3x-2x^2) \ge 0,$$

it follows that $f(a)$ is increasing, therefore $f(a) \ge f(b)$ and $f(a) \ge f(2c-d)$ (if $2c-d \ge b$). There are two cases to consider: $b \ge 2c-d$ and $c \le b \le 2c-d$.

Case 1: $b \ge 2c-d$. Since $f(a) \ge f(b)$, we need to show that $f(b) \ge 0$, that is equivalent to $g(b) \ge 0$, where

$$g(b) = b - 2c + 3d + 8\sqrt[4]{b^2cd} - 10\sqrt{bd}.$$

Since

$$g'(b) = 1 + 4\sqrt[4]{\frac{cd}{b^2}} - 5\sqrt{\frac{d}{b}} \ge 1 + 4\sqrt[4]{\frac{d^2}{b^2}} - 5\sqrt{\frac{d}{b}}$$

$$= 1 - \sqrt{\frac{d}{b}} \ge 0,$$

$g(b)$ is increasing, hence

$$g(b) \ge g(2c-d) = 2d + 8\sqrt[4]{(2c-d)^2cd} - 10\sqrt{(2c-d)d}$$

$$= 2d + 2\sqrt{2c-d}\left(4\sqrt[4]{cd} - 5\sqrt{d}\right).$$

So, we need to show that

$$d \ge \sqrt{2c-d}\left(5\sqrt{d} - 4\sqrt[4]{cd}\right).$$

For $d=0$, this inequality is an equality. For $d>0$, due to homogeneity, we may set $d=1$ (hence $c\ge 1$), when the inequality becomes

$$1\ge \sqrt{2c-1}\left(5-4\sqrt[4]{c}\right).$$

Since

$$\sqrt{2c-1}\le c,$$

it suffices to show that

$$1\ge c\left(5-4\sqrt[4]{c}\right).$$

By the AM-GM inequality, we have

$$1+4c\sqrt[4]{c}=1+c\sqrt[4]{c}+c\sqrt[4]{c}+c\sqrt[4]{c}+c\sqrt[4]{c}\ge 5\sqrt[5]{c^5}=5c.$$

Case 2: $c\le b\le 2c-d$. Since $f(a)\ge f(2c-d)$, we need to show that $f(2c-d)\ge 0$, that is equivalent to $h(b)\ge 0$, where

$$h(b)=-2b+4c+8\sqrt[4]{(2c-d)bcd}-10\sqrt{(2c-d)d}.$$

We will show that

$$h(b)\ge h(2c-d)\ge 0.$$

Since

$$h'(b)=-2+2\sqrt[4]{\frac{(2c-d)cd}{b^3}}\le -2+2\sqrt[4]{\frac{(2c-d)d}{c^2}}\le 0,$$

it follows that $h(b)$ is decreasing, hence $h(b)\ge h(2c-d)$. Further, the inequality $h(2c-d)\ge 0$ is equivalent to

$$2d+8\sqrt[4]{(2c-d)^2cd}-10\sqrt{(2c-d)d}\ge 0,$$

which was proved in the case 1.

The proof is completed. The equality occurs for $a=b=c=d$, and also for $a=b=2c$ and $d=0$.

□

P 3.164. *If a,b,c,d are nonnegative real numbers such that $a\ge b\ge c\ge d$ and*

$$a+kd\ge 2c,\qquad k=(3+2\sqrt{3}\,)^4\approx 1745.95,$$

then

$$a+b+c+d-4\sqrt[4]{abcd}\;\le\frac{5}{2}\left(\sqrt{a}-\sqrt{d}\right)^2.$$

(Vasile Cîrtoaje, 2020)

Solution. For fixed b, c and d, write the inequality as $f(a) \ge 0$, where

$$f(a) = 5\left(\sqrt{a} - \sqrt{d}\right)^2 - 2(a+b+c+d) + 8\sqrt[4]{abcd}$$

$$= 3a - 2b - 2c + 3d + 8\sqrt[4]{abcd} - 10\sqrt{ad}.$$

Denote

$$x = \sqrt[4]{\frac{d}{a}}, \qquad x \le 1.$$

Since

$$f'(a) = 3 + 2\sqrt[4]{\frac{bcd}{a^3}} - 5\sqrt{\frac{d}{a}} \ge 3 + 2\sqrt[4]{\frac{d^3}{a^3}} - 5\sqrt{\frac{d}{a}}$$

$$= 3 + 2x^3 - 5x^2 = (1-x)(3+3x-2x^2) \ge 0,$$

it follows that $f(a)$ is increasing, therefore $f(a) \ge f(b)$ and $f(a) \ge f(2c-kd)$ (if $2c - kd \ge b$). There are two cases to consider: $b \ge 2c - kd$ and $b \le 2c - kd$.

Case 1: $b \ge 2c - kd$, $b \ge c$. Since $f(a) \ge f(b)$, we need to show that $f(b) \ge 0$, that is equivalent to $g(b) \ge 0$, where

$$g(b) = b - 2c + 3d + 8\sqrt[4]{b^2cd} - 10\sqrt{bd}.$$

Since

$$g'(b) = 1 + 4\sqrt[4]{\frac{cd}{b^2}} - 5\sqrt{\frac{d}{b}} \ge 1 + 4\sqrt[4]{\frac{d^2}{b^2}} - 5\sqrt{\frac{d}{b}}$$

$$= 1 - \sqrt{\frac{d}{b}} \ge 0,$$

$g(b)$ is increasing. There are two sub-cases to consider: $c \ge kd$ and $c \le kd$.

Sub-case $c \ge kd$. We have

$$g(b) \ge g(2c-kd) = (3-k)d + 8\sqrt[4]{(2c-kd)^2cd} - 10\sqrt{(2c-kd)d}$$

$$\ge (3-k)d + 8\sqrt[4]{(2c-kd)^2kd^2} - 10\sqrt{(2c-kd)d}$$

$$= (3-k)d + 2\sqrt{(2c-kd)d}\left(4\sqrt[4]{k} - 5\right)$$

$$\ge \left[3 - k + 2\sqrt{k}\,(4\sqrt[4]{k} - 5)\right]d = 0.$$

Denoting

$$m = \sqrt[4]{k} = 3 + 2\sqrt{3},$$

we have

$$3 - k + 2\sqrt{k}\,(4\sqrt[4]{k} - 5) = 3 - m^4 + 2m^2(4m-5)$$

$$= 3 - 10m^2 + 8m^3 - m^4 = (1-m)^2(3+6m-m^2) = 0.$$

Sub-case $c \le kd$. Since

$$g(b) \ge g(c) = -c + 3d + 8\sqrt[4]{c^3 d} - 10\sqrt{cd},$$

we need to show that

$$-c + 3d + 8\sqrt[4]{c^3 d} - 10\sqrt{cd} \ge 0.$$

Due to homogeneity, we may set $d = 1$. So, we need to show that $c \le k$ yields

$$-c + 3 + 8\sqrt[4]{c^3} - 10\sqrt{c} \ge 0.$$

Denoting

$$x = \sqrt[4]{c}, \qquad x \le \sqrt[4]{k} = 3 + 2\sqrt{3},$$

we have

$$-c + 3 + 8\sqrt[4]{c^3} - 10\sqrt{c} = -x^4 + 3 + 8x^3 - 10x^2$$

$$= 3 - 10x^2 + 8x^3 - x^4 = (1-x)^2(3 + 6x - x^2) \ge 0.$$

Case 2: $b \le 2c - kd$, $b \ge c$. From $c \le b \le 2c - kd$, it follows that

$$c \ge kd.$$

Since $f(a) \ge f(2c - kd)$, we need to show that $f(2c - kd) \ge 0$, that is equivalent to $g(b) \ge 0$, where

$$g(b) = -2b + 4c + 3(1-k)d + 8\sqrt[4]{(2c-kd)bcd} - 10\sqrt{(2c-kd)d}.$$

We will show that

$$g(b) \ge g(2c - kd) \ge 0.$$

We have

$$g'(b) = -2 + 2\sqrt[4]{\frac{(2c-kd)cd}{b^3}} \le -2 + 2\sqrt[4]{\frac{(2c-kd)d}{c^2}}.$$

Since

$$c^2 - (2c - kd)d = (c-d)^2 + (k-1)d^2 \ge 0,$$

it follows that $g'(b) \le 0$, $g(b)$ is decreasing, hence $g(b) \ge g(2c - kd)$. Further, the inequality $g(2c - kd) \ge 0$ is equivalent to

$$(3-k)d + 8\sqrt[4]{(2c-kd)^2 cd} - 10\sqrt{(2c-kd)d} \ge 0,$$

$$(3-k)d + 2\sqrt{2c-kd}\left(4\sqrt[4]{cd} - 5\sqrt{d}\right) \ge 0.$$

Since $c \ge kd$, we have

$$4\sqrt[4]{cd} - 5\sqrt{d} \ge \left(4\sqrt[4]{k} - 5\right)\sqrt{d} \ge 0.$$

Thus, it suffices to prove the inequality for $c = kd$, that is

$$(3-k)+2\sqrt{k}\left(4\sqrt[4]{k}-5\right)\geq 0,$$

which is an equality.

The proof is completed. The equality occurs for $a = b = 2c$ and $d = 0$, and also for $a = b = c = kd$.

Remark. Let

$$k_1 \leq k = (3+2\sqrt{3}\,)^4.$$

Since $a + k_1 d \geq 2c$ yields $a + kd \geq 2c$, it follows that the inequality is true for all nonnegative real numbers a, b, c, d such that $a \geq b \geq c \geq d$ and $a + k_1 d \geq 2c$. For $k_1 < k$, the equality occurs when $a = b = 2c$ and $d = 0$. The particular cases $k_1 = 0$ and $k_1 = 1$ are relevant.

□

P 3.165. *If a, b, c, d are nonnegative real numbers such that $a \geq b \geq c \geq d$ and*

$$a+b+c+d=4,$$

then

$$(a-d)^2 \;\leq\; \frac{a^4+b^4+c^4+d^4}{4} - abcd \;\leq 4(a-d)^2.$$

(Vasile Cîrtoaje, 2020)

Solution. (a) Write the left inequality in the homogeneous form

$$2(a^4+b^4+c^4+d^4) \;-\; 8abcd \;\geq \frac{1}{2}(a-d)^2(a+b+c+d)^2.$$

Since

$$8 = \frac{1}{2}(a+b+c+d)^2 \leq (b+c)^2+(a+d)^2,$$

it suffices to prove the homogeneous inequality $F(a, b, c, d) \geq 0$, where

$$F(a,b,c,d) = 2(a^4+b^4+c^4+d^4) - 8abcd - (a-d)^2[(b+c)^2+(a+d)^2].$$

Denote

$$x = \frac{b+c}{2}, \qquad a \geq x \geq d,$$

and show that

$$F(a,b,c,d) \geq F(a,x,x,d) \geq 0.$$

Since $b^4 + c^4 \geq 2x^4$ and $x^2 \geq bc$, we have

$$F(a,b,c,d) - F(a,x,x,d) = 4(b^4+c^4-2x^4) + 16ad(x^2-bc) \geq 0.$$

Further, we have

$$F(a,x,x,d) = 4x^4 + 2(a^4+d^4) - 8adx^2 - (a-d)^2\left[4x^2+(a+d)^2\right]$$
$$= (2x^2-a^2-d^2)^2 \ge 0.$$

The equality occurs for $a=b=c=d=1$.

(b) Write the right inequality in the homogeneous form

$$a^4+b^4+c^4+d^4-4abcd \le (a-d)^2(a+b+c+d)^2.$$

Use the substitution

$$a=x+d,\quad b=y+d,\quad c=z+d,\qquad x\ge y\ge z\ge 0,$$

and, for fixed x,y,z, write the inequality in the homogeneous form $f(d)\ge 0$, where

$$f(d)=a^4+b^4+c^4+d^4-4abcd-(a-d)^2(a+b+c+d)^2.$$

Since

$$a'=b'=c'=d'=1,$$

we have

$$\frac{1}{4}f'(d)=a^3+b^3+c^3+d^3-(bcd+cda+dab+abc)-2(a-d)^2(a+b+c+d),$$

$$\frac{1}{4}f''(d)=3(a^2+b^2+c^2+d^2)-2\sum_{sym}ab-8(a-d)^2,$$

$$\frac{1}{4}f'''(d)=6(a+b+c+d)-6(a+b+c+d)=0,$$

therefore

$$\frac{1}{4}f''(d)=\frac{1}{4}f''(0)=3(a^2+b^2+c^2)-2(ab+bc+ca)-8a^2\le -5a^2+3(b^2+c^2)-ab$$
$$=3(b^2-a^2)+2(c^2-a^2)+c^2-ab\le 0,$$

$f'(d)$ is decreasing, hence

$$\frac{1}{4}f'(d)\le\frac{1}{4}f'(0)=a^3+b^3+c^3-abc-2a^2(a+b+c)$$
$$\le a^3+b^3+c^3-a^2(a+b+c)=b(b^2-a^2)+c(c^2-a^2)\le 0,$$

$f(d)$ is decreasing, hence

$$f(d)\le f(0)=a^4+b^4+c^4-a^2(a+b+c)^2\le a^4+b^4+c^4-a^2(a^2+^2b+c^2)$$
$$=b^2(a^2-b^2)+c^2(c^2-a^2)\le 0.$$

The equality occurs for $b=c=d=0$.

□

P 3.166. *Let a, b, c, d, e be nonnegative real numbers.*

(a) If $a+b+c = 3(d+e)$, then

$$4(a^4+b^4+c^4+d^4+e^4) \ge (a^2+b^2+c^2+d^2+e^2)^2;$$

(b) If $a+b+c = d+e$, then

$$12(a^4+b^4+c^4+d^4+e^4) \le 7(a^2+b^2+c^2+d^2+e^2)^2.$$

(Vasile Cîrtoaje, 2010)

Solution. (a) Let

$$E(a,b,c,d,e) = 4(a^4+b^4+c^4+d^4+e^4) - (a^2+b^2+c^2+d^2+e^2)^2.$$

We will show that

$$E(a,b,c,d,e) \ge E(a,b,c,d+e,0) \ge 0.$$

The left side inequality is equivalent to

$$de(a^2+b^2+c^2-3d^2-3e^2-5de) \ge 0.$$

This is true since

$$a^2+b^2+c^2-3d^2-3e^2-5de \ge \frac{1}{3}(a+b+c)^2-3d^2-3e^2-5de$$
$$= 3(d+e)^2-3d^2-3e^2-5de = de \ge 0.$$

Also, in virtue of the Cauchy-Schwarz inequality, we have

$$E(a,b,c,d+e,0) = 4[a^4+b^4+c^4+(d+e)^4] - [a^2+b^2+c^2+(d+e)^2]^2 \ge 0.$$

The equality holds for $a=b=c=d$ and $e=0$, or for $a=b=c=e$ and $d=0$.

(b) Let

$$E(a,b,c,d,e) = 7(a^2+b^2+c^2+d^2+e^2)^2 - 12(a^4+b^4+c^4+d^4+e^4).$$

We will show that

$$E(a,b,c,d,e) \ge E(a,b,c,d+e,0) \ge 0.$$

The left side inequality is equivalent to

$$de[12(d^2+e^2)+11de-7(a^2+b^2+c^2)] \ge 0.$$

This is true since

$$\begin{aligned}12(d^2+e^2)+11de-7(a^2+b^2+c^2) &\ge 12(d^2+e^2)+11de-7(a+b+c)^2\\ &= 12(d^2+e^2)+11de-7(d+e)^2\\ &= 5(d^2+e^2)-3de \ge 0.\end{aligned}$$

Also, we have

$$\begin{aligned}\frac{1}{4}E(a,b,c,d+e,0) &= \frac{1}{4}E(a,b,c,a+b+c,0)\\ &= \sum a^4+2\sum ab(a^2+b^2)+3\sum a^2b^2-8abc\sum a\\ &\ge \sum a^2b^2+4\sum a^2b^2+3\sum a^2b^2-8abc\sum a\\ &= 8\left(\sum a^2b^2-abc\sum a\right)\\ &= 4\sum a^2(b-c)^2 \ge 0.\end{aligned}$$

The equality holds for $a=b=c=d/3$ and $e=0$, or for $a=b=c=e/3$ and $d=0$.

□

P 3.167. *Let a,b,c,d,e be nonnegative real numbers such that*

$$a+b+c+d+e=5.$$

Prove that

$$a^4+b^4+c^4+d^4+e^4+150 \le 31(a^2+b^2+c^2+d^2+e^2).$$

(Vasile Cîrtoaje, 2007)

Solution. Write the inequality as

$$\sum(a^4-31a^2+30a) \le 0,$$

or

$$\sum(1-a)f(a) \le 0,$$

where

$$f(a)=a^3+a^2-30a.$$

Without loss of generality, assume that $a \ge b \ge c \ge d \ge e$. Since $a+b \le 5$, we have

$$\begin{aligned}f(a)-f(b) &= (a-b)(a^2+ab+b^2+a+b-30)\\ &\le (a-b)[(a+b)^2+a+b-30]\\ &= (a-b)(a+b-5)(a+b+6) \le 0.\end{aligned}$$

Similarly,

$$f(b)-f(c)\le 0,\quad f(c)-f(d)\le 0,\quad f(d)-f(e)\le 0.$$

Since

$$a-1\ge b-1\ge c-1\ge d-1\ge e-1$$

and

$$f(a)\le f(b)\le f(c)\le f(d)\le f(e),$$

by Chebyshev's inequality, we get

$$5\sum(a-1)f(a)\le\left[\sum(a-1)\right]\left[\sum f(a)\right]=0.$$

The equality holds for $a=b=c=d=e=1$, and for $(a,b,c,d,e)=(5,0,0,0,0)$ or any cyclic permutation.

Remark. Similarly, we can prove the following generalization:

- *If $a_1,a_2,\dots,a_n$ are nonnegative real numbers such that*

$$a_1+a_2+\cdots+a_n=n,$$

then

$$a_1^4+a_2^4+\cdots+a_n^4+n^2(n+1)\le(n^2+n+1)(a_1^2+a_2^2+\cdots+a_n^2).$$

□

P 3.168. *Let a,b,c,d,e be positive real numbers such that*

$$a^2+b^2+c^2+d^2+e^2=5.$$

Prove that

$$abcde(a^4+b^4+c^4+d^4+e^4)\le 5.$$

(Vasile Cîrtoaje, 2006)

First Solution. Without loss of generality, assume that

$$a\le b\le c\le d\le e.$$

First, we prove that the expression

$$E(a,b,c,d,e)=abcde(a^4+b^4+c^4+d^4+e^4)$$

is maximal for $a=d$. We need to show that

$$E(a,b,c,d,e)\le E\left(\sqrt{\frac{a^2+d^2}{2}},b,c,\sqrt{\frac{a^2+d^2}{2}},e\right).$$

This inequality is true if

$$4ad(a^4+b^4+c^4+d^4+e^4) \le (a^2+d^2)[(a^2+d^2)^2+2(b^4+c^4+e^4)],$$

which is equivalent to

$$(a^2+d^2)^3-4ad(a^4+d^4)+2(b^4+c^4+e^4)(a-d)^2 \ge 0.$$

Since

$$\begin{aligned}(a^2+d^2)^3-4ad(a^4+d^4) &= (a^2+d^2)^3-4ad(a^2+d^2)^2+8a^3d^3\\ &= (a^2+d^2)[(a^2+d^2)-2ad]^2-4a^2d^2(a^2+d^2)+8a^3d^3\\ &= (a^2+d^2)(a-d)^4-4a^2d^2(a-d)^2\\ &\ge -4a^2d^2(a-d)^2,\end{aligned}$$

it suffices to show that

$$b^4+c^4+e^4 \ge 2a^2d^2.$$

Indeed, we have

$$b^4+c^4+e^4-2a^2d^2 \ge b^4+a^4+d^4-2a^2d^2 = b^4+(a^2-d^2)^2 > 0.$$

Since $E(a,b,c,d,e)$ is maximal for $a=d$ and, on the other hand, $a \le b \le c \le d$, it follows that $E(a,b,c,d,e)$ is maximal for $a=b=c=d$. Then, it suffices to show that the desired homogeneous inequality

$$\left(\frac{a^2+b^2+c^2+d^2+e^2}{5}\right)^9 \ge (abcde)^2\left(\frac{a^4+b^4+c^4+d^4+e^4}{5}\right)^2$$

holds for $a=b=c=d=1$. Denoting e^2 by x, we need to show that $f(x) \ge 0$ for $x>0$, where

$$f(x) = 9\ln\frac{4+x}{5}-\ln x-2\ln\frac{4+x^2}{5}.$$

From

$$f'(x) = \frac{9}{4+x}-\frac{1}{x}-\frac{4x}{4+x^2} = \frac{4(x-1)(x-2)^2}{x(4+x)(4+x^2)},$$

it follows that $f(x)$ is decreasing for $0<x\le 1$ and increasing for $x \ge 1$. Therefore, $f(x) \ge f(1)=0$. This completes the proof. The equality holds if and only if $a=b=c=d=e=1$.

Second Solution. Replacing a,b,c,d,e by $\sqrt{a}, \sqrt{b}, \sqrt{c}, \sqrt{d}, \sqrt{e}$, we need to show the homogeneous inequality

$$\left(\frac{a+b+c+d+e}{5}\right)^9 \ge abcde\left(\frac{a^2+b^2+c^2+d^2+e^2}{5}\right)^2,$$

where a,b,c,d,e are positive real numbers. According to Remark 1 from P 3.57, it suffices to prove this inequality for $b=c=d=e=1$; that is, to show that

$$\left(\frac{a+4}{5}\right)^9 \geq a\left(\frac{a^2+4}{5}\right)^2.$$

Taking logarithms of both sides, we need to prove that $f(a)\geq 0$, where

$$f(a)=9\ln(a+4)-7\ln 5-\ln a-2\ln(a^2+4).$$

From

$$f'(a)=\frac{9}{a+4}-\frac{1}{a}-\frac{4a}{a^2+4}=\frac{4(a-1)(a-2)^2}{a(a+4)(a^2+4)},$$

it follows that $f(a)$ is decreasing for $0<a\leq 1$ and increasing for $a\geq 1$; therefore, $f(a)\geq f(1)=0$.

Remark. The following more general statement holds (*Vasile Cirtoaje*, 2006):

- *If $a_1,a_2,\ldots,a_n$ are positive real numbers such that*

$$a_1+a_2+\cdots+a_n=n,$$

then

$$(a_1a_2\cdots a_n)^{\frac{1}{\sqrt{n-1}}}(a_1^2+a_2^2+\cdots+a_n^2)\leq n.$$

□

P 3.169. *Let a,b,c,d,e be positive real numbers such that*

$$a+b+c+d+e=5.$$

Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{20}{a^2+b^2+c^2+d^2+e^2}\geq 9.$$

(Vasile Cîrtoaje, 2006)

Solution. Without loss of generality, assume that

$$a\leq b\leq c\leq d\leq e.$$

First, we prove that the expression

$$E(a,b,c,d,e)=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{e}+\frac{20}{a^2+b^2+c^2+d^2+e^2}$$

is minimal when $a=d$. If this is true, then $E(a,b,c,d,e)$ is minimal when $a=b=c=d$, and it suffices to prove the desired inequality for $a=b=c=d$, when it is equivalent to the obvious inequality

$$(a-1)^2(6a-5)^2\geq 0.$$

Therefore, it remains to show that

$$E(a,b,c,d,d) \ge E\left(\frac{a+d}{2}, b, c, \frac{a+d}{2}, e\right).$$

This inequality is equivalent to

$$\frac{(a-d)^2}{ad(a+d)} \ge \frac{20(a-d)^2}{(a^2+b^2+c^2+d^2+e^2)[(a+d)^2+2b^2+2c^2+2e^2]}.$$

Since

$$a^2+b^2+c^2+d^2+e^2 \ge \frac{1}{5}(a+b+c+d+e)^2 = a+b+c+d+e,$$

it suffices to show that

$$(a+b+c+d+e)[(a+d)^2+2b^2+2c^2+2e^2] \ge 20ad(a+d).$$

Since

$$a+b+c+d+e \ge a+a+a+d+d = 3a+2d$$

and

$$(a+d)^2+2b^2+2c^2+2e^2 \ge (a+d)^2+2a^2+2a^2+2d^2 = 5a^2+2ad+3d^2,$$

it is enough to prove that

$$(3a+2d)(5a^2+2ad+3d^2) \ge 20ad(a+d).$$

This is true, since

$$\begin{aligned}(3a+2d)(5a^2+2ad+3d^2)-20ad(a+d) &= 15a^3-4a^2d-7ad^2+6d^3\\ &> 5a^3-4a^2d-7ad^2+6d^3\\ &= (a-d)^2(5a+6d) \ge 0.\end{aligned}$$

The proof is completed. The equality holds for $a=b=c=d=e=1$, and also for $(a,b,c,d,e) = \left(\frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{6}, \frac{5}{3}\right)$ or any cyclic permutation.

Remark. The following more general statement holds (*Vasile Cirtoaje*, 2006):

- *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1+a_2+\cdots+a_n = n,$$

then

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}+\frac{2n\sqrt{n-1}}{a_1^2+a_2^2+\cdots+a_n^2} \ge n+2\sqrt{n-1}.$$

□

P 3.170. *If $a,b,c,d,e \ge 1$, then*

$$\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)\left(d+\frac{1}{d}\right)\left(e+\frac{1}{e}\right)+68 \ge$$

$$\ge 4(a+b+c+d+e)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right).$$

(Vo Quoc Ba Can and Vasile Cîrtoaje, 2011)

Solution. Write the inequality as $E(a,b,c,d,e) \ge 0$, and denote

$$A=\left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)\left(c+\frac{1}{c}\right)\left(d+\frac{1}{d}\right), \quad A \ge 16.$$

We claim that

$$E(a,b,c,d,e) \ge E(a,b,c,d,1).$$

If this is true, then (by symmetry)

$$E(a,b,c,d,e) \ge E(a,b,c,d,1) \ge \ldots \ge E(a,1,1,1,1) = 0,$$

and the proof is completed. Since

$$E(a,b,c,d,e) - E(a,b,c,d,1) = (e-1)\left(B-\frac{C}{e}\right),$$

we need to show that

$$B-\frac{C}{e} \ge 0,$$

where

$$B = A - 4\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right),$$

$$C = A - 4(a+b+c+d).$$

Since $A \ge 16$ and

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \le 4,$$

it follows that $B \ge 0$. For the non-trivial case $C \ge 0$, we have

$$B-\frac{C}{e} \ge B-C = 4\left(a-\frac{1}{a}\right)+4\left(b-\frac{1}{b}\right)+4\left(c-\frac{1}{c}\right)+4\left(d-\frac{1}{d}\right) \ge 0.$$

The equality holds for $a=b=c=d=1$ (or any cyclic permutation).

□

P 3.171. *If a,b,c and x,y,z are nonnegative real numbers such that*

$$x^3+y^3+z^3=a^3+b^3+c^3,$$

then

$$(a+b+c)(x+y+z)\geq xy+yz+zx+ab+bc+ca.$$

(Vasile Cîrtoaje, 2020)

Solution. Denote

$$t=x+y+z.$$

From the known inequalities

$$(x+y+z)^3\geq x^3+y^3+z^3, \qquad (x+y+x)^3\leq 9(x^3+y^3+z^3),$$

we get

$$x+y+z\geq \sqrt[3]{x^3+y^3+z^3}=\sqrt[3]{a^3+b^3+c^3},$$

$$x+y+z\leq \sqrt[3]{9(x^3+y^3+z^3)}=\sqrt[3]{9(a^3+b^3+c^3)},$$

hence

$$t_1\leq t\leq t_2,$$

where

$$t_1=\sqrt[3]{a^3+b^3+c^3}, \qquad t_2=\sqrt[3]{9(a^3+b^3+c^3)}.$$

On the other hand, it is enough to prove the inequality

$$(a+b+c)(x+y+z)\geq \frac{(a+b+c)^2}{3}+\frac{(x+y+z)^2}{3},$$

or, better, the inequality

$$(a+b+c)(x+y+z)\geq \frac{5(a+b+c)^2}{12}+\frac{(x+y+z)^2}{3},$$

which is equivalent to

$$5t^2-12(x+y+z)t+4(x+y+z)^2\leq 0,$$

$$[t-2(x+y+z))[5t-2(x+y+z)]\leq 0.$$

This is true if $t_1\geq \dfrac{2}{5}(x+y+z)$ and $t_2\leq 2(x+y+z$. Thus, we need to show that

$$x^3+y^3+z^3\geq \frac{8}{125}(x+y+z)^3$$

and

$$4(x^3+y^3+z^3)\leq 8(x+y+z)^3.$$

These follow immediately from the known inequalities

$$x^3+y^3+z^3 \ge \frac{1}{9}(x+y+z)^3$$

and

$$x^3+y^3+z^3 \le (x+y+z)^3.$$

The equality holds for $a=b=x=y=z=0$.

Remark 1. Using the same method, we can prove the stronger inequalities

$$3k(a+b+c)(x+y+z) \ge (xy+yz+zx)^2+(ab+bc+ca)^2,$$

$$k(a+b+c)(x+y+z) \ge xy+yz+zx+ab+bc+ca,$$

where

$$k=\frac{1}{\sqrt[3]{3}}+\frac{\sqrt[3]{3}}{9} \approx 0.8536.$$

Remark 2. In the same conditions, the following inequality holds:

$$k(a+b+c)(x+y+z) \ge xy+yz+zx+ab+bc+ca, \qquad k=\frac{1}{\sqrt[3]{3}} \approx 0.6934,$$

with equality for $a=b=c=x=y=z$. If $a \ge b \ge c$ and $x \ge y \ge z$, the equality occurs also for $a=b=c=\dfrac{x}{\sqrt[3]{3}}$ and $y=z=0$, and for $x=y=z=\dfrac{a}{\sqrt[3]{3}}$ and $b=c=0$.

□

P 3.172. *Let a,b,c,d,e,f be nonnegative real numbers such that*

$$a^2+b^2+c^2+d^2+e^2+f^2=6.$$

If $a \ge b \ge c \ge d \ge e \ge f$, then

$$1-abcdef \le \frac{3}{2}(a-f)^2.$$

(Vasile Cîrtoaje, 2019)

Solution. For $a=f$, the inequality is a trivial equality. Consider next that $a>f$, give up the condition $b \ge c \ge d \ge e$ (consider only that $b,c,d,e \in [f,a]$) and write the inequality in the homogeneous form $T \ge 0$, where

$$T=216abcdef+9(a-f)^2G^2-G^3, \qquad G=a^2+b^2+c^2+d^2+e^2+f^2.$$

For fixed a, c, d, e and f, T is a function of b, $b \in [f,a]$. We have

$$T'(b)=216acdef+36(a-f)^2bG-6bG^2=6bGE(b),$$

where

$$E(b)=\frac{36acdef}{bG}+6(a-f)^2-G.$$

Since $E(b)$ is a decreasing function, there are three possible cases: (1) $E(b)\ge 0$ for $b\in[f,a]$, hence $T(b)$ is increasing on $[f,a]$; (2) $E(b)\ge 0$ for $b\in[f,f_1]$ and $E(b)\le 0$ for $b\in[f_1,a]$, hence $T(b)$ is increasing on $[f,f_1]$ and decreasing on $[f_1,a]$; (3) $E(b)\le 0$ for $b\in[f,a]$, hence $T(b)$ is decreasing on $[f,a]$. In all these cases $T(b)$ is minimal when $b\in\{a,f\}$. As a consequence, we only need to prove the required inequality for $b\in\{a,f\}$. Similarly, we only need to prove the required inequality for $c,d,e\in\{a,f\}$. So, we need to show that

$$216a^kf^{6-k}+9(a-f)^2[ka^2+(6-k)f^2]^2-[ka^2+(6-k)f^2]^3\ge 0,$$

where

$$k\in\{1,2,3,4,5\}.$$

For $f=0$, the inequality reduces to

$$k(9-k)a^4\ge 0,$$

which is true for $k\in\{1,2,3,4,5\}$. Next, due to homogeneity, we may set $d=1$ (which involves $a>1$). The required inequality becomes

$$9(a-1)^2(ka^2+6-k)^2\ge(ka^2+6-k)^3-216a^k.$$

Case 1: $k=1$. We need to show that

$$9(a-1)^2(a^2+5)^2\ge(a^2+5)^3-216a,$$

$$9(a-1)^2(a^2+5)^2\ge(a-1)^2(a^4+2a^3+18a^2+34a+125),$$

$$2(a-1)^2(4a^4-a^3+36a^2-17a+50)\ge 0.$$

The last inequality is true because

$$2a^4-a^3+36a^2-17a+50>4a^2(a-1)^2+2(2a-5)^2>0.$$

Case 2: $k=2$. We need to show that

$$9(a-1)^2(a^2+2)^2\ge 2(a^2+2)^3-54a^2,$$

$$9(a-1)^2(a^2+2)^2\ge 2(a^2-1)^2(a^2+8),$$

$$(a-1)^2(7a^4-4a^3+18a^2-32a+20)\ge 0.$$

The last inequality is true because

$$7a^4-4a^3+18a^2-32a+20>2a^2(a-1)^2+16(a-1)^2>0.$$

Case 3: $k = 3$. We need to show that

$$3(a-1)^2(a^2+1)^2 \geq (a^2+1)^3 - 8a^3,$$

$$9(a-1)^2(a^2+2)^2 \geq (a-1)^2(a^4+2a^3+6a^2+2a+1),$$

$$2(a-1)^3(a^3-1) \geq 0.$$

Case 4: $k = 4$. We need to show that

$$9(a-1)^2(2a^2+1)^2 \geq 2(2a^2+1)^3 - 54a^4,$$

$$9(a-1)^2(2a^2+1)^2 \geq 2(a^2-1)^2(8a^2+1),$$

$$(a-1)^2(20a^4-32a^3+18a^2-4a+7) \geq 0.$$

The last inequality is true because

$$20a^4-32a^3+18a^2-4a+7 > 16a^2(a-1)^2+2(a-1)^2 > 0.$$

Case 5: $k = 5$. We need to show that

$$9(a-1)^2(5a^2+1)^2 \geq (5a^2+1)^3 - 216a^5,$$

$$9(a-1)^2(5a^2+1)^2 \geq (a-1)^2(125a^4+34a^3+18a^2+2a+1),$$

$$2(a-1)^2(50a^4-17a^3+36a^2-a+4) \geq 0.$$

The last inequality is true because

$$50a^4-17a^3+36a^2-a+4 > 2a^2(5a-2)^2+4(a-1)^2 > 0.$$

The proof is completed. The equality occurs for $a = b = c = d = e = f = 1$.

Remark. The following more general statement holds:

- *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that*

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n, \qquad a_1 \geq a_2 \geq \cdots \geq a_n,$$

then

$$1 - a_1 a_2 \cdots a_n \leq \frac{n}{4}(a_1 - a_n)^2,$$

with equality for $a_1 = a_2 = \cdots = a_n = 1$.

To prove this inequality, we need to show that

$$4 - 4a^k b^{n-k} \leq n(a-b)^2$$

for $a \geq 1 \geq b \geq 0$ and

$$ka^2 + (n-k)b^2 = n, \qquad k \in \{1, 2, \ldots, n-1\}.$$

□

P 3.173. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$be nonnegative real numbers such that*

$$a_1^2 + a_2^2 + \cdots + a_n^2 = b_1^2 + b_2^2 + \cdots + b_n^2.$$

Then, for $n = 3$ and $n = 4$, the following inequalities holds:

$$(n-1)(a_1 + a_2 + \cdots + a_n)(b_1 + b_2 + \cdots + b_n) \ge n\left(\sum_{i<j} a_i a_j + \sum_{i<j} b_i b_j\right).$$

(Vasile Cîrtoaje, 2020)

Solution. Denote

$$a = a_1 + a_2 + \cdots + a_n, \qquad b = b_1 + b_2 + \cdots + b_n, \qquad c = \sqrt{n(b_1^2 + b_2^2 + \cdots + b_n^2)},$$

and write the inequality as

$$2(n-1)ab + 2c^2 \ge na^2 + nb^2.$$

We have

$$a \le \sqrt{n(a_1^2 + a_2^2 + \cdots + a_n^2)} = \sqrt{n(b_1^2 + b_2^2 + \cdots + b_n^2)} = c$$

and

$$a \ge \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = \sqrt{b_1^2 + b_2^2 + \cdots + b_n^2} = \frac{c}{\sqrt{n}}.$$

For fixed b and c, we may write the required inequality as $f(a) \le 0$, where

$$f(a) = na^2 - 2(n-1)ab + nb^2 - 2c^2$$

is a quadratic convex function. Thus, it is enough to show that $f(c) \le 0$ and $f(c/\sqrt{n}) \le 0$. We have

$$f(c) = nb^2 - 2(n-1)bc + (n-2)c^2 = (b-c)[nb - (n-2)c] \le 0.$$

Since $b \le c$ and $nb \ge (n-2)c$ for $n = 3$ and $n = 4$, we have $f(c) \le 0$. The inequality $f(c/\sqrt{n}) \le 0$ is equivalent to

$$nb^2 - \frac{2(n-1)}{\sqrt{n}}bc - c^2 \le 0.$$

Since $c \ge b$, it is enough to show that

$$n - \frac{2(n-1)}{\sqrt{n}} - 1 \le 0,$$

which is true for $n = 3$ and $n = 4$.

The proof is completed. The equality occurs when all a_i and b_i are equal.

□

P 3.174. *Let a, b, c and x, y, z be positive real numbers such that*

$$(a+b+c)(x+y+z) = (a^2+b^2+c^2)(x^2+y^2+z^2) = 4.$$

Prove that

$$abcxyz < \frac{1}{36}.$$

(Vasile Cîrtoaje, 1997)

Solution. Using the AM-GM inequality, we have

$$4(ab+bc+ca)(xy+yz+zx) =$$

$$= [(a+b+c)^2 - (a^2+b^2+c^2)][(x+y+z)^2 - (x^2+y^2+z^2)]$$

$$= 20 - (a+b+c)^2(x^2+y^2+z^2) - (x+y+z)^2(a^2+b^2+c^2)$$

$$\le 20 - 2(a+b+c)(x+y+z)\sqrt{(a^2+b^2+c^2)(x^2+y^2+z^2)} = 4,$$

hence

$$(ab+bc+ca)(xy+yz+zx) \le 1.$$

On the other hand, multiplying the well-known inequalities

$$(ab+bc+ca)^2 \ge 3abc(a+b+c)$$

and

$$(xy+yz+zx)^2 \ge 3xyz(x+y+z),$$

we get

$$(ab+bc+ca)^2(xy+yz+zx)^2 \ge 36abcxyz,$$

hence

$$1 \ge (ab+bc+ca)^2(xy+yz+zx)^2 \ge 36abcxyz.$$

In order to have $36abcxyz = 1$, the following relations are necessary:

$$(ab+bc+ca)^2 = 3abc(a+b+c)$$

and

$$(xy+yz+zx)^2 = 3xyz(x+y+z).$$

These relations imply $a = b = c$ and $x = y = z$, which contradict the hypothesis

$$(a+b+c)(x+y+z) = (a^2+b^2+c^2)(x^2+y^2+z^2) = 4.$$

Consequently, we have $abcxyz < \dfrac{1}{36}$.

Remark. The following sharper result holds (*Vasile Cirtoaje* and *Vo Quoc Ba Can*, 2008):

- *If a, b, c and x, y, z are positive real numbers such that*

$$(a+b+c)(x+y+z)=(a^2+b^2+c^2)(x^2+y^2+z^2)=4,$$

then

$$abcxyz \le \frac{16}{729},$$

with equality for $(a,b,c)=\left(\frac{1}{3},\frac{1}{3},\frac{4}{3}\right)$ (or any cyclic permutation) and $(x,y,z)=\left(\frac{1}{3},\frac{1}{3},\frac{4}{3}\right)$ (or any cyclic permutation).

□

P 3.175. *Let $a_1, a_2, \cdots, a_n$ $(n \ge 3)$ be positive real numbers such that*

$$a_1+a_2+\cdots+a_n=a_1^2+a_2^2+\cdots+a_n^2=n-1.$$

Prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n} \ge \frac{n^2(2n-3)}{2(n-1)(n-2)}.$$

(Vasile Cîrtoaje, 2010)

Solution. By the Cauchy-Schwarz inequality, we have

$$n-1=a_1^2+a_2^2+\cdots+a_n^2 \ge \frac{(a_1+a_2+\cdots+a_{n-1})^2}{n-1}+a_n^2$$
$$=\frac{(n-1-a_n)^2}{n-1}+a_n^2,$$

which provides

$$a_n \le \frac{2(n-1)}{n}.$$

Similarly, $a_i \le 2(n-1)/n$ for all i. The hint for proving the given inequality is to apply the Cauchy-Schwarz inequality after we made the numerators nonnegative and as small as possible. So, since $2n-2-na_i \ge 0$, we have

$$\sum \frac{1}{a_1}=\sum\left(\frac{1}{a_1}-\frac{n}{2n-2}\right)+\frac{n^2}{2n-2}$$
$$=\frac{1}{2(n-1)}\sum\frac{2n-2-na_1}{a_1}+\frac{n^2}{2n-2}$$
$$\ge \frac{1}{2(n-1)}\cdot\frac{\left[\sum(2n-2-na_1)\right]^2}{\sum a_1(2n-2-na_1)}+\frac{n^2}{2n-2}$$
$$=\frac{1}{2(n-1)}\cdot\frac{\left[n(2n-2)-n\sum a_1\right]^2}{(2n-2)\sum a_1-n\sum a_1^2}+\frac{n^2}{2n-2}$$
$$=\frac{1}{2(n-1)}\cdot\frac{n^2(n-1)^2}{(n-1)(n-2)}+\frac{n^2}{2n-2}=\frac{n^2(2n-3)}{2(n-1)(n-2)},$$

from where the conclusion follows. The equality holds for $a_1 = a_2 = \cdots = a_{n-1} = 1-2/n$ and $a_n = 2-2/n$ (or any cyclic permutation).

□

P 3.176. *Let $a_1, a_2, \cdots, a_n$ be positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$n^2\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}-n\right) \ge 4(n-1)(a_1^2+a_2^2+\cdots+a_n^2-n).$$

(Vasile Cîrtoaje, 2004)

Solution. From

$$\frac{1}{n}(a_1+a_2+\cdots+a_n)^2 \le a_1^2+a_2^2+\cdots+a_n^2 < (a_1+a_2+\cdots+a_n)^2,$$

it follows that

$$n \le a_1^2+a_2^2+\cdots+a_n^2 < n^2.$$

Thus, we can use the substitution

$$a_1^2+a_2^2+\cdots+a_n^2 = n+n(n-1)t^2,$$

where $0 \le t < 1$. On the other hand, from

$$a_1^2+a_2^2+\cdots+a_n^2 \ge a_1^2+\frac{(a_2+\cdots+a_n)^2}{n-1} = a_1^2+\frac{(n-a_1)^2}{n-1},$$

we get

$$n+n(n-1)t^2 \ge a_1^2+\frac{(n-a_1)^2}{n-1},$$

which involves $1-(n-1)t \le a_1 \le 1+(n-1)t$; similarly, we get

$$1-(n-1)t \le a_i \le 1+(n-1)t$$

for any i. We will apply now the Cauchy-Schwarz inequality after we made the numerators nonnegative and as small as possible. Since

$$\begin{aligned}\sum\frac{1}{a_1} &= \sum\frac{1}{1+(n-1)t}+\sum\left[\frac{1}{a_1}-\frac{1}{1+(n-1)t}\right]\\ &= \frac{n}{1+(n-1)t}+\frac{1}{1+(n-1)t}\sum\frac{1+(n-1)t-a_1}{a_1}\end{aligned}$$

and

$$a_1^2+a_2^2+\cdots+a_n^2-n = n(n-1)t^2,$$

we can write the desired inequality as

$$\sum \frac{1+(n-1)t-a_1}{a_1} \ge n(n-1)t + \frac{4(n-1)^2t^2[1+(n-1)t]}{n}.$$

By virtue of the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum \frac{1+(n-1)t-a_1}{a_1} &\ge \frac{\left[\sum(1+(n-1)t-a_1)\right]^2}{\sum a_1(1+(n-1)t-a_1)}\\ &= \frac{\left[n+n(n-1)t-\sum a_1\right]^2}{(1+(n-1)t)\sum a_1 - \sum a_1^2}\\ &= \frac{n(n-1)t}{1-t},\end{aligned}$$

Therefore, it suffices to prove that

$$\frac{n(n-1)t}{1-t} \ge n(n-1)t + \frac{4(n-1)^2t^2[(1+(n-1)t])}{n}.$$

This inequality is true if

$$4(n-1)(1-t)[1+(n-1)t] \le n^2.$$

Indeed,

$$4(n-1)(1-t)[1+(n-1)t] \le [(n-1)(1-t)+1+(n-1)t]^2 = n^2.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = \frac{n}{2}$ and $a_2 = \cdots = a_n = \frac{n}{2n-2}$ (or any cyclic permutation).

□

P 3.177. *If $a_1, a_2, \ldots, a_n$ are positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n, \qquad a_2, a_3, \ldots, a_n \ge 1,$$

then

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \ge a_1^2 + a_2^2 + \cdots + a_n^2.$$

(Vasile C., 2021)

Solution. We will use the induction method. For $n = 2$, we need to show that $a_1 + a_2 = 2$ involves

$$\frac{1}{a_1} + \frac{1}{a_2} \ge a_1^2 + a_2^2.$$

This is equivalent to

$$(a_1a_2-1)^2 \ge 0.$$

Next, consider $n \ge 3$ and assume that

$$a_1 \le 1 \le a_2 \le \cdots \le a_n.$$

For fixed a_1 and $a_3, \ldots, a_{n-1}$, write the required inequality as $f(a_2) \ge 0$, where

$$f(a_2) = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} - (a_1^2 + a_2^2 + \cdots + a_n^2), \qquad a_n = n - a_1 - a_2 - \cdots - a_n.$$

We will show that

$$f(a_2) \ge f(1) \ge 0.$$

Since

$$2a_2^2a_n^2 \ge 2a_n^2 \ge 2a_n \ge a_2 + a_n,$$

we have

$$f'(a_2) = \frac{-1}{a_2^2} + \frac{1}{a_n^2} - 2(a_2 - a_n) = (a_n - a_2)\left(2 - \frac{a_2 + a_n}{a_2^2a_n^2}\right) \ge 0,$$

$f(a_2)$ is increasing, hence $f(a_2) \ge f(1)$. The inequality $f(1) \ge 0$ has the form

$$\frac{1}{a_1} + \frac{1}{a_3} + \cdots + \frac{1}{a_n} \ge a_1^2 + a_3^2 + \cdots + a_n^2.$$

where $a_n = n - 1 - a_1 - a_3 - \cdots - a_{n-1}$ and $a_1 \le 1 \le a_3 \le \cdots \le a_n$. Clearly, this is true by the induction hypothesis.

The equality occurs for $a_1 = a_2 = \cdots = a_n = 1$.

□

P 3.178. *Let $a_1, a_2, \cdots, a_n$ be nonnegative real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$(n+1)(a_1^2 + a_2^2 + \cdots + a_n^2) \ge n^2 + a_1^3 + a_2^3 + \cdots + a_n^3.$$

(Vasile Cîrtoaje, 2002)

First Solution. If $a_1 = a_2 = \cdots = a_n$, then the equality holds. Otherwise, as in the preceding proof, we will use the substitution

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n + n(n-1)t^2, \qquad 0 < t \le 1;$$

in addition, we have

$$1 - (n-1)t \le a_i \le 1 + (n-1)t, \qquad i = 1, 2, \ldots, n.$$

From the Cauchy-Schwarz inequality

$$\sum[1+(n-1)t-a_1]a_1^2 \geq \frac{[\sum(1+(n-1)t-a_1)a_1]^2}{\sum[1+(n-1)t-a_1]} = n(n-1)t(1-t)^2,$$

we get

$$\begin{aligned}\sum a_1^3 &\leq [1+(n-1)t]\sum a_1^2 - n(n-1)t(1-t)^2\\ &= n[(n-1)(n-2)t^3+3(n-1)t^2+1].\end{aligned}$$

Therefore, it suffices to show that

$$(n+1)[n+n(n-1)t^2] \geq n^2 + n[(n-1)(n-2)t^3+3(n-1)t^2+1],$$

which is equivalent to the obvious inequality

$$n(n-1)(n-2)t^2(1-t) \geq 0.$$

For $n=2$, the original inequality is an identity. For $n \geq 3$, the equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = n$ and $a_2 = \cdots = a_n = 0$ (or any cyclic permutation).

Second Solution. Assume that

$$a_1 \geq a_2 \geq \cdots \geq a_n.$$

Replacing n^2 by $n(a_1+a_2+\cdots+a_n)$, the desired inequality becomes as follows

$$\sum[(n+1)a_1^2 - na_1 - a_1^3] \geq 0,$$

$$\sum(a_1-1)(na_1-a_1^2) \geq 0.$$

Since

$$a_1 - 1 \geq a_2 - 1 \geq \cdots \geq a_n - 1$$

and

$$na_1 - a_1^2 \geq na_2 - a_2^2 \geq \cdots \geq na_n - a_n^2,$$

we apply Chebyshev's inequality to get

$$n\sum(a_1-1)(na_1-a_1^2) \geq \left[\sum(a_1-1)\right]\left[\sum(na_1-a_1^2)\right] = 0.$$

□

P 3.179. *Let $a_1, a_2, \cdots, a_n$ be nonnegative real numbers such that*

$$a_1+a_2+\cdots+a_n = n.$$

Prove that

$$(n-1)(a_1^3+a_2^3+\cdots+a_n^3)+n^2 \geq (2n-1)(a_1^2+a_2^2+\cdots+a_n^2).$$

(Vasile Cîrtoaje, 2002)

Solution. For $a_1 = a_2 = \cdots = a_n$, the equality holds. Otherwise, as in the proof of P 3.171, we will use the substitution

$$a_1^2 + a_2^2 + \cdots + a_n^2 = n + n(n-1)t^2, \qquad 0 < t \le 1;$$

in addition, for any i, we have

$$1 - (n-1)t \le a_i \le 1 + (n-1)t.$$

By the Cauchy-Schwarz inequality, we have

$$\sum [a_1 - 1 + (n-1)t]a_1^2 \ge \frac{[\sum (a_1 - 1 + (n-1)t)a_1]^2}{\sum [a_1 - 1 + (n-1)t]} = n(n-1)t(t+1)^2,$$

which yields

$$\sum a_1^3 \ge n(n-1)t(t+1)^2 + [1-(n-1)t]\sum a_1^2$$
$$= n[1 + 3(n-1)t^2 - (n-1)(n-2)t^3].$$

Therefore, it suffices to show that

$$(n-1)n[1 + 3(n-1)t^2 - (n-1)(n-2)t^3] + n^2 \ge (2n-1)[n + n(n-1)t^2],$$

which is equivalent to the obvious inequality

$$n(n-1)(n-2)t^2[1-(n-1)t] \ge 0.$$

The equality holds for $a_1 = a_2 = \cdots = a_n = 1$, and also for $a_1 = 0$ and $a_2 = \cdots = a_n = \dfrac{n}{n-1}$ (or any cyclic permutation).

□

P 3.180. *Let $a_1, a_2, \ldots, a_n$ ($n \ge 3$) be positive real numbers such that*

$$a_1 + a_2 + \cdots + a_n = n.$$

Prove that

$$a_1^2 + a_2^2 + \cdots + a_n^2 - n \ge \frac{n}{n-1}(1 - a_1a_2\cdots a_n).$$

Solution. According to Remark 1 from P 3.57, for $a_1^2 + a_2^2 + \cdots + a_n^2 = constant$, the product $a_1a_2\cdots a_n$ is minimal when one of $a_1, a_2, \ldots, a_n$ is zero or $n-1$ numbers of $a_1, a_2, \ldots, a_n$ are equal. Therefore, it suffices to consider these cases.

Case 1: $a_1 = 0$. We need to show that $a_2 + a_3 + \cdots + a_n = n$ involves

$$a_2^2 + a_3^2 + \cdots + a_n^2 \ge \frac{n^2}{n-1}.$$

Indeed, by the Cauchy-Schwarz inequality, we have

$$a_2^2+a_3^2+\cdots+a_n^2 \ge \frac{1}{n-1}(a_2+a_3+\cdots+a_n)^2 = \frac{n^2}{n-1}.$$

Case 2: $a_2=a_3=\cdots=a_n$. Setting $a_1=x$ and $a_2=y$, we need to show that

$$x+(n-1)y=n$$

involves

$$x^2+(n-1)y^2-n+\frac{n}{n-1}\left(xy^{n-1}-1\right)\ge 0.$$

By Bernoulli's inequality, we have

$$y^{n-1}=\left(1+\frac{1-x}{n-1}\right)^{n-1}\ge 1+(1-x)=2-x.$$

Therefore, it suffices to prove that

$$x^2+(n-1)y^2-n+\frac{n}{n-1}[x(2-x)-1]\ge 0,$$

which is an identity. The equality holds for $a_1=a_2=\cdots=a_n=1/n$, and also for $a_1=0$ and $a_2=\cdots=a_n=\dfrac{n}{n-1}$ (or any cyclic permutation).

□

P 3.181. *If $a_1,a_2,\ldots,a_n$ are positive numbers such that $a_1+a_2+\cdots+a_n=n$ and*

$$a_1a_2\cdots a_n \le \frac{1}{(n-1)^{n-2}},$$

then

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\ge a_1^2+a_2^2+\cdots+a_n^2.$$

(Vasile Cîrtoaje, 2018)

Solution. According to Remark 3 from P 3.58, the following statement is valid:

- *If $a_1,a_2,\ldots,a_n$ are positive real numbers so that*

$$a_1+a_2+\cdots+a_n=n\ , \qquad a_1a_2\cdots a_n=constant\ , \qquad a_1\ge a_2\ge\cdots\ge a_n\ ,$$

then the sums

$$\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}$$

and

$$-(a_1^2 + a_2^2 + \cdots + a_n^2)$$

are minimal when $a_1 \ge a_2 = \cdots = a_n$.

Thus, it is enough to consider the case $a_1 \ge a_2 = \cdots = a_n$, that means to show that

$$\frac{1}{x} + \frac{n-1}{y} \ge x^2 + (n-1)y^2$$

for

$$x + (n-1)y = n, \qquad xy^{n-1} \le \frac{1}{(n-1)^{n-2}}, \quad n > x \ge 1 \ge y > 0.$$

Write the inequality as follows:

$$\frac{(n-1)(1-y^3)}{y} \ge \frac{x^3-1}{x},$$

$$\frac{(n-1)(1-y)(1+y+y^2)}{y} \ge \frac{(x-1)(x^2+x+1)}{x},$$

$$\frac{(n-1)(1-y)(1+y+y^2)}{y} \ge \frac{(n-1)(1-y)(x^2+x+1)}{x},$$

$$(1-y)\left(\frac{1}{y} + y - x - \frac{1}{x}\right) \ge 0,$$

$$(1-y)(x-y)(1-xy) \ge 0,$$

$$(1-y)^2(1-xy) \ge 0.$$

So, we need to show that $1-xy \ge 0$. For $x = 1$, which implies $y = 1$, the inequality is an equality. Consider further $x > 1$. Since

$$1 - xy = 1 - \frac{(n-x)x}{n-1} = \frac{(x-1)(x-n+1)}{n-1},$$

we need to prove that $x \ge n-1$. Write the hypothesis $xy^{n-1} \le \dfrac{1}{(n-1)^{n-2}}$ as $f(x) \ge 0$, where

$$f(x) = n - 1 - x(n-x)^{n-1}.$$

From

$$f'(x) = n(n-x)^{n-2}(x-1) > 0,$$

it follows that f is strictly increasing. Since $f(n-1) = 0$, the hypothesis $f(x) \ge 0$ involves $x \ge n-1$.

The inequality is an equality for $a_1 = n-1$ and $a_2 = \cdots = a_n = \dfrac{1}{n-1}$ (or any cyclic permutation).

□

P 3.182. *If $a_1, a_2, \ldots, a_n$ are nonnegative real numbers such that*

$$a_1 \le a_2 \le \cdots \le a_n,$$

then

$$\frac{a_1 + a_2 + \cdots + a_n}{n} - \sqrt[n]{a_1 a_2 \cdots a_n} \le \left(1 - \frac{1}{n}\right)\left(\sqrt{a_n} - \sqrt{a_1}\right)^2.$$

(Vasile Cîrtoaje, 2005)

Solution. Based on Lemma below (or Remark from P 2.104 applied to the function $-\ln x$), it suffices to consider the case when $k-1$ of $a_2, a_3, \ldots, a_{n-1}$ are equal to a_1, and the other $n-1-k$ of $a_2, a_3, \ldots, a_{n-1}$ are equal to a_n, where $k \in \{1, 2, \ldots, n-1\}$. The required inequality becomes

$$ka_1 + (n-k)a_n - n\sqrt[n]{a_1^k a_n^{n-k}} \le (n-1)\left(\sqrt{a_n} - \sqrt{a_1}\right)^2,$$

$$(n-k-1)a_1 + (k-1)a_n + n\sqrt[n]{a_1^k a_n^{n-k}} \ge (2n-2)\sqrt{a_1 a_n}.$$

Clearly, this inequality follows from the AM-GM inequality applied to $2n-2$ numbers. The equality occurs when $a_1 = a_2 = \cdots = a_n$, and also when $a_1 = 0$ and $a_2 = a_3 = \cdots = a_n$.

Lemma (*Vasile Cîrtoaje*, 1990). *Let $a_1, a_2, \ldots, a_n$ be nonnegative real numbers such that*

$$a_1 \le a_2 \le \cdots \le a_n.$$

For fixed a_1 and a_n, the expression

$$E = a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n}$$

is maximum when $a_2, a_3, \ldots, a_{n-1} \in \{a_1, a_n\}$.

Proof. For fixed $a_1, a_3, \ldots, a_n$, define the function

$$f(a_2) = a_1 + a_2 + \cdots + a_n - n\sqrt[n]{a_1 a_2 \cdots a_n}.$$

Denote

$$b_2 = \sqrt[n-1]{a_1 a_3 \cdots a_n}.$$

From

$$f'(a_2) = 1 - \sqrt[n]{\frac{a_1 a_3 \cdots a_n}{a_2^{n-1}}} = 1 - \left(\frac{b_2}{a_2}\right)^{(n-1)/n},$$

it follows that $f'(a_2) \ge 0$ for $x_2 \in [a_1, b_2]$, and $f'(a_2) \le 0$ for $x_2 \in [b_2, a_n]$, $f(a_2)$ is increasing on $[a_1, b_2]$ and decreasing on $[b_2, a_n]$, therefore $f(a_2)$ is maximal when $a_2 \in \{a_1, a_n\}$. Similarly, the expression E is maximal when $x_3, \ldots, x_{n-1} \in \{a_1, a_n\}$.

□

P 3.183. *Let $a_1, a_2, \ldots, a_n$ ($n \geq 3$) be positive real numbers such that*

$$a_1 \leq a_2 \leq \cdots \leq a_n,$$

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) = k.$$

(a) If $n^2 < k \leq n^2 + \dfrac{i(n-i)}{2}$, $i \in \{2, 3, \cdots, n-1\}$, then a_{i-1}, a_i and a_{i+1} are the lengths of the sides of a non-degenerate or degenerate triangle;

(b) If $n^2 < k \leq \alpha_n$, where $\alpha_n = \dfrac{9n^2}{8}$ for even n, and $\alpha_n = \dfrac{9n^2-1}{8}$ for odd n, then there exist three numbers a_i which are the lengths of the sides of a non-degenerate or degenerate triangle.

(Vasile Cîrtoaje, 2010)

Solution. From the AM-HM inequality, we have

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2,$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$. Therefore, the hypothesis $k > n^2$ in (a) and (b) involves $a_1 < a_n$.

(a) For the sake of contradiction, assume that a_{i-1}, a_i and a_{i+1} are not the lengths of the sides of a triangle; that is,

$$a_{i+1} > a_{i-1} + a_i.$$

Let us denote

$$x = \frac{a_1 + \cdots + a_{i-1}}{i-1},$$

$$y = \frac{a_{i+1} + \cdots + a_n}{n-i},$$

$$A(x, y) = (i-1)x + a_i + (n-i)y,$$

$$B(x, y) = \frac{i-1}{x} + \frac{1}{a_i} + \frac{n-i}{y},$$

$$f(x, y) = A(x, y)B(x, y).$$

We have

$$x \leq a_{i-1} \leq a_i < a_{i+1} \leq y, \quad x < y,$$

$$A(x, y) = a_1 + a_2 + \cdots + a_n,$$

$$B(x, y) \leq \left(\frac{1}{a_1} + \cdots + \frac{1}{a_{i-1}}\right) + \frac{1}{a_i} + \left(\frac{1}{a_{i+1}} + \cdots + \frac{1}{a_n}\right) = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n},$$

$$f(x, y) \leq (a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) = k.$$

On the other hand, we claim that

$$f(x,y) > f(a_{i-1}, a_{i-1}+a_i).$$

This inequality is equivalent to

$$[a_{i-1}y-(a_{i-1}+a_i)x][(a_{i-1}+a_i)y-a_{i-1}x] > 0,$$

and is true since $y \ge a_{i+1} > a_{i-1}+a_i$ and $x \le a_{i-1}$ imply

$$a_{i-1}y-(a_{i-1}+a_i)x > a_{i-1}(a_{i-1}+a_i)-(a_{i-1}+a_i)a_{i-1} = 0.$$

Then, we have

$$\begin{aligned} k &\ge f(x,y) > f(a_{i-1}, a_{i-1}+a_i) \\ &= [(n-1)a_{i-1}+(n-i+1)a_i]\left(\frac{i-1}{a_{i-1}}+\frac{1}{a_i}+\frac{n-i}{a_{i-1}+a_i}\right) \\ &\ge n^2+\frac{i(n-i)}{2}, \end{aligned}$$

which contradicts the hypothesis $k \le n^2+\dfrac{i(n-i)}{2}$. Setting $a_{i-1}=1$ and denoting $a_i=t$, $t \ge 1$, the last inequality becomes

$$\frac{[n-1+(n-i+1)t][1+nt+(i-1)t^2]}{t(1+t)} \ge n^2+\frac{i(n-i)}{2},$$

or

$$(t-1)(Ct^2+Dt+E) \ge 0,$$

where

$$C = 2(i-1)(n-i+1), \quad D = (i-2)(n-i), \quad E = -2(n-1).$$

This is true, since

$$Ct^2+Dt+E \ge C+D+E = 3(i-2)(n-i) \ge 0.$$

(b) We apply the result of (a). If n is even, then $n^2+\dfrac{i(n-i)}{2}$ attains its maximum $\dfrac{9n^2}{8}$ for $i=\dfrac{n}{2}$. If n is odd, then $n^2+\dfrac{i(n-i)}{2}$ attains its maximum $\dfrac{9n^2-1}{8}$ for $i=\dfrac{n\pm 1}{2}$.

□

P 3.184. *Let $a_1, a_2, \ldots, a_n$ $(n \geq 3)$ be positive real numbers such that*

$$a_1 \leq a_2 \leq \cdots \leq a_n,$$

$$(a_1 + a_2 + \cdots + a_n)^2 = k(a_1^2 + a_2^2 + \cdots + a_n^2).$$

(a) If $\dfrac{(2n-i)^2}{4n-3i} \leq k < n$, $i \in \{2, 3, \cdots, n-1\}$, then a_{i-1}, a_i and a_{i+1} are the lengths of the sides of a non-degenerate or degenerate triangle;

(b) If $\dfrac{8n+1}{9} \leq k < n$, then there exist three numbers a_i which are the lengths of the sides of a non-degenerate or degenerate triangle.

(Vasile Cîrtoaje, 2010)

Solution. From the the Cauchy-Schwarz inequality, we have

$$(a_1 + a_2 + \cdots + a_n)^2 \leq n(a_1^2 + a_2^2 + \cdots + a_n^2),$$

with equality if and only if $a_1 = a_2 = \cdots = a_n$. Therefore, the hypothesis $k < n$ in (a) and (b) involves $a_1 < a_n$.

(a) For the sake of contradiction, assume that a_{i-1}, a_i and a_{i+1} are not the lengths of the sides of a triangle; that is,

$$a_{i+1} > a_{i-1} + a_i.$$

Let us denote

$$x = \frac{a_1 + \cdots + a_{i-1}}{i-1},$$

$$y = \frac{a_{i+1} + \cdots + a_n}{n-i},$$

$$A(x,y) = (i-1)x + a_i + (n-i)y,$$

$$B(x,y) = (i-1)x^2 + a_i^2 + (n-i)y^2,$$

$$f(x,y) = \frac{A^2(x,y)}{B(x,y)}.$$

We have

$$x \leq a_{i-1} \leq a_i < a_{i+1} \leq y, \quad x < y,$$

$$A(x,y) = a_1 + a_2 + \cdots + a_n,$$

$$B(x,y) \leq (a_1^2 + \cdots + a_{i-1}^2) + a_i^2 + (a_{i+1}^2 + \cdots + a_n^2) = a_1^2 + a_2^2 + \cdots + a_n^2,$$

$$f(x,y) \geq \frac{(a_1 + a_2 + \cdots + a_n)^2}{a_1^2 + a_2^2 + \cdots + a_n^2} = k.$$

On the other hand, from

$$\frac{\partial f(x,y)}{\partial x} = \frac{2(i-1)AC}{B^2} > 0$$

and

$$\frac{\partial f(x,y)}{\partial y} = \frac{2(n-i)AD}{B^2} < 0,$$

where

$$C = a_i(a_i - x) + (n-i)y(y-x) > 0,$$

$$D = a_i(a_i - y) + (i-1)x(x-y) < 0,$$

it follows that $f(x,y)$ is strictly increasing with respect to x and strictly decreasing with respect to y. Then, since $x \le a_{i-1}$ and $y \ge a_{i+1} > a_{i-1} + a_i$, we have

$$f(x,y) < f(a_{i-1}, a_{i-1} + a_i).$$

This involves

$$k < f(a_{i-1}, a_{i-1} + a_i),$$

hence

$$\begin{aligned} k &< \frac{[(i-1)a_{i-1} + a_i + (n-i)(a_{i-1}+a_i)]^2}{(i-1)a_{i-1}^2 + a_i^2 + (n-i)(a_{i-1}+a_i)^2} \\ &= \frac{[(n-1)a_{i-1} + (n-i+1)a_i]^2}{(n-1)a_{i-1}^2 + 2(n-i)a_{i-1}a_i + (n-i+1)a_i^2} \\ &\le \frac{(2n-i)^2}{4n-3i}, \end{aligned}$$

which contradicts the hypothesis $k \ge \dfrac{(2n-i)^2}{4n-3i}$. Setting $a_{i-1} = 1$ and denoting $a_i = t$, $t \ge 1$, the last inequality becomes

$$\frac{[n-1+(n-i+1)t]^2}{n-1+2(n-i)t+(n-i+1)t^2} \le \frac{(2n-i)^2}{4n-3i},$$

or

$$(t-1)(Et-F) \ge 0,$$

where

$$E = (n-i+1)[(3i-4)n - 2i^2 + 3i],$$

$$F = (n-1)[(4-i)n + i^2 - 3i].$$

Since

$$E = (n-i+1)[(3i-4)(n-i-1) + i^2 + 2i - 4] > 0,$$

we get

$$Et - F \ge E - F = 2(i-2)(n-i)(2n-i) \ge 0.$$

(b) According to (a), it suffices to show that there exists $i \in \{2, 3, \cdots, n-1\}$ such that

$$\frac{(2n-i)^2}{4n-3i} \le \frac{8n+1}{9};$$

that is,

$$4n-3i \ge (2n-3i)^2.$$

Since one of the numbers $\frac{2n-1}{3}$, $\frac{2n}{3}$ and $\frac{2n+1}{3}$ is integer, it suffices to prove this inequality for all $i \in \left\{\frac{2n-1}{3}, \frac{2n}{3}, \frac{2n+1}{3}\right\}$. Indeed, for these cases, the inequality reduces to $2n \ge 0$, $2n \ge 0$ and $2n-2 \ge 0$, respectively.

Appendix A

Bibliography

[1] Andreescu T., Cîrtoaje V., Dospinescu G., Lascu M., *Old and New Inequalities*, GIL Publishing House, 2004.

[2] Bin X., Boreico I., Can V.Q.B., Bulj A., Lascu M., *Opympiad Inequalities*, GIL Publishing House, 2015.

[3] Bin X., Boreico I., Can V.Q.B., Cîrtoaje V., Lascu M., *An Introduction to Inequalities*, GIL Publishing House, 2015.

[4] Can V.Q.B., Pohoaţă C., *Old and New Inequalities*, GIL Publishing House, 2008.

[5] Can V.Q.B., Anh T. Q., *Su Dung Phuong Phap Cauchy-Schwarz De Chung Minh Bat Dang Thuc*, Nha Xuat Ban Dai Hoc Su Pham, 2010.

[6] Cîrtoaje V., *Asupra unor inegalitati cu restrictii*, Revista Matematica din Timisoara, Nr. 1, 1990, 3-7.

[7] Cîrtoaje V., *Two Generalizations of Popoviciu's Inequality*, Crux Mathematicorum, Issue 5, 2005, 313 - 318.

[8] Cîrtoaje V., *A Generalization of Jensen's Inequality*, Gazeta Matematica-A, Nr. 2, 2005, 124-138.

[9] Cîrtoaje V., *The Equal Variable Method*, Journal of Inequalities In Pure and Applied Mathematics, Volume 8, Issue 1, 2007.

[10] Cîrtoaje V., *Arithmetic Compensation Method*, Mathematical Reflections, 2, 2006, 1 - 5.

[11] Cîrtoaje V., *Algebraic Inequalities-Old and New Methods*, GIL Publishing House, 2006.

[12] Cîrtoaje V., *Four Applications of RCF and LCF Theorems*, Mathematical Reflections, 1, 2007.

[13] Cîrtoaje V., *On Jensen Type Inequalities with Ordered Variables*, Journal of Inequalities In Pure and Applied Mathematics, Volume 9, Issue 1, 2008.

[14] Cîrtoaje V., *The Proof of Three Open Inequalities*, Crux Mathematicorum, Volume 34, Issue 4, 2008, 231-238.

[15] Cîrtoaje V., Can V. Q. B., Anh T. Q., *Inequalities with Beautiful Solutions*, GIL Publishing House, 2009.

[16] Cîrtoaje V., *On Some Inequalities with Power-Exponential Functions,* Journal of Inequalities In Pure and Applied Mathematics, Volume 10, Issue 1, 2009.

[17] Cîrtoaje V., *On the Cyclic Homogeneous Polynomial Inequalities of Degree Four*, Journal of Inequalities in Pure and Applied Mathematics, Volume 10, Issue 3, 2009.

[18] Cîrtoaje V., *The Best Lower Bound Depended on Two Fixed Variables for Jensen's Inequality with Ordered Variables*, Journal of Inequalities and Applications, Volume 2010.

[19] Cîrtoaje V., Baiesu A., *An Extension of Jensen's discrete inequality to half convex functions*, Journal of Inequalities and Applications, Volume 2011.

[20] Cîrtoaje V., *Proofs of Three Open Inequalities with Power-Exponential Functions*, Journal of Nonlinear Sciences and Applications, Volume 4, no. 2, 2011.

[21] Cîrtoaje V., *The Best Upper Bound for Jensen's Inequality*, Australian Journal of Mathematical Analysis and Aplications, Volume 7, Issue 2, Art. 22, 2011.

[22] Cîrtoaje V., Zhou Y., *Necessary and Sufficient Conditions for Cyclic Homogeneous Polynomial Inequalities of Degree Four in Real Variables*, The Australian Journal of Mathematical Analysis and Applications, Volume 9, Issue 1, Article 15, 2012.

[23] Cîrtoaje V., Zhou Y., *Some Strong Sufficient Conditions for Cyclic Homogeneous Polynomial Inequalities of Degree Four in Real Variables*, Journal of Nonlinear Analysis and Applications, Volume 2012.

[24] Cîrtoaje V., Can V.Q.B., *On Some Cyclic Homogeneous Polynomial Inequalities of Degree Four in Real Variables*, International Journal of Pure and Applied Mathematics, Volume 80, No. 3, 2012.

[25] Cîrtoaje V., *On the Arithmetic Compensation Methods*, International Journal of Pure and Applied Mathematics, Volume 80, No. 3, 2012.

[26] Cîrtoaje V., *Necessary and Sufficient Conditions for Symmetric Homogeneous Polynomial Inequalities of Degree Four and Six in Real Variables*, Journal of Nonlinear Science and Applications, Volume 5, Issue 5, 2012.

[27] Cîrtoaje V., *Necessary and Sufficient Conditions for Symmetric Homogeneous Polynomial Inequalities in Nonnegative Real Variables*, Mathematical Inequalities and Applications, Volume 16, No. 2, 2013.

[28] Cîrtoaje V., *The Best Lower Bound for Jensen's Inequality with three fixed ordered variables Journal*, Banach Journal of Mathematical Analysis, Volume 7, Issue 1, 2013.

[29] Cîrtoaje V., *An Extension of Jensen's discrete inequality to partially convex functions*, Journal of Inequalities and Applications, Volume 2013:54.

[30] Cîrtoaje V., *A Strong Method for Symmetric Homogeneous Polynomial Inequalities of Degree Six in Nonnegative Real Variables*, British Journal of Mathematical and Computers Science, 4(5), 2014.

[31] Cîrtoaje V., *On the Equal Variables Method Applied to Real Variables*, Creative Mathematics and Informatics, No. 2, 2015.

[32] Cîrtoaje V., *Three Extensions of HCF and PCF Theorems*, Advanced Inequalities and Applications, 2016-14, 2016.

[33] Cîrtoaje V., *Extensions of Jensen's Discrete Inequality with Ordered Variables to Half and Partially Convex Functions*, Journal of Inequalities and Special Functions, 8(3), 2017.

[34] Cvetkovski Z., *Inequalities: Theorems, Techniques and Selected Problems*, Springer-Verlag Berlin Heidelberg, 2012.

[35] Drimbe M.O., *Inegalitati - Idei si Metode*, GIL Publishing House, 2003.

[36] Engel A., *Problem-Solving Strategies*, Springer, 1998.

[37] Hung P. K., *Secrets in Inequalities*, Volume 1: Basic Inequalities, GIL Publishing House, 2007.

[38] Hung P. K., *Secrets in Inequalities*, Volume 2: Advanced Inequalities, GIL Publishing House, 2008.

[39] Lee Hojoo, *Topics in Inequalities-Theorems and Techniques*, http://www.artofproblemsolving.com/Forum/viewtopic.php?t=602698

[40] Littlewood G. H., Polya, J. E., *Inequalities*, Cambridge University Press, 1967.

[41] Mitrinovič D. S., Pecarič J. E., Fink A. M., *Classical and New Inequalities in Analysis*, Kluwer, 1993.

[42] Panaitopol L., Băndilă V., Lascu M., *Inegalităţi*, GIL Publishing House, 1995.

[43] Zhou Y., Cîrtoaje V., *Some strong sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in nonnegative variables*, Journal of Nonlinear Science and Applications, Volume 6, Issue 2, 2013.